Introduction to Riemann Surfaces and Algebraic Curves

201.2.5101 Fall 2025 (Dmitry Kerner)

Homework 9.

Submission date: 5.01.2026.

Questions to submit: 2.b. 2.d. 2.e. 3.c. 5.a. 5.b.

(Either typed or in readable handwriting and scanned in readable resolution.)

1. Below $X \subset \mathbb{C}^n$ is a Riemann surface, while $M(\mathbb{C}^n, o) = Frac(\mathcal{O}(\mathbb{C}^n, o))$.

a. For $(X, o) \subset (\mathbb{C}^n, o)$ and $f, u \in M(\mathbb{C}^n, o)$, with $\infty \neq u(o) \neq 0$ verify: $div_{(X,o)}(f) = div_{(X,o)}(u \cdot f)$.

b. Prove: for any $f \in M(\mathbb{C}^n, o)$ there exists a linear form $l(\underline{z})$ and $d \in \mathbb{Z}$ satisfying: $div_{(X,o)} f = div_{(X,o)} l(\underline{z})^d$.

c. For $X \subset \mathbb{P}^n$ and homogeneous polynomials $p(\underline{z}), q(\underline{z}) \in \mathbb{C}[\underline{z}]$ we have defined $div_X \frac{p}{q}$ via certain $\tilde{p}, \tilde{q}$. Verify: $div_X \frac{p}{q}$ does not depend on the choice of $\tilde{p}, \tilde{q}$.

2. Take two smooth curve-germs $(X_j, o) = \{f_j(x, y) = 0\} \subset (\mathbb{C}^2, o)$. Verify: $ord_{(X_1,o)}(f_2) = ord_{(X_2,o)}(f_1)$. Assuming $(X_1, o) \neq (X_2, o)$, define the tangency order as $i_o(X_1, X_2) := ord_{(X_1,o)}(f_2)$. (Another name: the intersection multiplicity.)

a. Verify: the tangency order is preserved under analytic coordinate changes on $(\mathbb{C}^2, o)$.

b. Compute the tangency order of $X_\pm$ defined by $f_\pm(x, y) = y + x \pm x^d$.

c. Verify: the tangency order of X_1, X_2 is 1 iff ...

d. In the polynomial case prove: $i_o(X_1, X_2) \leq deg(f_1) \cdot deg(f_2)$.

e. Fix a point $q \in \mathbb{P}^2 \setminus X$ and a line $q \notin L \subset \mathbb{P}^2$. Take the projection $\mathbb{P}^2 \setminus \{q\} \supset X \xrightarrow{\pi} L$. For a point $p \in X$ what is the relation between $mult_p \pi$ and $i_p(X, \overline{pq})$? [$\overline{pq}$ =the line through p, q]

3. Let L be the tangent line of (X, o). Verify: $i_o(X, L) \geq 2$. The point o is called an inflection point (or "a flex") if $i_o(X, L) > 2$.

a. Find all the flexes of the curve $\{y = x^d\} \subset \mathbb{C}^2$, and their orders. Are these preserved in local coordinate changes?

b. Prove: a smooth conic (i.e. deg=2) $X \subset \mathbb{P}^2$ has no flexes. (Hint: no computations are needed here)

c. Prove: $p \in \{f(x, y) = 0\} \subset \mathbb{C}^2$ is an inflection point iff the Hessian matrix $f''|_p$ satisfies

$$(-\partial_y f, \partial_x f)|_p \cdot f''|_p \cdot \begin{pmatrix} -\partial_y f \\ \partial_x f \end{pmatrix} |_p = 0.$$

4. Take a hypersurface $X = \{f(\underline{z}) = 0\} \subset \mathbb{P}^n$, where $f \in \mathbb{C}[\underline{z}]$ is homogeneous.

a. Verify: if df has no zeros on X, then X is smooth.

b. Prove the converse statement, assuming f is locally square free at all points of X.

c. Take a line $X \not\supseteq L \subset \mathbb{P}^n$. Verify (without using Bezout): $\sharp(L \cap X) = deg(f)$, counted with multiplicities.

d. For $deg(f) = 2$ X is called *a quadric hypersurface*. Verify: $X \xrightarrow[\sim]{GL(n+1)} \{\sum_{j=0}^r z_j^2 = 0\} \subset \mathbb{P}^n$. Here $0 \leq r \leq n$ is called the rank of the quadric. When is X smooth?

5. a. Take a conic $X \subset \mathbb{P}^2$ and two lines $L_j = \{l_j(\underline{z}) = 0\} \subset \mathbb{P}^2$ satisfying: $L_1 \cap L_2 \cap X$ =one point. Prove: the function $\frac{l_1}{l_2}|_X \in M(X)$ has just one zero and one pole. Deduce: $X \approx \mathbb{P}^1$.

b. For the projection of q.2.e we have proved $Ram(\pi) = div_X(\partial_y f)$ in the affine case, $X \subset \mathbb{C}^2$. Prove this in the projective case, $X \subset \mathbb{P}^2$.

c. Consider all the lines in $\mathbb{P}^2$ passing through a point $p \in \mathbb{P}^2 \setminus X$. Prove: exactly $d(d-1)$ of them are tangent to X. [Here we count with multiplicities.]

d. Take a smooth (connected) curve $X \subset \mathbb{P}^2$ of degree $d > 1$. Prove: no $(d+1)$ points of X lie on one line.