Introduction to Riemann Surfaces and Algebraic Curves

201.2.5101 Fall 2025 (Dmitry Kerner)

Homework 7.

Submission date: 22.12.2025.

Questions to submit: 1.d. 2.e. 3.a. 3.d. 4.c. 4.d.

(Either typed or in readable handwriting and scanned in readable resolution.)

1.a. For any $\mathcal{O}$-submanifold $X \subset \mathbb{C}^N$ verify: $TX \subset X \times \mathbb{C}^N_v$ is an $\mathcal{O}$-submanifold, and the fibres of the projection $TX \to X$ are complex subspaces of $\mathbb{C}^N_v$.

b. Prove: TX is ($\mathcal{O}$) trivial iff there exist analytic vector fields $\xi_1, \dots, \xi_n$ on X whose values $\xi_1|_{x_o}, \dots, \xi_n|_{x_o}$ are linearly independent at all points $x_o \in X$.

c. Prove: the vector space of analytic vector fields on $\mathbb{P}^1$ is of $dim_{\mathbb{C}} = 3$, while for $\mathbb{C}/_L$ the dimension is 1.

d. For any non-zero meromorphic vector field on $\mathbb{P}^1$, $\mathbb{C}/_L$ compute $\sharp(zeros) - \sharp(poles)$.

e. The pushforward $TX \xrightarrow{f_*} TY$ is defined for *any* map $X \xrightarrow{f \in \mathcal{O}} Y$. Is the pushforward of vector fields defined for any map as well?

2.a. Prove: $dim_{\mathbb{C}} \Omega^1(\mathbb{P}^1) = 0$ and $dim_{\mathbb{C}} \Omega^1(\mathbb{C}/_L) = 1$.

b. Obtain explicit presentations of meromorphic differential forms on $\mathbb{P}^1$, $\mathbb{C}/_L$. Compute $\sharp(zeros) - \sharp(poles)$.

c. Prove: the bundles $T(\mathbb{C}/_L)$ and $T^*(\mathbb{C}/_L)$ are trivial.

d. The pullback $\Omega^1(X) \xleftarrow{f^*} \Omega^1(Y)$ is defined for *any* map $X \xrightarrow{f \in \mathcal{O}} Y$. Is the pullback $T^*X \xleftarrow{f^*} T^*Y$ defined for any map as well?

e. Suppose there exists $const \neq f \in M(X)$. Verify: $\Omega^1_M(X) = M(X) \cdot df$. (A vector space of $dim_{M(X)} = 1$.)

3. Take a Riemann surface $X = \{f(x, y) = 0\} \subset \mathbb{C}^2$.

a. Define the vector field on X as follows:

- At points where $\partial_y f|_p \neq 0$ take x as the local coordinate, and define $\xi = \partial_y f \cdot \partial_x$.
- At points where $\partial_x f|_p \neq 0$ take y as the local coordinate, and define $\xi = -\partial_x f \cdot \partial_y$.

Verify: ξ is a well-defined analytic vector field on X, and it has no zeros. Conclude: TX is trivial.

b. Conclude: the space of analytic, resp. meromorphic, vector fields on X is $\mathcal{O}(X) \cdot \xi$, resp. $M(X) \cdot \xi$.

c. Similarly, define $\omega \in \Omega^1(X)$ by $\frac{dx}{\partial_y f}$, resp. $-\frac{dy}{\partial_x f}$. Obtain the corresponding versions of a. and b.

d. Suppose the compactification $\bar{X} \subset \mathbb{P}^2$ is smooth, of $deg = d \geq 3$. Prove: ω of c. extends to $\omega \in \Omega^1(\bar{X})$. Moreover, all the forms $p(x, y) \cdot \omega$, for $p(x, y) \in \mathbb{C}[x, y]_{deg \leq d-3}$ extend to $\Omega^1(\bar{X})$. Conclude: $dim_{\mathbb{C}} \Omega^1(\bar{X}) \geq \binom{d-1}{2}$.

4.a. Verify: the exterior derivative $\mathcal{O}(X) \xrightarrow{d} \Omega^1_{C^\infty}(X)$ is compatible with its real version $C^\infty(X) \xrightarrow{d_{\mathbb{R}}} \Omega^1(X)$ (which was defined in Infi.4).
For any Riemann surface X and any $\omega \in \Omega^1_M(X)$ verify: $d_{\mathbb{R}} \omega = 0$.

b. For meromorphic forms/vector fields verify: $ord_p \omega$ and $ord_p \xi$ do not depend on coordinate choices.

c. Given $X \xrightarrow{f} Y$ and $\omega \in \Omega^1_M(Y)$, verify: $ord_p(f^*\omega) = ord_{f(p)}\omega \cdot mult_p f + (mult_p f - 1)$.

d. Take the smooth curve $X = \{y^2 = h(x)\} \subset \mathbb{C}^2$, where $deg(h) = d \geq 3$, and the corresponding hyperelliptic curve $\tilde{X} \to \bar{X}$.

i. Prove: the form $\frac{dx}{y}$ extends to a form $\omega \in \Omega^1(\tilde{X})$. (Find its zeros and poles, with their multiplicities.) Moreover, any form $q(x)\frac{dx}{y}$ for $deg(q(x)) \leq \lfloor \frac{d-3}{2} \rfloor$ extends to a form $\omega \in \Omega^1(\tilde{X})$.

ii. Take the ramified covering $\tilde{X} \xrightarrow{f} \mathbb{P}^1_{[x:z]}$ and a form $\omega \in \Omega^1_M(\mathbb{P}^1)$. Compute $\sharp(zeros(f^*\omega)) - \sharp(poles(f^*\omega))$.

5. Fix a complex torus $\mathbb{C} \xrightarrow{\pi} \mathbb{C}/_L$, a point $z_o \in L$ and the path $[0, 1] \xrightarrow{\gamma} \mathbb{C}$, $\gamma(t) = t z_o$. Compute $\int_{\pi\gamma} dz$.