A combinatorial presentation of the variety of complete quadrics

1. INTRODUCTION

We shall identify the space of hyperquadrics in $\mathbb{P}^N = \mathbb{P}(E)$ with the projective space $\mathbb{P}^N = \mathbb{P}(\mathbb{P}(O^N(2)))$ associated to the vector space of symmetric maps, which we interpret initially for a fixed reference as symmetric matrices $M = M_q$ of order $n+1$. Let us consider the correspondence $\lambda_{r+1}$ which associates to each $[W]$ the class of the $(n+1)$-matrix $M^{r+1} \lambda$ modulo multiplication by a nonzero scalar

This correspondence is obviously not injective on $\mathbb{P}^N - V_p$, $V_p$ being the variety parametrising the quadrics of rank less than or equal to $r$ (note that $\mathbb{P}^N = V_{n+1}$).

A classical problem of enumerative geometry has been to extend the correspondences $\lambda_r$ to other ones such that the new ones are one-to-one over some transformation of the projective space $\mathbb{P}^N$. In other words, to obtain a "natural" compactification $B$ for the graph $(\lambda_2, \ldots, \lambda_n)$.

For $n = 2$ we refer to the suggestive survey of Kleiman [4]. The interesting work of Casas and Xambó [1] contains a complete and up-to-date revision of a different approach based on Halphen's theory. One can find also in these papers a great number of references which will not be repeated by us. Nevertheless, we shall quote the classical works (cf. Schubert [6], [7], Van der Waerden [14], Semple and Roth [9], Semple [8] and Tyrrell [11]), which may also serve as motivation for some general constructions of $B$ presented below.

From a modern viewpoint, the interest has been centered on the explicit computations of the Chow rings of the variety $B$ or in classical language, the determination of "characteristikzahlen" of Schubert's book ([6], p. 104). These have been determined recently by Vainsencher [12] and De Concini and Procesi [2] by using quite different methods. We have no new method for obtaining this Chow ring and thus the aim of this work is to show the geometrical aspects of the variety $B$ in the vein of classical approaches which are not explicitly presented in the quoted works.
This paper is organised as follows. In §2 we introduce the set of complete quadrics following the classical approach in the vein of works of Severi, Van der Waerden and Semple. We give a first approach to the enveloping quadrics founded on the classical polarity with respect to a form. The main point is Lemma 2.5 showing the incidence variety between quadrics and linear r-spaces.

The approach based on the resolution of singularities of \( V_n \) is treated in §3. We give there an analytical local presentation of monoidal transforms. The crucial point is the choice of the regular system of parameters explained in 3.2. The iteration of these monoidal transformations gives a model \( B^{(n-1)}_n \) for the complete quadrics which is essentially B (Corollary 3.6). The idea is to identify the successive monoidal transforms \( B^{(1)}_n \) of \( P^n \) with the closure of graphs of some correspondence products parametrizing incidence varieties.

In the following Chapter, we recover the combinatorial viewpoint and give two presentations of the types of complete quadrics by means of multiranks of quadrics (this is possible by virtue of remark 2.7) and partitions of \( n+1 \). One sees easily that it is possible to give a representation of both partially ordered sets by means an ordered n-cube (see e.g. the figure appearing in 4.8). We introduce an action on the set \( B_n \) extending the projectivization of the linear action (monoidal transforms are equivariant, in this case). This allow us to interpret such presentations in terms of projectivizations of linear representations associated to all the possible unordered decompositions of a vector space of dimension \( n+1 \). This will be the main point for §5.

Next, in §5, we make the explicit construction of the n-hypercube representing orbits of \( B_n \) from the n-simplex representing orbits of \( P^n \). We introduce the permissible transform on an ordered graph. Then, we verify that the orbits appearing in each step of the resolution of singularities of \( V_n \) are represented by the vertices of the polyhedra appearing from n-simplex to n-hypercube.

Finally in §6, I sketch another proof of the formulas given by Schubert [7] relating the primary condition (of being tangent to linear spaces) with the classes of the components of the exceptional divisors. This proof is founded on the geometry of parabolic subgroups. The approach is simpler (I think) than De Concini and Procesi's [2]. It is not so general but it has the advantage that it makes available the geometry of divisors on \( B^{(1)}_n \) in each step of the monoidal transformations. Roughly speaking, the central idea is to study the cellular decomposition associated to each monoidal transform \( B^{(1)}_n \) and to observe that each transformation separates cells which are unions of orbits into smaller cells to obtain cells which have a single orbit. At last, one justifies the degeneration formulas of Schubert's book.

All these results make part of my doctoral dissertation (Valladolid, 1983).
The author is grateful to J.M. Aroca, his advisor, for all his encouragement and helpful discussions. Thanks also to D. Laksov who very attentively looked through a preliminary version of this paper. His suggestions have been very useful for the actual form of this paper.

2. NOTATIONS, DEFINITIONS AND ELEMENTARY CONSTRUCTIONS

(2.1) Let \( E \) be a vector space of dimension \( n+1 \) over an algebraically closed field \( k \) of characteristic \( \neq 2 \). Let \( \mathfrak{p}^N = \mathfrak{p}(E) \) and let us denote by \( \mathfrak{p}^N = \mathfrak{p}(\mathfrak{h}^0(\mathcal{O}, (2))) \) the projectivization of the vector space of symmetric maps \( \mathfrak{e}^r \rightarrow \mathfrak{e} \). Then we have a natural stratification of \( \mathfrak{p}^N \) by the rank

\[
\mathfrak{v}_1 \subset \mathfrak{v}_2 \subset \ldots \subset \mathfrak{v}_n \subset \mathfrak{v}_{n+1} = \mathfrak{p}^N
\]

where \( \mathfrak{v}_r \) denotes the subvariety of \( \mathfrak{p}^N \) of quadric loci of rank \( \leq r \) which are parametrised by the set of symmetric maps \( \mathfrak{m}_q \) of rank at most \( r \) up to a proportional factor (obviously \( \mathfrak{v}_1 \) is the image of \( \mathfrak{p}^N \) by the Veronese embedding). The description of \( \mathfrak{v}_n \) as a determinantal variety gives its natural structure as closed subscheme of \( \mathfrak{v}_{n+1} \).

Each stratum \( \mathfrak{v}_{r+1} \rightarrow \mathfrak{v}_r \) for \( r = 1, \ldots, n \), is an open orbit for the natural action of \( \mathfrak{p}(E) \) on \( \mathfrak{p}^N \). Furthermore \( \mathfrak{v}_1 \) is the only closed orbit of this action.

(2.2) Let \( F_r = \text{Sym}^2(\Lambda^r \mathfrak{e}^*) \) be the space of symmetric bilinear forms over the \( r \)-th exterior power \( \Lambda^r \mathfrak{e}^* \) of the dual vector space \( \mathfrak{e}^* \) of \( \mathfrak{e} \). Consider the morphism \( \lambda_r : F_r \rightarrow \mathfrak{p}_r \) given by

\[
(\lambda_r(F)e_1 \wedge \ldots \wedge e_q, e_1 \wedge \ldots \wedge e_q) = \det F(e_1, \ldots, e_q)(1, k, 1, \ldots, r)
\]

for any fixed basis of \( \mathfrak{e}^* \). In terms of matrices \( \lambda_r(F) \) is given by a symmetric matrix \( M_r \) of size \( (n+1)^r \) whose generic entry \( M_{r,1} \) is the minor of order \( r \) of \( M \) indexed by the pair of \( r \)-multisubset of \( (1, \ldots, n) \).

The space of hyperquadrics in \( \mathfrak{p}^N \) corresponds to the projectivization of \( \mathfrak{p}(F_r) \) which can be identified with \( \mathfrak{p}(\text{Hom}(\mathfrak{e}, \mathfrak{e}^*)) \). The projectivization of \( \lambda_r \) gives a rational map \( \mathfrak{p}^N \rightarrow \mathfrak{p}^r \), which we shall also denote by \( \lambda_r \) and which associates to each "quadric locus" the \( r \)-th "enveloping quadric" \( \mathfrak{v}_r \) which is the envelope by \( r \)-spaces of \( a \) as we shall see in 2.7. The map \( \lambda_{r+1} \) is only geometrically defined outside of \( \mathfrak{v}_r \) for each \( r = 1, \ldots, n \). Our aim is to extend \( \lambda_{r+1} \), replacing \( \mathfrak{p}^N \) by a certain variety in such a way that the extension \( \lambda_{r+1} \) becomes one-to-one.

(2.3) Before realising this extension let us recall some well known facts (at least for \( n = 2, 3 \)). These allow us to justify the geometric language I choose deliberately the classical language of polarity with respect to \( \mathfrak{v}_n \) for showing the connection with the work of Seimple and Roth [9].

Let \( L_r \) be a fixed \( r \)-dimensional subspace of \( \mathfrak{p}^N \) and let us denote by \( \mathfrak{p}_r \) its Plücker coordinates in the Grassmanian \( G(r, n) \) i.e. we consider \( L_r \) as the intersection...
of \( n-r \) hyperplanes. Let us remember that in coordinate terms, a quadric \( q \) is tangent to \( L_r \) if and only if

\[ P(\Lambda^{r+1} q) p^t = 0 \]  

(2.4)

where \( \Lambda^{r+1} q \) is the \((r+1)\)-th exterior product of the matrix \( M_q \) representing the quadric \( q \). We shall represent this more synthetically by writing \( L_r \in \Lambda^r q \).

(2.5) **Lemma-Definition.** - The closure in \( \mathbb{P}_k^N \) of the set of point-quadrics \( P \in V_{n+1-k} \) such that \( q \) is tangent to a fixed space \( L_r \), is a divisor of degree \( r+1 \) which we shall call the \( r-th \) primary condition, and which we shall denote by \( W_r \).

**Proof:** Observe that if \( u \) denotes the \((n+1)x(n-r)\)-matrix given by the coefficients of \( n-r \) hyperplanes defining \( L_r \), then we may write the condition for a quadric to be tangent to \( L_r \) as follows (see Seemle and Roth [9] p. 311):

\[
\begin{vmatrix}
M_q & u^t \\
u & 0
\end{vmatrix} = 0
\]

If \( \{z_i\}_{1 \leq i \leq n} \) are the local coordinates of \( \mathbb{P}_k^N \) and \( P = (p^t_{j_1 \ldots j_{n-r}}) \) the Plücker coordinates of the Grassmannian we may write this condition in a more descriptive form as

\[
\sum_{j_1, \ldots, j_{n-r}} \frac{\partial^{n-r} p}{\partial x_{j_1} \partial x_{j_2} \ldots \partial x_{j_{n-r}}} p_{j_1 \ldots j_{n-r}} \cdot \frac{\partial^{n-r} p}{\partial y_{l_1} \partial y_{l_2} \ldots \partial y_{l_{n-r}}} l_{l_1 \ldots l_{n-r}} = 0
\]  

(2.5.1)

which is the \((n-r)\)-th polar variety induced by the determinant \( f \) of a general symmetric matrix of order \( n+1 \).

(2.6) As a by-product one obtains that in a linear generic pencil the number of tangent hyperquadrics to a general \( L_r \) is \( r+1 \). (For each fixed space \( L_r \), (2.5.1) is a form of degree \( r+1 \).

Moreover, if \( V \) denotes the variety of quadrics less than or equal to \( r \), then \( V \subseteq W_k \) for \( k \geq r \). This generalizes the well known classical facts for conics in \( \mathbb{P}^2 \) (resp. quadrics in \( \mathbb{P}^3 \)) that the condition to be tangent to a line (resp. plane) is satisfied at all elements of the Veronese variety \( V_1 \). Analogously the condition for a quadric in \( \mathbb{P}^3 \) to be tangent to a line is satisfied for all the quadrics of rank less than or equal to 2.

(2.7) **Remark.** - variety of \( r \)-spaces tangent to \( L_r \) shall call the \((r+1)\)-th envelope of the corresponded quadric an \((r+1)\)-th quadric. This gives a corresponding quadric in \( \mathbb{P}_k^N \) of degree \( r+1 \).

(2.8) We define the set of \( n \)-tuples \( p = (p^t_{j_1 \ldots j_{n-r}}) \) of quadrics which \( p \) is\( \in \Lambda^r q \). Set-theoretically

\[
\text{quadric loci } \{q_1, \ldots, q_{r+1}\}\text{ enveloping quadric of } q = \frac{A^r q + B}{A^r q - B}
\]

If \( A_{r+1} = q \) as the image of \( Q \) be the \( n\)-tuple \( (q_1, \ldots, q_{r+1}) \), then \( \Lambda^r q \) represents the matrix of the \( (r+1) \)-th quadric in \( (n+1) \), and \( p \) is the \( (n+1) \)-tuple \( (p^t_{j_1 \ldots j_{n-r}}) \).

**Set-theoretically**

3. **MONOIDAL TRANSFORMS**

(3.1) Let us define \( \mathbb{P}_k^N \) with center a subvariety of \( \mathbb{P}^N \). A straightforward argument extending that of \( \text{PGL}_{n+1} \) shows that this is also a monoidal transform.

First we need a little discussion. Consider \( V \) as a determinantal \( n \)-tuple for \( L_r \), the idea is the following: \( \mathbb{P}^N \), the variety \( V \) becomes of order one less, as \( n \) becomes regular system of parameters.
(2.7) Remark. Conversely, by (2.5.1) if we fix the quadric $q \subset \mathbb{P}^n$, the variety of $r$-spaces tangent to $q$ is given by a quadric section of $G(r, n)$, which we shall call the $(r+1)$-th generalised quadric associated to $q$, which clearly is the same as the $(r+1)$-th enveloping quadric introduced in (2.2). With this interpretation, the correspondence which to each quadric-locus associates the $(r+1)$-th generalised quadric gives a non-zero quadric section of $G(d, n)$ when $r > (M)_{(r+1)}$. This gives a correspondence between the sections of $Q_r(2)$ and the sections of $Q_r(2)_G$ where $G = G(r, n)$, which we shall denote by $\lambda_{r+1}$.

(2.8) We define (as in Tyrrell [11] or Vainsencher [13]) the variety of complete quadrics which we shall denote $B_r$ to be the closure of the image of the product correspondence $\lambda_2 \times \ldots \times \lambda_n$ in the Segre product $\mathbb{P}^n \times \cdots \times \mathbb{P}^n$.

Set-theoretically every $n$-complete quadric $Q \subset B_r$ is given by an $n$-tuple of quadric loci $(q_1, \ldots, q_n)$, where $q_{r+1}$ is a quadric in $\mathbb{P}^n$, $E$ called the $(r+1)$-th enveloping quadric of $q = q_1 \times \cdots \times q_n$.

If $\lambda_{r+1} : B_r \to \mathbb{P}^{n+1}$ is the $(r+1)$-th natural projection, we can obtain $q_{r+1}$ as the image of $Q$ via $\lambda_{r+1}$. The multirank or simply rank of $Q$ is defined to be the $n$-tuple $(r_1, \ldots, r_n)$ given by $r_i = \text{rank} (\lambda_i(Q))$ for $i = 1, \ldots, n$. Naturally if $M$ represents the matrix of a quadric-locus $q$ of maximal rank $n+1$ its associated complete quadric is represented by $(M, M^2, \ldots, M^n)$ and its rank is given by $(n+1, \ldots, (n+1))$. The results of the following section allow us to extend this description for quadrics of rank less than or equal to $n$.

3. MONOIDAL TRANSFORMS ON $\mathbb{P}^n$

(3.1) Let us denote generically by $\pi : \mathbb{P}^n_k \to \mathbb{P}^n_k$ the monoidal transform of $\mathbb{P}^n_k$ with center a subvariety $V$ which is stable under the action of $G_{n+1}(k)$. By a straightforward argument one shows directly that there exists an action on $\mathbb{P}^n_k$ extending that of $G_{n+1}(k)$ on $\mathbb{P}^n_k$ such that the projection $\pi$ is $G_{n+1}(k)$-equivariant (this is also a consequence of Prop. 2.4 of Wright [15]). The aim of this paragraph is to give a description of the orbits of the model of complete quadrics which uses successive monoidal transforms.

First we need a simple description of the monoidal transforms. If we consider $V_r$ as a determinantal variety, e.g., the locus of zeros of minors of order $r+1$, the idea is the following: After making the monoidal transform with center $V_q$, the variety $V_r$ becomes a "cylinder with directrix" a determinantal variety $V_r$ of order one less, as locus of zeros of minors of order $r$ with respect to a new regular system of parameters (for a more general presentation see Vainsencher.
(3.2) Proposition.- Let $V_{11}^{(n)}$ be the strict transform of $V_1$ by the monoidal transform $p_i: B_1 \rightarrow \mathbb{P}_k^N$ of $\mathbb{P}_k^N = V_{n+1}$ with center at $V_1$. Then $V_{11}^{(n)}$ is given locally by $V_{11}^{(n-1)} \times \mathbb{P}_k^N$ for $i = 1, \ldots, n$. In particular $V_{11}^{(n)}$ is non-singular.

Proof: The action of $\text{PGL}_{n+1}(k)$ on $\mathbb{P}_k^N$ allows us to reduce the study of the monoidal transform $p_i$ to the situation of a generic point $P_q$ of each orbit $V_r - V_{r-1}$. Therefore we may suppose that $\sum x_i^2 = 0$ is the equation of the quadric locus corresponding to $p_q$ with respect to a system $(x_i^0)_{0 \leq i \leq n}$ (resp. $(z_i^j)_{0 \leq i \leq n}$) of homogeneous coordinates of $\mathbb{P}_k^N$ (resp. $\mathbb{P}_k^N$). Then it is clear that the minors of order $r+1$ given by

$$f_{i,j} = \begin{vmatrix} z_{i,0}^0 & \cdots & z_{i,r-1}^0 & z_{i,j}^0 \\ \vdots \\ z_{i,0}^{r-1} & \cdots & z_{i,r-1}^{r-1} & z_{i,j}^{r-1} \\ z_{i,0}^1 & \cdots & z_{i,1}^{r-1} & z_{i,1}^1 \end{vmatrix}$$

and the $(\binom{n}{2}) + (n-r)^2$ coordinates $(z_{ij}^j)_{0 \leq i \leq n}$ give rise to a regular system of parameters for $V_r$ in an affine neighbourhood of $P_q$. In particular if $P_q \in V_1$ (by convention one supposes $V_0 = \emptyset$), the regular system of parameters for $V_1$ is given in the neighbourhood $D(z_{00}^0)$ by

$$y_{ij}^j = z_{ij}^j - z_{ij}^0 z_{ij}^0$$

for $j = 0, \ldots, n$, $y_{ij}^j = z_{ij}^j - z_{ij}^0 z_{ij}^0$ for $1 \leq i \leq n$.

where $z_{ij}^j = z_{ij}^j / z_{00}^0$ for every $(i,j)$. From the affine viewpoint we may hence write in $D(z_{00}^0)$ the generic minor $f_{L,K}$ associated to the pair of $(r+1)$-multi-indices $(L,K) = ((0,1,\ldots,1_r), (0,k_1,\ldots,k_r))$ with respect to the new regular system of parameters given $y_{ij}^j$ as follows:

$$f_{L,K}(y_i^j) = \ldots$$

To obtain the corresponding ideal $I$ of $V_1$. So, for each $k = 1, \ldots, m$ we formed into

$$f_{L,K}(y_i^j) = \ldots$$

multiplied by $y_{i1}^{k_1}$ when

$$f_{L,K}(y_i^j) = \ldots$$

Hence we have a corresponding $f_{L,K}(y_i^j)$, the minor of order $r$ follows by glueing together

By iterating this argument we also contained in Vainshtein's proof.

(3.3) Theorem.-
\[ f_{(L,K)}(y') = \begin{vmatrix} 1 & y_{0,1}^r & \cdots & y_{0,k_1}^r \\ y_{1,1,0}^r & y_{1,1,1}^r & \cdots & y_{1,1,k_1}^r \\ \vdots & \vdots & \ddots & \vdots \\ y_{1,r,0}^r & y_{1,r,1}^r & \cdots & y_{1,r,k_1}^r \end{vmatrix} \]

To obtain the stated result it suffices to blow-up \( \mathbb{P}^N \) with respect to the ideal \( I \) of \( V_1 \). So, for example, we have in \( D_+(y_{1,1,k_1}^r) \) that \( f_{(L,K)}(y') \) is transformed into

\[ F_{(L,K)}(x') = \begin{vmatrix} 1 & y_{1,1,k_2}^r & \cdots & y_{1,1,k_r}^r \\ \vdots & \vdots & \ddots & \vdots \\ y_{1,k_1}^r & y_{1,k_2}^r & \cdots & y_{1,k_r}^r \end{vmatrix} \]

multiplied by \( y_{1,1,k_1}^r \) where the monoidal transform is given in this open set by

\[
\begin{align*}
y_{0,i} & = y_{0,i} \\
y_{1,j} & = y_{1,j}y_{1,k_1}^r \\
y_{1,k_1} & = y_{1,k_1}^r
\end{align*}
\]

Hence we have a correspondence which associates to each monoid of order \( r \), \( f_{(L,K)}(x') \), the minor of order \( r+1 \), \( F_{(L,K)}(x') = F_{1,k}(x') \), where \( x' = (x_{0,0}^r) \) on \( D_+(x_{0,0}^r) \). The result then follows by gluing together the affine pieces.

By iterating this argument one obtains the following result (essentially also contained in Vainsencher [12]).

(3.3) Theorem.- Let us denote by \( y_{1,j}^{(n)} \) the strict transform of \( y_{1,j}^{(n)} \) by the monoidal transformation \( p : B_n^{(j)} \rightarrow B_n^{(j-1)} \) of \( B_n^{(j-1)} \) with center the determi-
nontal ideal corresponding to the minimal orbits of the action on $B_{n-j}^{(j-1)}$ extending the action on $B_{n-j}^{(j-2)}$ (one set $B_{n}^{(0)} = \mathbb{P}^n$ and $B_{n}^{(1)} = \emptyset$ for $i < 0$). In particular $V^{(n)}_{j, j-1}$ is a component of the center of the blow-up. Then one has

\[ V^{(n)}_{i, j} = V^{(n-j)}_{1, j-i} \times \mathbb{P}^{n+i-1} \quad i \leq j-1 \]

b) $V^{(n)}_{j, j}$ is non-singular and we shall consider it as the non-singular model of $V_j$. Hence in particular, the composition $\mathfrak{M}$ of monoidal transformations

\[ B_{n}^{(n-1)} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_{j+1}} B_{n}^{(j+1)} \xrightarrow{p_{j+1}} B_{n}^{(j)} \xrightarrow{p_{j}} \mathbb{P}^N \]

is a resolution of the singularities of $V_n$.

(3.4) Finally we shall prove that the variety of complete quadrics $B$ coincides with the total transform $B_{n}^{(n-1)}$ of $\mathbb{P}^N$ by the composition $\mathfrak{M}$ of the successive monoidal transformations considered just above.

Let us observe that each $\lambda_{r}$ is a pseudo-G-morphism (EGA IV, 20.2.1) with $\text{dom}(\lambda_{r}) = \mathbb{P}^{N-r-1}$. Hence applying theorem 4.1.1 of Wright [15], one obtains that the closure of the product correspondence $\lambda_{2} x \cdots x \lambda_{n}$ is a closed $G$-subscheme.

(3.5) Proposition. 1) The mappings $h_{i}$ appearing in the diagram

\[
\begin{array}{cccc}
\text{graph}(\lambda_{2}) & \xrightarrow{h_{1}} & \text{graph}(\lambda_{2}, x, \lambda_{3}) & \xrightarrow{h_{2}} \cdots \xrightarrow{h_{n}} \text{graph}(\lambda_{2}, \cdots, \lambda_{n}) \\
\mathbb{P}^N & \xrightarrow{p_{1}} & B_{n}^{(1)} & \xrightarrow{p_{2}} B^{(2)} & \cdots & \xrightarrow{p_{n-1}} B_{n}^{(n-1)}
\end{array}
\]

induced by the universal property of blowing-up are $G$-isomorphisms.

2) The G-scheme $\text{graph}(\lambda_{2}, \cdots, \lambda_{n})$ is a non-singular minimal model of all the G-schemes $\text{graph}(\lambda_{i})$ for $i = 2, \ldots, n$.

Proof: Let us observe first that the composed blowing-up $p_{2} \cdots p_{1}$ is a $G$-morphism whose orbits we have given in paragraph 4. Moreover the successive monoidal transformations resolve the "singularities" of the morphism $\lambda_{r+1}$ i.e. the variety $V_{r}$. Let us observe that $V_{r}$ is the $(r-1)$-multiple locus of $V_{r}$ for $r > 1$ and hence, after $j$ monoidal transformations, $V_{1, j}$ is the $(r-1-j)$-multiple locus of $V_{r, j}$ but they may be contained in the singular locus of the lifted morphism. For example, $V_{1, 1}$ and $V_{2, 1}$ are smooth but they are contained in the $(k-1)$-multiple locus of $V_{k}$.

(3.6) Corollary

4. RECURRENT DESCRIPTION

The main goal of this section is to describe the complete quadrics $B$ as a $G$-scheme, see De Concini and Procesi [11]. The natural action of $POL_{k}$ by means of monoidal transformations $B_{n-r}$ is as follows.

(4.1) First, we consider the complete quadrics $B_{n-r}$ by means of monoidal transformations, as in Theorem 3.4, whose fiber over each point $p_{r}$ of $B_{n-r}$ is associated to a $G$-scheme.

(4.2) Remark. Van der Waerden [14].

Proof: Let us recall it briefly (see 3.3) the result obtained in paragraph 4. Then one chooses a $G$-scheme of $B_{n-r}$.

(4.3) Therefore, the complete quadric in $\mathbb{P}^{N}$ is

\[ \dim (\text{Vert}(q)) = 0, 1, 2 \quad \text{on a point, line, plane, respectively (i.e. their singularities)} \]

a) 1 for $\dim (\text{Vert}(q)) = 0, 1, 2$

b) 2 and 1 for $\dim (\text{Vert}(q)) = 0, 1, 2$

c) $\cdots (3, 3), (2, 1)$

d) $\cdots (4, 6), (3, 3)$
(1.1) Consider it as the monoidal transformation $\pi$ of the complete quadrics $B^{n-1}_n$ extending $B^{n-1}_i$ for $i < 0$. In particular, one has

$B^{n-1}_n = \pi(B^{n-1}_i) \subset B^{n-1}_i$ for i = 0.

4. Recurrent Description of the Types of Complete Quadrics

The main goal of this paragraph is to show the structure of the variety of complete quadrics $B$ as a homogeneous (in fact, symmetric) space. For another construction see De Concini and Procesi's work [12]. We will do this by lifting to $B$ the natural action of $\text{PSL}_k(n+1)$ given on $\mathbb{P}^N$.

(4.1) First, we shall give two representations of the type of complete quadrics by means of multiranks and partitions of $n+1$. Let us recall that, with the same notations as in Theorem 3.3, we have a fiber bundle $\pi^{-1}(V_{r-r}^{-1}) \rightarrow V_{r-r}^{-1}$ whose fiber over each point-quadric $P_q$ is isomorphic to the variety of complete quadrics $B_{n-r}$ associated to quadric loci in $\mathbb{P}^r$.

(4.2) Remark.- This claim was implicitly presented in earlier works (see Van der Waerden [14], Severi [15], Semple [8], Tyrell [11]). The classical reasoning was founded for $n = 2$, 3 on a careful analysis of degradation, and we recall it briefly. The existence of a lifted action with good properties (see 3.3) allows us to reduce the question to a study of the neighborhood of a point. Hence, one chooses a canonical form for the quadric locus q as a sum of squares. Then one considers an analytic arc meeting transversally the orbit containing the point $P_q$ in all possible ways such that all the $i$-th enveloping quadrics are well defined for $i \geq \text{rank}(q)$. A simple argument of specialization on the matrices representing the lifting of the matrix $M_q$ gives us the solution. Actually this choice is possible thanks to Luna's theorem [5].

(4.3) Therefore by decreasing induction we may associate to each type of complete quadric in $\pi^{-1}(V_{r-r}^{-1})$ its multirank. In particular if $\dim(\text{Vertex}(q)) = 0, 1, 2$ or 3 there are $2^0, 2^1, 2^2$ or $2^3$ types of complete quadrics on a point, line, plane or 3-space, as is well known. Their multiranks on the vertex (i.e. their singular locus) are

a) 1 for $\dim(\text{Vertex}(q)) = 0$

b) 2 and 1 for $\dim(\text{Vertex}(q)) = 1$

c) $(3, 3), (2, 1), (1, 2), (1, 1)$ for $\dim(\text{Vertex}(q)) = 2$

d) $(4, 6, 4), (3, 3, 1), (2, 2, 1), (2, 1, 2), (1, 3, 3), (1, 2, 1), (1, 1, 2)$ and
\[(1,1,1) \text{ for } \dim(\text{Vertex}(q)) = 3.\]

The next result allow to us give a more general description.

**Lemma 4.4.** - One can represent the set of possible types of an n-complete quadric by means of the vertices of an ordered hypercube \(I^n \cap \mathbb{R}^n\) where \(I = [0, 1]\).

Hence there are exactly \(2^n\) possible types or possible values for the multirank of an n-complete quadric \(Q\).

**Proof.** - For low values of \(n\) the last assertion is a well known fact, as we have recalled above. If we suppose the claim is true for \(n-1\) we have that \(B\) is the disjoint union of \(\Pi^{-1}(V_{n+1}V_1)\) and \(\Pi^{-1}(V_n)\). By applying Cor. 3.4 and the induction hypothesis we obtain that there are exactly \(2^{n-1} + 2^{n-1} = 2^n\) possible types of n-complete quadrics as claimed. In other words, if \((r_1, \ldots, r_n)\) is the rank of an n-complete quadric \(Q\) and if \(t\) is the first index such that \(r_t = 1\), then \((r_1, \ldots, r_n)\) is the rank of an \((n-t-1)\)-complete quadric. This gives \(2^0 + 2^0 + 2^1 + \ldots + 2^{n-1}\) possible values for the rank. (The first \(2^0\) corresponds to the smooth quadrics and the second one to the cones). The correspondence which to each rank of any n-complete quadric associates the n-tuple obtained making zero all the entries different from 1 in its multirank gives the representation by means of the vertices of the hypercube.

4.5. Next we will show that the inverse of lexicographical order on the vertices of the hypercube corresponds to the specialization of complete quadrics. Before explaining this more precisely, we give another elementary description of an n-complete quadric. This is presented in the classical literature (see Severi [10] or Van der Waerden [14]) from the topological viewpoint, but it is possible to give a combinatorial description suggested also by Lemma 3.6 and which completes the preceding description. For let us consider each n-complete quadric as a pair given by a complete quadric locus \(q_{r_1} \subset \mathbb{P}^n\) and a complete quadric \(Q'\) in the singular locus of \(q\) which we shall call the vertex of \(q_{r_1}\). If we iterate this decomposition for the complete quadric \(Q'\), we have that for each fixed complete quadric \(Q\) one obtains a representation as a k-tuple of quadric-loci \((q_{r_1}, \ldots, q_{r_k})\) such that every \(q_{r_i}\) is a quadric-locus in the vertex of \(q_{r_{i+1}}\). Hence one trivially has \(r_1 + \ldots + r_k = n+1\).

4.6. **Definition.** - We shall say that a complete quadric represented by \((q_{r_1}, \ldots, q_{r_k})\) is a degeneration of \((q_{r_1}, \ldots, q_{r_k})\) if \(k \leq k'\) and for every \(i = 1, \ldots, t\) there exists an \(1(i)\) such that \(q_{r_{1(i)}}^i\) is a degeneration of the quadric-locus \(q_{r_1}\), which of \(q_{r_{1(i)}}^i\).

4.7. **Proposition.** - The ranks of n-complete quadrics are naturally subordinated by this map to the.

4.8. **Example.** -

<table>
<thead>
<tr>
<th>Lifting the Act</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 0, 1, 0</td>
</tr>
<tr>
<td>0, 1, 1, 1</td>
</tr>
<tr>
<td>0, 1, 0, 1</td>
</tr>
</tbody>
</table>

4.9. The last paragraph of the proof of the lifted action of the hypercube on the vertices of the hypercube.

For this, we shall consider the quadric-loci following.

Let us associate a quadric of maximal rank \(n+1\) such that \(L_1 + \ldots + L_k = n\).

We define the lifting of \(L\) on \(t\). For \(t = 1\), let \(L = [q_{r_1} q_{r_2}]\), where \(q_{r_1} \subset \mathbb{P}^n\).

For each \(t > 1\) the lifting is given by a quadric locus.
quadratic-locus $q^{'}_r$, which means that the vertex of $q^{'}_r$ is contained in the vertex of $q^{'}_1$.

4.7. Proposition.- There exists a bijective mapping between the set of ranks of $n$-complete quadrics and the set of partitions of $n+1$. The relation of natural subordination in the partially ordered set of partitions of $n+1$ corresponds by this map to the degeneration of complete quadrics.

4.8. Example.- For $n = 3$ we therefore have the three following cubes.

- Lifting the Action

4.9 The last part of this paragraph is devoted to showing an explicit description of the lifted action on the variety of complete quadrics and to prove that the vertices of the hypercube correspond in fact to the orbits under the lifted action. For this, we shall consider a complete quadratic $Q$ as a $k$-tuple $(q_1, ..., q_k)$ of quadric loci following the notation introduced in 3.7.

Let us associate to each quadratic-locus $q_i$, the matrix representing it as a quadratic of maximal rank in $(r_i-1)$-projective space $L_i \subset P^n$, for $i = 1, ..., t$, such that $L_1 + ... + L_t = P^n$.

We define the action of the product $G = \prod_{r=1}^n \text{PGL}_k(r)$ on $B$ by induction on $t$. For $t = 1$, let us consider the natural action defined by $[q_1] [q_1] = [q_1 q_1]$, where $q_1 \in \text{GL}_k(n+1)$.

For each $t > 1$, if we consider the complete quadratic $Q$ as a pair $(q_1, Q_2)$ given by a quadratic locus $q_1$ and a complete quadratic on vert $[q_1]$, we may apply...
the induction hypothesis on t and define

\[(q_{r_1}^1, g_1) g: = (q_{r_1}^1, g_1, q_2 g_2, \ldots, q_n g_n, g_{r_1}).\]

In explicit terms, \([q_{r_2}^1] \subset \text{vert} [q_{r_1}^1]\) and if we set \([q_{r_2}^1]: = [(q_{r_2}^1, g_2, g_2)]\) it is clear that \([q_{r_2}^1] \subset \text{vert} [q_{r_1}^1 g_1]\) where \(g_2, g_2\) denotes the restriction to \(\text{vert} [q_{r_1}^1]\) of \(g_2, g_2\). If we fix two complete quadrics \(Q\) and \(Q'\) with the same associated partition \((r_1, \ldots, r_t)\) of \(n+1\) there exist \(g_n, g_2\) such that

\[q_{r_1}^1 = g_{r_1} \cdot q_n, q_{r_2}^1 = (q_{r_2}^1, g_2) g_2.\]

(the converse assertion is obvious). Hence by iterating the process we may represent this action symbolically as

\[Q.g = (q_{r_1}^1, g_n, (q_{r_2}^1, g_2) g_2, \ldots, (q_{r_t}^1, g_n, g_{r_2}^1, g_{r_2}) g_{r_2}).\]

4.10 Theorem.- Let us suppose that char \((k) = 0\) and let us denote by \(G(r_1, \ldots, r_t)\) the subgroup of matrices given by \(t\) boxes of size \(r_i\) for \(i = 1, \ldots, t\) defined up to multiplication by scalar. Then there exists a bijective correspondence between the types of complete quadrics and the set of linearizations of representations

\[j: (r_1, \ldots, r_t): G(r_1, \ldots, r_t) \rightarrow \text{PGL}_{n+1}(k)\]

associated to ranks \((r_1, \ldots, r_t)\) of complete quadrics.

Proof: It suffices to take into account that the types of complete quadrics are parametrized by the partitions of \(n+1\) and if \(\text{char}(k) = 0\), that every linear representation associated to the direct sum \(\bigoplus_{i=1}^n V_i\) of vector spaces is a product of \(t\) linear representations \(G_i \rightarrow \text{GL}(V_i)\). Hence the subspaces \(L_i\) in the decomposition \(\mathbb{P}^n = L_1 + \ldots + L_t\) are invariant under the action of the representation given by \(j(r_1, \ldots, r_t)\).

4.11. Corollary.- The topological space of orbits on \(B\) by the above lifted action is a partially ordered lattice, which we denote by \(\mathcal{B}_n\). It can be represented by an \(n\)-hypercube \(I^n\). Note that \(\mathcal{B}_n\) is a spectral topological space, so we may consider \(\mathcal{B}_n\) as an affine scheme.

Proof: The last assertion is a consequence of the result of Hochster [3], the others ones being obvious by the construction of \(B\).

5. Obtaining the Hypergraph

5.1. By the real exceptional divisor \(E = \Sigma \text{action when the center.}\)

For obtaining an explicit geometrical interpretation the geometrical interpretation.

The basic facts one is the following.

5.2. Permissible...

Let us consider \(\mathbb{P}^N\) of hyperquadrics structures of the standard simplex \(S_k\) of hyperquadrics of permissible degenerations. Let \(\tau_{n-1}(V_{i})\) be the \((n-1)\)-simplex. We may consider \(\{0\} \times \tau_{n-1}(V_{i})\) with \(\tau_{n-1}(V_{i})\) as this process is a \(V_i\). (In fact this corner graph). By selecting a \(\{i\} \times \tau_{n-1}(V_{i})\), with \(i\) question that remains invariant gives in \((2.4)\).

It suffices to the relation of specialization depends only on the rank the same orientation hypergraph \(\{0,1\} \times \tau_{n-1}\) on order (where natural vertices for each for each minimal vertex a monoidal transformation connecting the two minimal exercise of combinatorial a set of \(2^i\) linear sim.
5. OBTAINING THE HYPERCUBE

5.1. By the reasoning made at the beginning of §3, one has that the exceptional divisor $E = \mathbb{P}^{-1}(V)$ and $\mathbb{P}^{-1}(F_{k}^{-1}(V))$ are unions of orbits by the lifted action when the center $V$ is non-singular and stable under the action of $\text{PGL}_{n+1}(k)$. For obtaining an explicit description of the orbits in each step we shall need the geometrical interpretation of the transformations involved and its combinatorial representation.

The basic facts for the first ingredient are contained in (3.5). The second one is the following construction.

5.2. Permissible transforms on a ordered graph

Let us consider all the possible degenerations in the projective space $\mathbb{P}^{n}$ of hyperquadrics stratified by the rank. Then we may interpret all the vertices of the standard simplex $\sigma_{n}$ on $\mathbb{P}^{n}$ as the orbits by the action of $\text{PGL}_{n+1}(k)$ on the space of $k$-dimensional hyperquadrics in $\mathbb{P}^{n}$ and its oriented edges as all the possible degenerations. Let $v_{1}$ be a vertex and $\sigma_{n-1}(v_{1})$: $= \sigma_{n} - (v_{1})$ the complementary $(n-1)$-simplex. We may define a mapping $\pi: \{0,1\} \times \sigma_{n-1}(v_{1}) \rightarrow \sigma_{n}$ by identifying $\{0\} \times \sigma_{n-1}(v_{1})$ with $\sigma_{n-1}(v_{1})$ and collapsing $\{1\} \times \sigma_{n-1}(v_{1})$ to $v_{1}$. We shall say that this process is a quadratic transformation of $\sigma_{n}$ with center at the vertex $v_{1}$. (In fact this corresponds analytically to a quadratic transformation in a graph). By selecting a vertex $v_{2} \in \sigma_{n-1}$, we have two vertices for each copy $\{i\} \times \sigma_{n-1}(v_{1})$, with $i = 0, 1$, and we repeat the process for both vertices. The question that remains is how to select the vertices in order to obtain the representation given in (2.4).

It suffices to consider the $n$-simplex $\sigma_{n}$ as an ordered hypergraph by the relation of specialization of hyperquadrics representing the orbits. This relation depends only on the rank of the hyperquadric. Then, if we give to $\{i\} \times \sigma_{n-1}$ the same orientation as that of $\sigma_{n-1}$ for $i = 0, 1$, we may orient all the hypergraph $\{0,1\} \times \sigma_{n-1}$ by means of the orientation induced by the lexicographic order (where naturally, one considers $0 < 1$). Hence one has two minimal vertices for each for each "floor" or $\{0,1\} \times \sigma_{n-1}$ and the iteration of the process for each minimal vertex in each "floor" is equivalent (analytically too!) to the monoidal transformation of the hypergraph $\{0,1\} \times \sigma_{n-1}$ with center the line connecting the two minimal vertices. By iterating this process it is an easy exercise of combinatorial topology to prove that at the $i$-th step we are adjoining a set of $2^{i}$ linear simplices of dimension $n-i-1$ at the minimal vertices in such a
way that the adjunction extends the orientations of the former simplices. Last I must show how one arrives at the ordered $n$-hypercube. The following example helps to understand the idea.

(5.3) Example.- Let us consider these transformations for orbits of hyper-quadrics in $\mathbb{P}^3$. In this case we have only two transformations from the tetrahedron into the cube:

The first one is the substitution of the minimal vertex $(1)$ of the tetrahedron by the opposite side (a 2-simplex $\sigma_2$). So we obtain a triangular prism with two minimal vertices, one for each floor. The second transformation consists of the substitution of each minimal vertex by the opposite side (a 1-simplex $\sigma_1$). So we arrive at a 3-cube whose vertices parametrize the orbits of complete quadrics. Let us note that each arrow denotes a specialization. Each vertex of the prism is denoted as described in the following result.

5.4 Proposition. - 1) The exceptional divisor $E_1 = p^{-1}_1(V_1)$ of the monoidal transformation $p_1 : B_1^{(1)} \rightarrow \mathbb{P}^N$ with center the Veronese variety $V_1$ given in (3.2) is the union of $n$ orbits under the lifted action of $(3.1)$ which we denote $(1,1)$, $i = 1, \ldots, n$, such that

   a) The orbit $(1,n)$ is open and dense in $E_1$.

   b) $(1,k) = \overline{(1,j)} \times (1,j)$. In particular $(1,k)$ is locally closed and $(1,1)$ is the only closed orbit (the bar denotes the closure of the orbit).

   2) For each $i \geq 2$, $V_i^{(1)} = V_i^{(1)} + V_i^{(1)}$ is the union of two orbits which we shall denote $(1,i)$ and $(1,1)$ (one justifies this notation from 2.5).

   3) $(1,i) = \overline{(1,n)} \cap \overline{(1,1)}$. Hence, the monoidal transform $B_1$ is represented by the first permissible transform of the tetrahedron.

Proof. - Let us recall that $p_1$ is $\text{PGL}(n+1)$-equivariant and thus the problem is local. By (3.5) the closure of the orbit $(1,k)$ parametrizes all the pairs given by a double hyperplane as quadric-locus and a quadric of rank less than or equal to direction in the tangent bundle.

On the other hand, the correspondence $\lambda_2$, e.g., meeting $V_1$ transversally, is a simple construction.

Lastly, the third diagram between the natural representation $(p_2, p_1) : \mathbb{P}^N \rightarrow \mathbb{P}^N$. Let us observe the coordinates of the points of the orbit of vertices whose sum in lifting successively, the $\overline{(1,n)}$ direction.

5.5 Proposition. - 1) The exceptional divisor $E_1 = p^{-1}_1(V_1)$ corresponds to the one defined in (3.6).

2) $\overline{(1,n)}$ is the closure of the orbits of the first $n$; the orbits of the first $n - 1$.

3) The closure of the orbits of the first $n - 1$.

4) One can replace the divisor by the hyperplane...
the tetrahedron by
the prism with two
prisms of the substi-
d (a 1-simplex $\sigma_1$)
of complete
of each vertex of the
of the monoidal
$V_q$ given in (3.2)
denote (1,1),
its which we
2.5).
first permissible
and thus the
(k) parametrizes
quadric of rank
less than or equal to \( k \) as quadric-envelope by straight lines (in other words, as a
direction in the tangent bundle). This gives the second part b) of 1). The first
part a) is a simple consequence.

On the other hand, also by (3.5), for each non-regular point of the
correspondence $\lambda_2^*$, e.g. for each point $P_q$ of $V_q$, there exists an analytic arc
meeting $V_q$ transversally at $P_q$ and such that all the points of the arc in a punct-
tured neighborhood of $P$ belong to $V_i \setminus V_i'$ for each $i$. Hence this gives that the
points of the exceptional divisor of birank $(1,1)$ are infinitely close points to
"regular" points of birank $(i, (i-1)/2)$ (the word regular is relative to the corre-
dence $\lambda_2^*$).

Lastly, the third assertion is consequence of the two first parts.

(5.5) Remark. - Let us note that there exists a relation $(r_0, r_1) \Rightarrow (r_0', r_1')$ between the closures of the orbits if and only if their corresponding points in
the natural representation relative to coordinates in $\mathbb{R}^n$, satisfy the inverse
relation $(p_0, p_1) \Leftrightarrow (p_0', p_1')$ relative to the lexicographical order in the product
space $\mathbb{R}^n$. Let us observe that this representation uses only 0 and 1 for giving
the coordinates of the vertices. Accordingly we shall call the $k$-th level the set
of vertices whose sum of coordinates is equal to $k$ in the last representation. By
lifting successively, the action in the same way as that considered in (3.1) we
arrive at the following result:

(5.6) Theorem. - Let us denote by $\Pi$ the mapping $B \rightarrow \mathbb{P}^N$ which is the com-
position of the monoidal transforms given at (3.3). Then

1) $\Pi^{-1}(B^N \setminus V_q)$ is the only maximal open dense orbit under the lifted action
corresponding to the 0-th level.

2) $\Pi^{-1}(V_q)$ is a divisor with normal crossings whose components are the
closures of the orbits adjacent to the maximal orbit whose interiors correspond to the
orbits of the first level.

3) The closure of each orbit of the $k$-th level is the intersection of the
closure of the orbits of the $(k-1)$-th level containing it for $k = 2, \ldots, n$.

4) One can represent the set of orbits corresponding to the exceptional
divisor by the hypergraph corresponding to the pointed $n$-hypercube $I_n \setminus \{0\}$ in $\mathbb{R}^n$.  

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6. CELLULAR DECOMPOSITION OF $B$

(6.1) This last part is founded on some aspects of the geometry of the lattice of the parabolic subgroups containing the group of triangular matrices. Let us interpret for a quadric the condition of passing through a fixed point as the condition of being "tangent to such a point". The following result is the general solution of the "Charakteristikenproblem" for hyperquadrics (Schubert, [7]).

(6.2) Basis Theorem. - The inverse images $m_r$ by $\Pi$ of the divisors $W_r$ of $F^N$ corresponding to the condition for a quadric-locus to be tangent to a fixed $r$-subspace $L_r$ for $r = 0, \ldots, n-1$ (see 2.5) give a basis of the cohomology ring $H^*(B, \mathbb{Z})$. Furthermore in Pic($B$) we have $\psi_r = -m_{r-1} + 2m_r - m_{r+1}$ for $r = 0, \ldots, n-1$ (Convention: $m_k = 0$ for $k > n$ or $k < 0$) where $\psi_0, \ldots, \psi_{n-1}$ represent the classes of the exceptional divisor components.

Sketch of proof

Let us denote by $G$ the group acting on $B$ which we have introduced in 2.9 for a fixed complete quadric $Q$. If rank $(Q) = [r_1, \ldots, r_k]$ then there exists some element $g \in G$ such that $Q, g$ admits a representation as a pseudo-diagonal matrix given by $k$ disjoint boxes of size $r_1, \ldots, r_k$ on the diagonal, which correspond to the $k$ quadric-loci $q_{r_1}, \ldots, q_{r_k}$ mentioned above in 2.5.

The stabilizer subgroup $P(r_1, \ldots, r_k)$ of the flag $F$ of the vertices $L_i = \text{ver}(q_i), i = 1, \ldots, k$, is also the stabiliser subgroup of the orbit of the complete quadric $Q \in B$. Obviously $P(r_1, \ldots, r_k)$ acts transitively on this orbit.

Let us denote by $\mathcal{P}$ the set of conjugacy classes of standard parabolic subgroups $P$ containing the Borel subgroup $B_0$ of $GL_k(n+1)$ and which we also denote by $P(r_1, \ldots, r_k)$. One may think of the elements of $P(r_1, \ldots, r_k)$ (respectively, $B_0$) as given by the upper pseudo-triangular matrices (respectively, upper triangular matrices). We introduce a partial order relation on $\mathcal{P}$ by setting $P(r_1, \ldots, r_{ik}) \leq P(r_1', \ldots, r_{ik}')$ if and only if $1 \leq k$ and for each $i = 1, \ldots, k$ there exists a $k(i) \leq k'$ such that $r_{ki} = r_{k(i)-1} + \ldots + r_{ki}'$.

Then $\mathcal{P}$ is a "poset" (partially ordered set) represented by an $n$-cube whose maximal element is $P(n+1) = GL_k(n+1)$ and whose minimal element is $P(1, \ldots, 1) = B_0$.

On the other hand, if $H'$ and $H''$ are algebraic subgroups of an algebraic group $G$ such that $H'' \leq H'$, the natural epimorphism $f: G/H'' \longrightarrow G/H'$ exhibits $G/H''$ as a quotient variety of $G/H'$. In particular this allows us to relate the $(G/P)$-equi-...
variant cellular decomposition and the \((G/B_0)\)-equivariant cellular decomposition for each parabolic subgroup \(P\) containing the Borel subgroup \(B_0\). The basic idea in the following reasoning is devoted to interpreting the transformation of the \(n\)-simplex into the \(n\)-cube given at \(54\) as an adjunction of cells.

First of all, let us observe that two adjacent parabolic subgroups of the poset \(\mathcal{P}\) differ only in the elements of some box of size \(r\). Thus, the problem is reduced to a study of the difference between the cellular decompositions associated to \(GL_k(r)\) and those associated to the subgroups of type \(GL_k(i,r-1)\) for each \(i = 1, \ldots, \left[\frac{r-1}{2}\right]\).

It is easy to see that the set of orbits corresponding to the actions of the subgroups \(GL_k(i,r-1)\) and \(GL_k(r)\) differ only at one Schubert-Bruhat cell. This last is, in more classical language, the complement of the condition for an \(i\)-plane to be tangent to a quadric, which is a Zariski open set.

As a by-product, if we iterate this process then by virtue of Proposition 5.6 we have obtained an explicit description of the lifted actions on each successive transform \(B_n^{(i)}\) of Corollary (3.4). Let us now consider the natural mappings in the following diagram

\[
\begin{align*}
\text{graph}(A_2 \times \cdots \times A_r) & \xrightarrow{b_{n}^{(r-1)}} \text{graph}(A_r) \\
\Lambda_{r} & \xrightarrow{\gamma} \Lambda_{r}^N
\end{align*}
\]

Observe that the primary condition mentioned above is the inverse image by \(\Lambda_r\) of the restriction of the cycle whose dual generates the first cohomology group \(H^1(G(i,r), \mathcal{O})\) of the Grassmannian for each \(i = 1, \ldots, r-1\). Then, the inverse image by \(g\) of the class of the hyperplane sections of \(\mathcal{O}^i\) for \(i = 1, \ldots, r\) gives a basis for \(H^1(B_n^{(r-1)}, \mathcal{O})\). This gives the first half of the result.

On the other hand remember now that \(V_k\) is a divisor on \(F_k\) of degree \(k+1\) such that \(V_k \subseteq W_k\) for \(4 \leq k\) (see 2.5) is contained at the \((k-r)\)-multiple locus of \(W_k\).

Let us consider the composition \(q_{k-1} = p_{k-1} \circ \cdots \circ p_1\) of the first \(k-1\) blow-ups for \(k \geq 2\). Let us denote by \(m_k\) the inverse image \(q_{k-1}^{-1}([W_k])\) on \(\text{Pic}(B_n^{(k-1)})\) and let us denote by \(\Psi_0, \ldots, \Psi_{k-1}\) the classes of the components of the exceptional divisor.

By using standard facts on the Picard group of the \(\sigma\)-process one can easily prove by induction on \(k\) the following formula:

\[
(k+1)m_0 = (k\Psi_0 + (k+1)\Psi_1 + \cdots + (k+1)\Psi_{k-1}) + m_k
\]

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Hence, if we develop \(2(k+1)m_o - (k+2)m_o - km_o = 0\) we obtain

\[\psi_k = -m_{k-1} + 2m_k - m_{k+1}\]

where one makes \(m_k = 0\) for \(k \neq n\) or \(k < 0\). We may write this in more synthetic form as

\[A_1 = A_n \cdot m\]

where \(\psi = (\psi_0, \ldots, \psi_{n-1})\), \(m = (m_0, \ldots, m_{n-1})\) and

\[
A_n = \begin{pmatrix}
2 & -1 & 0 & \cdots & 0 \\
-1 & 2 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -1 & 2
\end{pmatrix}
\]

is the Cartan matrix.

(6.3) Corollary.- Let us denote by \(\alpha_{k-1} = \psi_k \cap \psi_1\) for \(k = 2, \ldots, n\) some intersections of components of exceptional divisor on \(B\) corresponding to the closure of the dense open set of complete quadrics of rank equal to \((1, k, \ldots)\). Then with the above notation one has:

\[-2m_0 + 2m_1 - m_2 = \alpha_1\]

\[-m_{k-1} + 2m_k - m_{k+1} = \alpha_k\]

for \(k = 2, \ldots, n\).

(6.4) Remark.- In particular, for \(n = 3\) we obtain \(-2m_0 + 2m_1 - m_2 = \alpha_1\) and \(-m_2 + 2m_2 = \alpha_2\) which gives the two reduction formulas

\[m_1 = \frac{1}{3} (4m_0 + 2\alpha_1 + \alpha_2)\]

\[m_2 = \frac{1}{3} (2m_0 + \alpha_1 + 2\alpha_2)\]

This gives one justification of Schubert's initial computation of Charakteristikzahlen ([6]) based on these reduction formulas and the use of the degeneration principle.
REFERENCES


