

Introduction to Riemann Surfaces and Algebraic Curves

201.2.5101 Fall 2025 (Dmitry Kerner)

Homework 8.

Submission date: 29.12.2025.

Questions to submit: 1.a. 1.e. 2.c. 2.d. 3.b. 3.c. 5.b. 5.c.

(Either typed or in readable handwriting and scanned in readable resolution.)



Everywhere below X is a Riemann surface.

1. **a.** (Local normal form) Prove: for every $\omega \in \Omega_M^1(X, x_o)$ of $ord_{x_o}\omega = p$, with $Res_{x_o}\omega = 0$, there exists \mathcal{O} -local coordinate with $\omega = z^p \cdot dz$. Establish the corresponding fact for meromorphic vector fields.
b. Let $\omega = a(z)dz \in \Omega_M^1(\mathbb{P}^1)$, for some $0 \neq a(z) \in M(\mathbb{P}^1)$. Verify the theorem of residues for ω .
[Hint: split $a(z)$ into “basic” fractions.]
c. Take $f \in M(X)$ and the corresponding map $X \xrightarrow{F} \mathbb{P}^1$. Verify: $div_0(f) = F^*(o) \in Div(X)$ and $div_\infty(f) = F^*(\infty) \in Div(X)$.
d. Take a map $X \xrightarrow{f} Y$, its ramification divisor $Ram_f \in Div(X)$, and its branching divisor $Br_f = \sum_{y \in f(Ram_f)} (y) \in Div(Y)$. Disprove: $Supp(f^*(Br_f)) = Supp(Ram_f)$. For which maps do we have $Ram_f = f^*(Br_f)$?
e. For X -compact, consider the map $Div(X)/PDiv(X) \xrightarrow{deg} \mathbb{Z}$. (Verify: this is a homomorphism of groups.)
Prove: this is an isomorphism iff $X = \mathbb{P}^1$.

2. Take $X = \{y^2 = \prod_{j=1}^d (x - x_j)\} \subset \mathbb{C}^2$, and the corresponding hyperelliptic curve $\tilde{X} \xrightarrow{\phi} \bar{X} \subset \mathbb{P}^2$.
a. For $p(x) \in \mathbb{C}[x]$ describe $div(\phi^*(p(x))) \in Div(\tilde{X})$. The same for $p(y) \in \mathbb{C}[y]$.
b. Compute $div(\phi^*dx), div(\phi^*dy) \in Div(\tilde{X})$ and the residues of ϕ^*dx, ϕ^*dy at all the points of \tilde{X} .
Verify the residue theorem in this case.
c. For $d \in 2\mathbb{Z} + 1$ and $\tilde{X} \xrightarrow{\pi} \mathbb{P}_{[x:z]}^1$ verify: $Ram_\pi = div(\pi^*dx) + 3 \cdot \infty \in Div(\tilde{X})$. What happens for $d \in 2\mathbb{Z}$?
d. Verify: $div(\phi^*dx) - div(\phi^*dy) \in PDiv(\tilde{X})$.

3. **a.** We have stated the lemma “ $\mathfrak{Riem.Surf.} \xrightarrow{Div} \mathfrak{Ab}$ is a functor”. Prove all those claimed properties.
b. For every $0 \neq \omega \in \Omega_M^1(X_{compact})$ we have proved: $deg(div(\omega)) = 2g(X) - 2$. State and prove the corresponding version for meromorphic vector fields.
c. Prove: for $X_{compact}$ the set of all canonical divisors is a subgroup of $Div(X)$ iff $X = \mathbb{C}/L$.
(Identify this subgroup in this case. You can assume here: any $X_{g=1}$ is a complex torus.)

4. **a.** Prove: TFAE for a subset $\emptyset \neq L \subseteq \mathbb{P}^n$. (Then L is called a *plane* in \mathbb{P}^n)
i. Each intersection with a standard chart, $L \cap \mathcal{U}_j$, is a plane in $\mathcal{U}_j = \mathbb{C}^n$.
ii. For the projection $\mathbb{C}^{n+1} \setminus \{o\} \xrightarrow{\pi} \mathbb{P}^n$ the preimage $\pi^{-1}(L) \cup \{o\} \subset \mathbb{C}^{n+1}$ is a plane.
iii. $L \subset \mathbb{P}^n$ is defined by a system of homogeneous linear equations, $\{l_j(z) = 0\}_j$.
Accordingly give (many) definitions of $dim_{\mathbb{C}}L$, and verify their equivalence.
b. For two planes $L_1, L_2 \subseteq \mathbb{P}^n$ prove: $dim_{\mathbb{C}}(L_1 \cap L_2) \geq dim_{\mathbb{C}}L_1 + dim_{\mathbb{C}}L_2 - n$.
Deduce: if $dim_{\mathbb{C}}L_1 + dim_{\mathbb{C}}L_2 \geq n$ then $L_1 \cap L_2 \neq \emptyset$.
c. Fix a point $p \in \mathbb{P}^n$ and a hyperplane $p \notin L \subset \mathbb{P}^n$. Take the projection from p to L . Prove: this defines an analytic map $\mathbb{P}^n \setminus \{p\} \rightarrow L$.

5. Define the map $\mathbb{C}^1 \rightarrow \mathbb{C}^n$ by $z \rightarrow (z, z^2, \dots, z^n)$.
a. Prove: it extends to an analytic map $\mathbb{P}^1 \xrightarrow{\phi} \mathbb{P}^n$.
b. Prove: this map is a parametrization of its image, which is a smooth curve $X \subset \mathbb{P}^n$.
c. Compute the degree of this curve.
d. Prove: any distinct points $p_o \dots p_n \in X$ do not lie in any hyperplane $L_{n-1} \subset \mathbb{P}^n$. [Vandermonde]