IN VariantS of WeIGHTED hOMOGENEOUS sINGULARITIES

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1.- Introduction:

In singularity theory one has first studied hypersurface singularities, where a holomorphic map germ \( f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0) \) is given. Especially simple is the example \( f(z) = z_1^a_1 + \ldots + z_m^a_m \) (Brieskorn polynomial). A more general class is given by the weighted homogeneous polynomials: Let \( d, w_1, \ldots, w_m \) be positive integers and \( f \in \mathbb{C}[z_1, \ldots, z_m] \) a polynomial.

Definition:

\[ f \text{ is weighted homogeneous of degree } d \text{ with respect to the weights } w_1, \ldots, w_m \text{ if } f \text{ is a linear combination of monomials } z_1^{j_1} \ldots z_m^{j_m}, \]

\[ j_1 w_1 + \ldots + j_m w_m = d. \]

Let us assume that \( f \) is such a weighted homogeneous polynomial which has an isolated singularity at \( 0 \) and that \( n := m - 1 > 0 \). Let \( \varepsilon \) be a positive real number (in general, \( \varepsilon \) should be small, but since \( f \) is weighted homogeneous this is not necessary here), \( t \in \mathbb{C}, 0 < |t| < \varepsilon \), \( Y_t = f^{-1}(t) \), \( Y_0 = f^{-1}(0) \). Then \( B_\varepsilon \cap Y_t \) is the Milnor fibre of \( f \) and \( \Sigma = \partial B_\varepsilon \cap Y_0 \) the link of \( Y_0 \) at \( 0 \). It is well-known that \( B_\varepsilon \cap Y_t \) has the homotopy type of a bouquet of spheres of dimension \( n \), their number is called the Milnor number \( \mu \), and the singularity of \( Y_0 \) at \( 0 \) is determined by \( \Sigma \) since \( B_\varepsilon \cap Y_0 \) is homomorphic to the cone over \( \Sigma \). There is an endomorphism \( h^* \) of \( H^n(B_\varepsilon \cap Y_t, \mathbb{Z}) \) – the (Picard-Lefschetz) monodromy – such that we have an exact sequence (with coefficients \( \mathbb{Z} \)):

\[ 0 \rightarrow H_n(\Sigma) \rightarrow H^n(B_\varepsilon \cap Y_t) \xrightarrow{h^* - \text{id}} H^n(B_\varepsilon \cap Y_t) \rightarrow \cdots \rightarrow 0 \]

Cf. [6].

As we are looking at the weighted homogeneous case \( B_\varepsilon \cap Y_t \) and \( B_\varepsilon \cap Y_0 \), are deformation retracts of \( Y_t \) and \( Y_0 \), respectively. Therefore we will consider \( Y_t \) instead of \( B_\varepsilon \cap Y_t \), the condition \( |t| < \varepsilon \) is no longer necessary then.
Let us list some invariants which have been calculated in the weighted homogeneous case already long ago:

\[ u = \text{rk\ } H^1(Y_t; \mathbb{Z}) : \text{Milnor-Orlik} [7] \]
characteristic polynomial \( \chi(x) \) of \( h^* : \text{Milnor-Orlik} [7] \)
\[ \text{rk } \mathcal{R}_{n-1}(\Sigma; \mathbb{Z}) : \text{Orlik} [8] \text{ (as a consequence of [7])} \]
\[ \sigma = \text{signature of } Y_t : \text{Steenbrink} [11] \]
Hodge numbers of \( Y_t \) (with respect to the mixed Hodge structure): Steenbrink [11].

In the next paragraph we will briefly recall the methods used in the hypersurface case and go over in the third paragraph to complete intersections. As a by-product we will get a further result for the hypersurface case, by determining the group \( \mathcal{R}_{n-1}(\Sigma; \mathbb{Z}) \).

2.- Methods in the hypersurface case:

Let us shortly discuss how the computations of the invariants cited above have been performed:

a) The Milnor number \( u \) can be calculated via the mapping degree of \( (z_1, \ldots, z_m) \to \left( \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_m} \right) \), because of the formula \( u = \dim \mathfrak{m} \mathfrak{m}_{0} / \left( \frac{\partial f_1}{\partial z_1}, \ldots, \frac{\partial f_m}{\partial z_m} \right), \) cf. [7].

b) For the study of the endomorphism \( h^* \) of \( H^n(Y_t; \mathbb{Z}) \) it is useful to note that \( f : \mathbb{C}^m \to \mathbb{C} \) is equivariant with respect to \( \mathbb{C}^* \) acting on \( \mathbb{C}^m \) and \( \mathbb{C} : c \cdot z := (c \cdot z_1, \ldots, c \cdot z_m) \), \( c \cdot t := c \cdot t' \) for \( c \in \mathbb{C}^*, z \in \mathbb{C}^m, t' \in \mathbb{C} \)
Then \( h^* \) is induced by \( h : Y_t \to Y_t : h(z) = e^{2\pi i/d} z \). From the Euler characteristics of the fixed point sets \( (z \in Y_t | h^*(z) = z) \), \( v = 1, 2, \ldots, \), one can compute the characteristic polynomial of \( h^* \), cf. [7].

c) The signature \( \sigma \) of \( Y := Y_t \) can be computed from the Hodge numbers, cf. [11]. In order to understand the mixed Hodge structure on \( Y \), let us begin with the special case \( w_1 = \ldots = w_m = 1 \) (\( f \) homogeneous): then \( Y \) is the affine part of some smooth projective hypersurface \( Y \) of degree \( d \) in \( \mathbb{P}^m(\mathbb{C}) \)
In general one takes \( \mathcal{P}_W := \text{Proj} \mathfrak{m} \{ z_0, \ldots, z_m \} \), where \( \deg z_j = w_j \), \( j = 0, \ldots, m \), \( w_0 = 1 \), instead of \( \mathbb{P}^m(\mathbb{C}) \). The underlying topological space of \( \mathcal{P}_W \) is the quotient of \( \mathbb{C}^{m+1} \setminus \{0\} \) by the \( \mathbb{C}^* \) action which corresponds to

the weights \( w_1, \ldots, w_m \), hence a rational \( f(z_0, \ldots, z_m) = f^{(w)} \) with respect to \( w_1, \ldots, w_m \) and \( Y_0 \) of \( Y \) is compact, according to [11] so we can speak for our purposes \( Y_0 \) denotes the jth

0 \leq p \leq \dim Y

3.- Complete intersections.

Let us list some invariants which have been calculated for \( \text{ polynomials sucribing } w_i, \ldots, w_m \) with respect to \( Y_0 \) of \( Y \) of degree \( d \) and \( \Sigma = Y_0 \cap \mathbb{A}^m \), a regular value of \( f \) and that \( Y_0 \) is a bouquet of \( Y \), invariants not
the weights \( w_0, \ldots, w_m \). Now \( \mathbb{P}_w \) is no longer smooth but still a \( V \)-manifold, hence a rational homology manifold. Let \( \mathcal{Y} \in \mathbb{C}[z_0, \ldots, z_m] \) be defined by \( \mathcal{Y}(z_0, \ldots, z_m) = f(z_1, \ldots, z_m) - t z_0^d \). Then \( \mathcal{Y} \) is weighted homogeneous of degree \( d \) with respect to \( w_0, \ldots, w_m \). The equation \( \mathcal{Y} = 0 \) defines subvarieties \( \mathcal{Y} \) and \( Y_\infty \) of \( \mathbb{P}_w \) and \( \{(z) \in \mathbb{P}_w | z_0 = 0 \} \) which are \( V \)-manifolds. Since \( \mathcal{Y} \) and \( Y_\infty \) are compact algebraic varieties they have canonical mixed Hodge structures according to Deligne [1], but in fact these are pure because we have \( V \)-manifolds, so we can speak of Hodge numbers \( \emph{h}^{pq}(\mathcal{Y}) = \dim_\mathbb{C} \emph{Gr}_n^0 \emph{H}^{pq}(\mathcal{Y}; \mathbb{C}) \). More important for our purpose are the numbers \( \emph{h}^{pq}(Y_\infty) = \dim_\mathbb{C} \emph{Gr}_n^0 \emph{H}^{pq}(Y_\infty; \mathbb{C}) \), where \( \emph{H}^{j}(Y_\infty; \mathbb{C}) \) denotes the \( j \)-th primitive cohomology group. Then \( \emph{h}^{pq}(Y_\infty) = \emph{h}^{pq}(\mathcal{Y}) - \delta_{pq} \), \( 0 \leq p \leq \dim \mathcal{Y} \). From the Gysin sequence for \( Y_\infty \subset Y \) one gets:

\[
\begin{align*}
\dim \emph{Gr}_n^0 \emph{H}^0(Y_\infty; \mathbb{C}) &= \emph{h}^0, \\
\dim \emph{Gr}_n^0 \emph{H}^{n-1,n-p}(Y_\infty) &= \emph{h}^{n-1,n-p}(Y_\infty) \\
\emph{Gr}_n^0 \emph{H}^n(Y_\infty; \mathbb{C}) &= 0, \quad i = n, \quad n+1.
\end{align*}
\]

See [11] for details. By generalizing Griffiths' description of the Hodge structure for a projective hypersurface [3] Steenbrink was able to describe the spaces \( \emph{F} \emph{Gr}_n^0 \emph{H}^n(Y; \mathbb{C}) \) explicitly, see [11].

3.- Complete intersections:

Let us leave the hypersurface case now and assume that \( f_1, \ldots, f_k \) are polynomials such that \( f_j \) is weighted homogeneous of degree \( d_j \) with respect to \( w_1, \ldots, w_m \) for \( j = 1, \ldots, k \). Here \( w_1, \ldots, w_m, d_1, \ldots, d_k \) are positive integers. Let \( \emph{F} : \mathbb{C}^m \to \mathbb{C}^k \) be defined by \( \emph{F} = (f_1, \ldots, f_k) \), let \( t \) be a regular value of \( \emph{F} \), \( Y_t = \emph{F}^{-1}(\{t\}) \), \( Y_0 = \emph{F}^{-1}(\{0\}) \), \( n = \dim Y_0 \), \( \Sigma = Y_0 \cap \mathfrak{A}_\infty \). In order to be able to compute invariants from the weights and degrees alone we assume that \( n = m-k \) (i.e. \( Y_0 \) is a complete intersection) and that \( Y_0 - \{0\} \) is non-singular, \( n > 0 \). Again \( Y_t \) has the homotopy type of a bouquet of spheres of dimension \( n \) [5]. Let us discuss the calculation of invariants now.
a) \( u = \text{rk} H^n_0(Y_t; \mathbb{Z}) \): This invariant has been calculated using differential forms in [2], the method is not just a generalization of the method described in the hypersurface case.

b) The monodromy is a more complicated object in the case \( k \geq 2 \) than for \( k = 1 \): one has an action of \( \pi_1(\mathbb{C}^k,0) \) on \( H^n(\mathbb{C}^k,0) \), \( D \) being the discriminant of \( f \). For \( k = 1 \), \( \pi_1(\mathbb{C}^1,0) \cong \mathbb{Z} \), and the endomorphism \( h^* \) of \( H^n(Y_t; \mathbb{Z}) \) introduced in this case corresponds to the action of the canonical generator. In the weighted homogeneous case, however, we have another possibility of generalizing the definition of \( h^* \). There is a \( \mathbb{C}^k \) action on \( \mathbb{C}^k \) defined by \( c \cdot t_j = (c \cdot t_j, \ldots, c \cdot t_j) \), \( c \in \mathbb{C}^k \), such that \( f: \mathbb{C}^m \to \mathbb{C}^k \) is equivariant. Let \( d \) be a positive integer such that \( t = e^{2\pi i/d} \cdot t \), i.e. \( d | j \) for all \( j \) with \( t_j = 0 \). Let \( h^*: H^n(Y_t; \mathbb{Z}) \to H^n(Y_t; \mathbb{Z}) \) be induced by \( h(z) = e^{2\pi i/d} \cdot z \). Now the characteristic polynomial \( \chi(x) \) of \( h^* \) can be calculated by the results of [2].

c) The algebraic variety \( Y = Y_t \) has a mixed Hodge structure which can be described just as in the hypersurface case: Let \( \mathbb{C}^n_1(\mathbb{C}^n_2, \ldots, \mathbb{C}^n_m) \) be defined by \( \mathbb{C}^n_1(z_1, \ldots, z_m) = \mathbb{C}^n_1(z_1, \ldots, z_m) - t_j z_j \), then \( \mathbb{C}^n_1 = \mathbb{C}^n_1 = \ldots = \mathbb{C}^n_k = 0 \) defines subvarieties \( \mathbb{V} \) and \( Y_0 \) of \( \mathbb{P}_1 \) and \( (\mathbb{C}^n_1(\mathbb{C}^n_2, \ldots, \mathbb{C}^n_m) = 0) \), and the results on \( \text{Gr}_P \text{Gr}_W H^n(Y; \mathbb{C}) \) are the same as in the hypersurface case. In order to state the formulae it is convenient to use the abbreviation

\[
Q(x,y) = \prod_{x=0}^{m} \prod_{y=0}^{k} \frac{1+y}{1-x} \prod_{x=0}^{w} \frac{1-y}{1-x} \prod_{y=0}^{d} \frac{1-x}{1-x}.
\]

Theorem 1 (cf. also formula (1) in [4]):
\[
h^n_0(Y) = (-1)^{n-p} \text{res}_{x=0} \text{res}_{y=0} \frac{x^{-1} y^{-p} - 1 + y x}{1 - x} Q(x,y),
\]

\[
h^{p-1}_0(Y) = (-1)^{-p} \text{res}_{x=0} \text{res}_{y=0} \frac{x^{-1} y^{-p} Q(x,y)}{1 - x}.
\]

The proof uses the description of the pure Hodge structure of projective varieties which are \( V \)-manifolds due to Steenbrink [1], it will be published elsewhere, as the proofs of the following theorems.
The same technique also yields more information about $h^*$ than that obtained in b). Note that $h^2 = \text{id}$, therefore $h^*: H^0(Y;\xi)_0 \to H^0(Y;\xi)_0$ is diagonalizable, and the eigenvalues are of the form $e^{2\pi ir/d}$, $r \in \mathbb{Z}$. Let $h^0_{r,p}(e^{2\pi ir/d})$ be the dimension of the subspace of $Gr_F^p H^0(Y;\xi)_0$ on which $h^*$ operates as multiplication by $e^{2\pi ir/d}$.

Theorem 2:

\[
h^0_{r,p}(e^{2\pi ir/d}) = (-1)^{n-p} \text{res}_{z=0} \text{res}_{x=0} \text{res}_{y=0} x^{-1} y^{-p-1} z^{r-1} \frac{z^d}{1-z^d} \frac{1+yzx}{1-zx} Q(x,y).
\]

Note that $h^*$ acts on $H^{n-1}(Y;\xi)_0$ as the identity. Now let $n$ be even, let $S$ be the intersection form on $H^0_c(Y;\xi)$ and $S^h$ the hermitian form on $H^0_c(Y;\xi)$ defined by $S^h(x,y) = S(x,y)$. Let $u^+(e^{2\pi ir/d})$ be the dimension of a maximal linear subspace of $H^0(Y;\xi)$ on which $S^h$ is positive definite and $h^*$ acts as multiplication by $e^{2\pi ir/d}$. Let $u^-(e^{2\pi ir/d})$ and $u^0(e^{2\pi ir/d})$ be defined in an analogous way, with "negative definite" resp. "identically zero" instead of "positive definite". From Theorem 2 and [11] one obtains

Theorem 3:

If $n$ is even we have:

\[
u^+(e^{2\pi ir/d}) = \text{res}_{z=0} \text{res}_{x=0} \text{res}_{y=0} x^{-1} y^{-n-1} z^{r-1} \frac{1}{1-y^2} \frac{z^d}{1-z^d} \frac{1+yzx}{1-zx} Q(x,y),
\]

\[
u^-(e^{2\pi ir/d}) = \text{res}_{z=0} \text{res}_{x=0} \text{res}_{y=0} x^{-1} y^{-n} z^{r-1} \frac{-1}{1-y^2} \frac{z^d}{1-z^d} \frac{1+yzx}{1-zx} Q(x,y),
\]

$u^0(e^{2\pi ir/d}) = 0$ if $d \not| r$.

\[
u^0(1) = \dim H^{n-1}(Y;\xi)_0 = \rk \tilde{H}^n_{n-1}(\xi;\mathbb{Z}) \quad \text{(see Theorem 4)}.
\]
d) Calculation of $\overline{H}_{n-1}(\Sigma;\mathbb{Z})$ : As Steenbrink pointed out to me, $\overline{H}_{n-1}(Y_0 - \{0\}%;\mathbb{Z}) \cong H_{n-1}(Y_0;\mathbb{Z})_0$ because of the existence of a rational Gysin sequence for the map $Y_0 - \{0\} \to \Sigma$ which is the canonical map to the orbit space with respect to the $\mathbb{C}^*$ action. Using this and the homotopy equivalence between $Y_0 - \{0\}$ and $\Sigma$ we obtain

Theorem 4 :

$$\text{rk } \overline{H}_{n-1}(\Sigma;\mathbb{Z}) = \text{res}_{x=0} \text{res}_{y=0} x^{-1} y^{-n} \frac{1}{1+y} Q(x,y).$$

So it remains to calculate the torsion subgroup of $\overline{H}_{n-1}(\Sigma;\mathbb{Z})$. There is an explicit way of doing this for $n$ odd by looking at certain fixed point sets similarly to section b) of the second paragraph.

In the special case where $f_1, \ldots, f_k$ are Brieskorn polynomials the torsion has been calculated for all $n$ by Randell [10].

The following object is related to $\overline{H}_{n-1}(\Sigma;\mathbb{Z})$ : let us assume that $\mathcal{Y}_0 = \{z \in \mathbb{C}^n | f_1(z) = \ldots = f_{k-1}(z) = 0\}$ has also an isolated singularity at $0$. Then let us consider $H_n(\mathcal{X};\mathbb{Z})$, where $\mathcal{X} = 3B \cap \mathcal{Y}_0$.

e) Calculation of $H_n(\mathcal{X};\mathbb{Z})$ : Because of the assumption just made we may choose $t = (0, \ldots, 0, t_k)$ and $d = d_k$. Then the exact sequence of the introduction has the following analogue (cf. [5], coefficients : $\mathbb{Z}$) :

$$0 \to H_{n+1}(\mathcal{X};\mathbb{Z}) \to H^n(\mathcal{Y}) \xrightarrow{h^* - \text{id}} H^n(\mathcal{Y}) \to H_n(\mathcal{X};\mathbb{Z}) \to 0.$$

As we can compute the characteristic polynomial $\chi(x)$ of $h^*$ (cf. b)), we can deduce a formula for $\text{rk } H_n(\mathcal{X};\mathbb{Z})$.

Note that there is an exact sequence

$$0 \to H_n(\mathcal{X};\mathbb{Q}) \to H_n(\mathcal{X};\mathbb{Q}) \to H_{n-1}(\Sigma;\mathbb{Q}) \to 0$$

(since $\overline{H}_{n-1}(\Sigma;\mathbb{Q}) \cong H_{n-1}(Y_0;\mathbb{Q})_0$ etc...) so that $\text{rk } H_n(\mathcal{X};\mathbb{Z})$ can also be calculated from Theorem 4. On the other hand, one can prove Theorem 4 inductively using a suitable formula for $\text{rk } H_n(\mathcal{X};\mathbb{Z})$.

But in fact, the whole group $H_n(\mathcal{X};\mathbb{Z})$ can be calculated from $\chi(x)$ : Write $x = m_1 \cdot \ldots \cdot m_\mu$ where $m_1 | m_2, \ldots, m_{\mu-1} | m_\mu$ and $m_\mu$ is square-free, $m_1, \ldots, m_\mu \in \mathbb{Z}$ [x].

Theorem 5 :

$$H_n(\mathcal{X};\mathbb{Z})$$

In the case

Corollary :

$$\overline{H}_{n-1}(\Sigma;\mathbb{Z})$$

The corollary by Orlik and Randell...
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polynomials. 

260-269.


Theorem 5:

\[ H_n(\Sigma; \mathbb{Z}) = (\mathbb{Z}/m_1(1)\mathbb{Z}) \oplus \ldots \oplus (\mathbb{Z}/m_\mu(1)\mathbb{Z}). \]

In the case \( k = 1 \) we have \( \Sigma = \mathfrak{g} \mathfrak{b} \), so \( H_n(\Sigma; \mathbb{Z}) \cong \tilde{H}_{n-1}(\Sigma) \):

Corollary:

\[ \tilde{H}_{n-1}(\Sigma; \mathbb{Z}) = (\mathbb{Z}/m_1(1)\mathbb{Z}) \oplus \ldots \oplus (\mathbb{Z}/m_\mu(1)\mathbb{Z}) \] if \( k = 1 \).

The corollary has been conjectured by Orlik [8] and proved in special cases 
by Orlik and Randell (see [8], [9], [10]).