NUMERICAL CHARACTERS OF A CURVE IN PROJECTIVE n-SPACE

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§1. Introduction

The enumerative properties of a curve embedded in projective n-space can be studied by considering curves in lower-dimensional projective spaces, derived from the given curve in a natural way. Such was the point of view of Veronese (13), who used the 'principle of projections and sections' to establish relations among the numerical characters of a curve in n-space by applying Pliicker's formulas to n-1 plane curves derived from the given one. This method was first used by Cayley (13) in the case of a curve in 3-space.

The numerical characters we shall consider are the ranks - the number of osculating spaces to the curve which satisfy a given Schubert condition - and the stationary indices - the number of hyperosculating spaces (i.e., osculating spaces which have 'too high' order of contact with the curve). The points of hyperosculatation were called singular points of higher order by Pohl (13); in fact, he represented the osculating spaces by vector bundles (dua to the jet bundles) and got the points of hyperosculatation as the singularities of maps of bundles. In this paper we show bow (the dual of) his representation can be used to prove the geometric statements about projections and sections made by Veronese. Another application of Pohl's viewpoint was given by Kato

1 An earlier version of this paper forms part of the author's dissertation, written under the direction of Professor S. L. Kleiman at MIT. While a graduate student at MIT (1974-76), the author was supported by the University of Oslo and the Norwegian Research Council for Science and the Humanities.
(9), who described the higher order singularities of an elliptic curve of degree $n+1$ in $\mathbb{P}^1$ as the points of order $n$ of the curve considered as abelian variety.

We shall assume, for simplicity, that the base field $k$ is algebraically closed, rather than just finite. By a curve we mean a proper 1-dimensional, reduced, proper, algebraic scheme. Any embedding $f: X_0 \to P$ of a curve $X_0$ in projective $n$-space is assumed to be such that $X_0$ spans $P$. Given an embedding $f$, we let $f : X \to P$ denote the finite map obtained by composing $f$ with the normalization $X \to X_0$. In §2 we define, for each integer $m$, $0 \leq m \leq n$, the osculating $m$-space $S(X, m)$ to the branch of $X_0$ at $f(x)$ corresponding to a point $x \in X$. We show that these spaces, for ordinary (non-hyperosculating) points $x$, are the projective fibers of the bundle $\mathcal{P}(2)$ of principal parts (or jets) of order $m$ of the canonical line bundle $\mathcal{O} = \mathcal{O}(1)$. A modification $\mathcal{P}^m$ of this bundle, called the osculating bundle of order $m$, gives (if $\operatorname{char} k = 0$ or $\operatorname{char} k$ is sufficiently large) also the hyperosculating $m$-spaces.

The numerical characters of the curve—the ranks and the stationary indices—are defined and interpreted in §3, and the formulas relating them (and the genus of the curve) are deduced (3.2). We then proceed to treat the principle of projections and sections, in §4. A $q$-projection of $f$ is defined as a generic linear projection of $f$ to $\mathbb{P}^q$, while an $m$-section $X_m \to \mathbb{P}^{n+m}$ of $f$ is defined to be the intersection of a (general) subspace $\mathbb{P}^{n+m}$ of $P$ with the $m$th osculating developable of $X_0$, i.e., with the ruled $(n+1)$-dimensional variety generated over $X_0$ by the $m$-spaces $S(X, m)$. We prove associativity (4.2, 4.4) and commutativity (4.6) of these two operations and relate the numerical characters of the derived curves to those of $X_0$ (4.7). Finally, in §5, Weyl's proof of duality for analytic curves is adapted. Our key result is an isomorphism of exact sequences (5.2) which shows that all the natural duality statements for a curve and its derived curves hold. As an application we deduce the Cayley-Bacharach equations for a curve in 3-space.

Three examples are carried through: the first is the rational $n$-ic $R_n$ in $n$-space. This curve is particularly simple: it has no points of hyperosculating and it is self-dual. The second example is a linearly normal, elliptic curve of degree $d \leq 3$. The third curve, $T_n$, was discussed by Dye [6] (5). It is the complete intersection of $n-1$ hypersurfaces of degree $d$ which are in special position.

In the appendix (86) we recall the definition of the functor of principal parts. We show the existence, as well as some general properties, of a natural transformation of functors which will be used to define the osculating bundles.

As far as the characteristic of $k$ is concerned, let us remark that in small, positive characteristic the osculating bundles $\mathcal{P}^m$ (though well defined) may not represent the hyperosculating spaces. For $n = \deg \mathcal{O}^m$ still to hold the osculating spaces could be defined as the fibers of $\mathcal{P}^m$. This definition also makes sense for any finite map (not necessarily birational onto its image) $f : X \to P$ such that $\mathcal{E}^* : X_0 \to \mathcal{P}(2)$ (where $\mathcal{E} = H^2(P, \mathcal{O}(1))$, see §2) is generically surjective. If $f : X' \to f(X)$ denotes the normalization and $g : X \to X'$ the induced map one can show that in this case one has $r_m(f) = (\deg g)^r_m(f')$; the restriction to the case of an embedded curve (deg $g = 1$) is made in order to simplify notations.

§2. The osculating bundles

Let $X_0 \to P = \mathbb{P}(V)$ be a curve embedded in, and spanning, projective $n$-space. Let $X \to X_0$ denote the normalization and $f : X \to P$ the induced map. For each point $x \in X$ we let $B(x)$ denote the corresponding branch at $f(x)$ of $X_0$. For each integer $m$, $0 \leq m \leq n$, we define the osculating $m$-space $S(x, m)$ to $B(x)$ to be the linear $m$-dimensional subspace of $P$ which has the highest order of contact with $B(x)$ at $f(x)$. To be more precise, put $A = S(X, m)$ and let $x \in A$ denote a uniformizing parameter. There is a parametrization $x_i = a_i t^i + o(t^i)$ (higher order terms),

$$i = 0, 1, \ldots, n, \text{ with } 0 < a_i \equiv a_i \equiv \ldots \equiv a_n \text{ (where the } a_i \text{ are considered as elements of the completion } \hat{A} \text{ of } A), \text{ and } a_n \neq 0. \text{ With this choice of coordinates in } P, \text{ the space defined by } X_{n+1} = X_{n+2} = \ldots = X_0 = 0 \text{ and has } (n+1+m+1)-\text{order contact with } B(x) \text{ at } f(x).$$

If $i = 0$ (and hence $i = 0, 0 \leq i \leq n$) holds, we say that $x$ is an ordinary point. The points of hyperosculating are points with $a_i = 0$ for some $m$. Pohl called such a point singular of order $m$ ([13]). Classically, a hyperosculating space was also called stationarity, e.g., a cusp is a stationary point, at a flex there is a stationary tangency, a stall is a stationary osculating hyperplane.

The osculating spaces $S(x, m)$ are the (projective) fibers of an $(m+1)$-bundle on $X$, which we will call the osculating bundle of order $m$; the formulation we use is dual to Pohl's ([13]). Let $\mathcal{P}(2)$ denote the functor of principal parts (or jets) of order $m$ (86). Put $\mathcal{E} = \mathcal{O}(2)$. Since $X$ is a smooth curve, $\mathcal{P}(2)$ is an $(m+1)$-bundle; in fact, there are exact sequences, for $m \geq 1$,

$$0 \to \mathcal{P}(2) \to \mathcal{P}(1)(2) \to \mathcal{P}(2) \to 0$$

([7], 16.10.1, 16.7.3). Let $g : x \to \mathbb{P}(k)$ denote the structure map. There are functorial maps (86)

$$a^m* : \mathcal{P}(2) \to \mathcal{P}(2).$$
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Three examples are carried through in §6; the first is the rational $n$-ic $R_n$ in $n$-space. This curve is particularly simple: it has no points of hyperosclusion and it is self-dual. The second example is a linearly normed, elliptic curve of degree $d \geq 3$. The third curve, $\Gamma_n$, was discussed by Dye ([4], [5]). It is the complete intersection of $n - 1$ hypersurfaces of degree $d$, which are in special position.

In the appendix (§6) we recall the definition of the functor of principal parts. We show the existence, as well as some general properties, of a natural transformation of functors which will be used to define the osculating bundles.

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\[ \begin{array}{c}
\varphi = 0, 1, \ldots, n, \quad 0 = 0 \leq \varphi \leq \cdots \leq \varphi_n \quad (\text{where the } x_0 \text{'s are considered as elements of the completion } \hat{A} \text{ of } A), \quad a_0, \ldots, a_n, \quad \end{array} \]

with this choice of coordinates in $P, S(x, m)$ is the space defined by $X_{m+1} = X_{m+2} = \cdots = X_n = 0$ and has $(m+1)$-order contact with $B(x)$ at $f(x)$. If $\varphi_n = 0$ (and hence $\varphi = 0, 0 \leq m \leq n$) holds, we say that $x$ is an ordinary point. The points of hyperosclusion are points with $\varphi_n > 0$ for some $m$. Pohl called such a point singular of order $m$ ([13]). Classically, a hyperoscluding space was also called stationary, e.g., a cusp is a stationary point, at a flex there is a stationary tangent, a stall is a stationary osculating hyperplane.

The osculating spaces $S(x, m)$ are the (projective) fibers of an $(m+1)$-bundle on $X$, which we will call the osculating bundle of order $m$; the formulation we use is dual to Pohl’s ([13]). Let $\mathcal{P}^m(\mathcal{E})$ denote the functor of principal parts (or jets) of order $m$ (§6). Put $\mathcal{E}' = f^*\mathcal{O}_P(1)$. Since $X$ is a smooth curve, $\mathcal{P}_m(\mathcal{E})$ is an $m$-bundle—indeed, there are exact sequences, for $m \geq 1$,

\[ \begin{array}{c}
\mathcal{P}_m(\mathcal{E}) \xrightarrow{\alpha^m} \mathcal{P}_{m+1}(\mathcal{E}) \xrightarrow{\beta^m} \mathcal{P}_{m+2}(\mathcal{E}) \xrightarrow{\gamma^m} \cdots \xrightarrow{\delta^m} \mathcal{P}_{m+n}(\mathcal{E}) \xrightarrow{\epsilon^m} 0
\end{array} \]

(7), 16.10.1, 16.7.3). Let $g : X \rightarrow \text{spec } (k)$ denote the structure map. There are functorial maps (§6)

\[ \begin{array}{c}
\alpha^m : \mathcal{E} \xrightarrow{g^*} g_* \mathcal{E} \xrightarrow{\alpha^m} \mathcal{P}_m(\mathcal{E}) \xrightarrow{\delta^m} \mathcal{P}_{m+n}(\mathcal{E}) \xrightarrow{\epsilon^m} 0
\end{array} \]
compatible with the maps \( b^m \). Let \( a^m : V_x \rightarrow \mathcal{O}^m(\mathcal{L}) \) denote the composition of \( a^m(\mathcal{L}) \) with the map \( V_x = g^m H^m(X, \mathcal{L}) \rightarrow g^m H^m(X, \mathcal{L}) = g^m A_4 \).

**Proposition (2.1).** Let \( a^m \in \mathcal{O}^m(\mathcal{L}) \) denote the image of the homomorphism \( a^m \). For each ordinary point \( x \in X \), the embedded linear space \( P(a^m(x)) = P(V_x(x)) = P \) is equal to the osculating \( m \)-space \( S(x, m) \); if \( \text{char} \ k = 0 \) or sufficiently big, then this holds for all points of \( X \).

**Proof.** Let \( x \in X \) and choose a parametrization \( \theta \) of \( S(x) \). Locally, the map \( a^m \) consists of taking the \( m \)-jets (or the Taylor series development up to order \( m \)) of the coordinate functions \( \theta \); consider \( \theta \) the basis \( (1, d\theta, \cdots, (d\theta)^m) \) for the free \( A \)-module \( \mathcal{O}^m(\mathcal{L}) \otimes A = \mathcal{O}^m(\mathcal{L}) \); let \( (\theta^0, \theta^1, \cdots, \theta^m) \) denote the dual basis for \( \mathcal{O}^m(\mathcal{L}) \). Then

\[ a^m \otimes A : V_x \otimes A = A^{m+1} \rightarrow \mathcal{O}^m(\mathcal{L}) \]

is given by the matrix

\[
\begin{pmatrix}
  \theta_0 & \theta_1 & \cdots & \theta_m \\
  d\theta_0 & d\theta_1 & \cdots & d\theta_m \\
  \vdots & \vdots & \ddots & \vdots \\
  d^m \theta_0 & d^m \theta_1 & \cdots & d^m \theta_m 
\end{pmatrix}
\]

Since \( d^j \) is a truncated Taylor series, we have \( d^j(\theta^i) = (i_d)^{m+1} \), for all \( i, j \). Hence we get (considering the entries as elements of \( A \)),

\[
\begin{pmatrix}
  1 & a_1 \theta^{m+1} & \cdots & a_m \theta^{m+1} & \cdots \\
  (i+1) a_1 \theta^{m+1} & \cdots & (i+1) a_m \theta^{m+1} & \cdots \\
  \vdots & \vdots & \ddots & \vdots \\
  \frac{i+m}{m} a_1 \theta^{m+1} & \cdots & \frac{i+m}{m} a_m \theta^{m+1} & \cdots 
\end{pmatrix}
\]

(where \( + \cdots \) indicates terms of higher powers of \( \theta \), and \( * \) indicates elements in the maximal ideal of \( A \), see also [92], p. 10.).

**Case 1.** \( x \) is an ordinary point, i.e., \( l_1 = l_2 = \cdots = l_m = 0 \). Then \( a^m \) is surjective at \( x \), so that \( \mathcal{O}^m(x) = \mathcal{O}^m(\mathcal{L}(x)) \), and \( a^m(x) \) is given by the matrix

\[
\begin{pmatrix}
  1 & 0 & \cdots & 0 \\
  0 & 1 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 1 \\
  0 & 0 & \cdots & 0 
\end{pmatrix}
\]

hence it defines \((\mathcal{E}, \mathcal{E})\) the subspace of \( P \) given by \( \mathfrak{X}_{m+1} = \mathfrak{X}_{m+2} = \cdots = X = 0 \).

**Case 2.** \( l_1 > 0 \) and \( \left( \frac{l_1}{l_2} \right) = 0 \) (char \( k \)) for \( i = 1, \cdots, n \). The inclusion \( \mathcal{O}^m \subset \mathcal{O}^m(\mathcal{L}) \) is given, locally at \( x \), by the matrix

\[
\begin{pmatrix}
  1 & 0 & \cdots & 0 \\
  0 & i_1 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & i_n 
\end{pmatrix}
\]

and \( a^m(x) \) is given by

\[
\begin{pmatrix}
  1 & 0 & \cdots & 0 \\
  0 & 1 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 1 \\
  0 & 0 & \cdots & 0 
\end{pmatrix}
\]

Hence it defines the same linear \( m \)-space as in case 1.

The bundle \( \mathcal{O}^m = \text{Im}(a^m) \) will be referred to as the osculating bundle of order \( m \) of the curve \( f : X \rightarrow P \). From now on we assume char \( k \) to be such that \( \mathcal{O}^m \) represents the osculating \( m \)-spaces at all points of \( X \).

**Example 1.** The rational \( n \)-ic in \( n \)-space.

The curve \( R_n \subset P \) of degree \( n \) and genus 0 has only ordinary points. Every point \( x \in R_n \) has a parametrization \( (1, i, i^2, \cdots, i^n) \), hence \( a^{m} : V_x \rightarrow \mathcal{O}^m(\mathcal{L}) \) is everywhere surjective and the osculating bundle of order \( m \) is equal to \( \mathcal{O}^m(\mathcal{L}) \) for \( m = 0, 1, \cdots, n \).
compatible with the map \( b^m \). Let \( a^m : V_X \rightarrow \mathcal{P}_m(X) \) denote the composition of \( a^m(\mathcal{E}) \) with the map \( V_X = g^m H^m(P, \mathcal{E}(1)) \rightarrow g^m H^m(X, \mathcal{E}) = g^m \mathcal{E} \).

**Proposition (2.1).** Let \( \mathcal{P}^m \sqsubseteq \mathcal{P}(\mathcal{E}) \) denote the image of the homomorphism \( a^m \). For each ordinary point \( x \in X \), the embedded linear space \( \mathcal{P}(\mathcal{P}^m(x)) = P \) is equal to the osculating \( m \)-space \( S(x, m) \); if \( k \leq m \) it is sufficiently big, then this holds for all points of \( X \).

**Proof.** Let \( x \in X \) and choose a parametrization \((\theta(x))\) of \( \mathcal{P}(\mathcal{E}) \). Locally, the map \( a^m \) consists of taking the \( m \)-jets (or Taylor series up to order \( m \)) of the coordinate functions \((\theta(x))\) for the free \( A \)-module \( \mathcal{P}(\mathcal{E}) \otimes A = P^m \). Let \((d^0, d^1, \cdots, d^m)\) denote the dual basis for \( P^m \). Then

\[
a^m \otimes A : V_X \otimes A \rightarrow P^m = A^{n+1}
\]

is given by the matrix

\[
\begin{pmatrix}
  x_0 & x_1 & \cdots & x_n \\
  d^0 x_0 & d^0 x_1 & \cdots & d^0 x_n \\
  \vdots & \vdots & \cdots & \vdots \\
  d^m x_0 & d^m x_1 & \cdots & d^m x_n 
\end{pmatrix}
\]

Since \( d^i \) is a truncated Taylor series, we have \( d^i(r^j) = (i)^r d^i \), for all \( i, j \). Hence we get (considering the entries as elements of \( A \)):

\[
\begin{pmatrix}
  1 & \cdots & a_0 r^{i+1} & \cdots & a_i r^{i+1} & \cdots \\
  \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\
  (i+1) a_i r^{i+1} & \cdots & (i+1) a_i r^{i+1} & \cdots & \vdots & \cdots \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\
  (m+1) a_i r^{i+1} & \cdots & (m+1) a_i r^{i+1} & \cdots & \vdots & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
  (m+1) a_i r^{i+1} & \cdots & (m+1) a_i r^{i+1} & \cdots & \ddots & \ddots \\
\end{pmatrix}
\]

(where \( \cdots \cdot \cdots \) indicates terms of higher powers of \( r \); and \( * \) indicates elements in the maximal ideal of \( A \), see also \((\mathcal{E}), p. 10\)).
EXAMPLE 2. A linearly normal, elliptic curve.

Let $E_0 \subset P$ be a curve of genus 1 and degree $d$ in 3, which is linearly normal. From Riemann-Roch it follows that at any point $x$ the parametrization (*) satisfies

$$i = 0, \quad i = 0, \ldots, n - 1,$$

$$l_i \leq 1.$$  

Hence the curve is non-singular up to order $n - 1$.

EXAMPLE 3. Dye’s special curve (4), (5).

Suppose $\Gamma_0 \subset P$ is the complete intersection of 1 - 1 smooth surfaces $S_j$ of degree $d$ with a common self-intersecting simplex $S$, i.e., assume $S_j$ has equation $\sum a_j^m X_j^m = 0$, where all $(n - 1)$-minors of the matrix $(a_j^m)$ are non-zero. Note that $\Gamma_0$ has degree $d^{n - 1}$, and its genus $g$ is given by

$$2g - 2 = d^{n - 1} - (d - 1)(n - 1) - 2.$$  

The points of hyperosculating on $\Gamma_0$ are its $(n + 1)d^{n - 1}$ intersections with the faces of the simplex $S$; the osculating $m$-spaces at these points have order of contact $l_m + m + 1 = mld$ and so are hyperosculating (except when $m = 1$ and $d = 2$).

§3. Numerical characters

Given a curve in $3$-space, we define its rank to be the number of tangents that meet a given line. The number of osculating planes that pass through a given point is called the class of the curve. In general, for a curve $X_0 \subset P = \mathbb{P}(V)$ in $n$-space, we define its $m$-rank to be the number $r_m$ of its osculating $m$-spaces that meet a given (general) sub-$(n - m - 1)$-space of $P$. As in §2, let $X \rightarrow X_0$ denote the normalization and $f : X \rightarrow P$ the induced map. We define (82) the osculating bundle of order $m$ of $f$ as the $(m + 1)$-quasient $f : V_0 \rightarrow \mathbb{P}_m^n$ also defines the $m$th osculating bundle of $X_0$ as the image of the map $f_0 : V_0 \rightarrow \mathbb{P}_m^n \rightarrow P$ obtained by composing $f_0 : (x^m)^n = f_0(X_0) \rightarrow P$. The corresponding interpretations of the ranks are given in the next proposition.

PROPOSITION (3.1). The following numbers are equal (for $0 \leq m \leq n - 1$):

(i) the $m$-rank $r_m$,

(ii) the degree $d(\mathbb{P}_m^n)$ of the osculating bundle of order $m$,

(iii) the degree of the $m$th osculating developable $X_0 = \sum_i \mathbb{P}(\mathbb{P}_m^n) \subset P$.

(iv) the degree of the $m$th associated curve $f_m : X \rightarrow P\mathbb{P}(\mathbb{P}_m^n \subset V)$.  

PROOF. Since the quotient $a^m : V_0 \rightarrow \mathbb{P}_m^n$ represents the osculating $m$-spaces of the curve, we see that the $m$-rank $r_m$ is equal to the degree of the pullback to $X$ of the Schubert cycle $a \subset \text{Grass}_{n - m}(V)$ of $P = \mathbb{P}(V)$ which meet a given $(n - m - 1)$-plane, via the map $G(a^m) : X \rightarrow \text{Grass}_{n - m}(V)$ defined by $a^m$. Consider the Plücker embedding, Grass$_{n - m}(V) \subset \text{Grass}(\wedge^m V)$; then $a$ is a hyperplane section, and the $m$th associated map $f_m$ is the composition of $G(a^m)$ with $\iota$. Hence we get $r_m = \deg(a^m) = \deg(\wedge^m V) - \deg(\mathbb{P}_m^n) - \deg(\mathbb{P}(V))$. This proves (i) = (ii).

We have \( \deg(Y_n) = (c_1[G(a^m)])^{n - m} \cdot \iota_\ast Y_n \), and since $X_1 \rightarrow X_0$ is birational, so is $\mathbb{P}(V^n) \rightarrow Y$, therefore we have $\deg(Y_n) = (c_1[\iota_\ast G(a^m)])^{n - m} \cdot \iota_\ast \deg(\mathbb{P}(V^n)) = (c_1[G(a^m)])^{n - m} \deg(\mathbb{P}(V))$. By Chern class theory this is equal to $c_1(\mathbb{P}_m^n) \cdot c_1(G(a^m)) = \deg(\mathbb{P}(V))$, which, by the projection formula, equals $(c_1(\mathbb{P}_m^n))_{X_0} = \deg(\mathbb{P}(V))$.

EXAMPLE 1 (n = 3). The twisted cubic $R_3 \subset \mathbb{P}^3$. We can write $R_3$ as the intersection of three quadrics $Q_1, Q_2, Q_3$ with its tangent developable defined by $Q_1^2 - 4Q_2Q_3 = 0$. Hence $R_3$ has 1-rank $r_1 = 4$.

Let $x \in X$ and consider, as in §2, a parametrization $x = x^1, \ldots, x^m, a \neq \emptyset(i), i = 0, \ldots, n, 0 = l_0 \leq l_1 \leq \cdots \leq l_m.$ The integer $l_m - 2$ is called the $m$th stationary index of the curve $f_0 : X \rightarrow P$ at $x$, and we define the stationary index of $f$ to be $k_n = \sum_{i=0}^m (l_m - l_i)$, $k_0$. Hence, in the case that $f$ is non-singular of order $m = 1$, $k_0$ is the number (counted properly) of hyperosculating (or stationary) $m$-spaces.

THEOREM (3.2). There are formulas

$$r_m = (m + 1)(l_m + m(g-1)) - \sum_{i=0}^m (m-i)k_n$$

for $m = 0, 1, \ldots, n - 1$, and

$$\sum_{i=0}^m (n - i)k_n = (n + 1)(n(g - 1)) - \sum_{i=0}^m (m-i)k_n,$$

where $g$ denotes the genus of $X$ (see [2], p. 200).

PROOF. Put $\Delta_n = Coker(a^m)$. The exact sequences $0 \rightarrow \mathbb{P}(\mathbb{P}_m^n) \rightarrow \Delta_n \rightarrow 0$ gives $r_m = \deg(\mathbb{P}(\mathbb{P}_m^n)) = \deg(\mathbb{P}(\mathbb{P}_m^n)) - \deg(\mathbb{P}_m^n)$, $x$,

where $\Delta_n$ is the 0th Fitting ideal $F_0(\Delta_n)$ of $\Delta_n$, i.e. $\Delta_n$ is the image of the homomorphism $(\mathbb{P}(\mathbb{P}_m^n)) \otimes (\mathbb{P}(\mathbb{P}_m^n)^\vee) \rightarrow \mathbb{P}$. Locally at a point $x \in X$, in terms of a parametrization $a^m$, $a^m$ is generated by an element of the form

$$\sum_{i=0}^m a^m_i = \sum_{i=0}^m a^m_i (m-i)k_n.$$
EXAMPLE 2. A linearly normal, elliptic curve.

Let $E_n = P$ be a curve of genus 1 and degree $d$ in $P$, which is linearly normal. From Riemann-Roch it follows that at any point $s$ the parametrization $\pi$ satisfies

$$ i = 0, \quad i = 0, \ldots, n - 1, $$

$$ l_s \leq 1. $$

Hence the curve is non-singular up to order $n - 1$.

EXAMPLE 3. Dye's special curve [4], [5].

Suppose $\Gamma_0 \subset P$ is the complete intersection of $n - 1$ smooth surfaces $S_i$ of degree $d$ with a common self-polar simplex $S$, i.e., assume $S$ has equation $\sum a_i^{10}X_i = 0$, where all $(n - 1)$-minors of the matrix $(a_i^{10})$ are non-zero. Note that $\Gamma_0$ has degree $d^{n-1}$ and its genus $g$ is given by

$$ 2g - 2 = d^{n-1}(d - 1)(n - 1) - 2. $$

The points of hyperosculation on $\Gamma_0$ are its $(n + 1)d^{n-1}$ intersections with the faces of the simplex $S$; the osculating $m$-spaces at these points have order of contact $l_s + m + 1 = md$ and so are hyperosculating (except when $m = 1$ and $d = 2$).

§3. Numerical characters

Given a curve in $3$-space, we define its rank to be the number of tangents that meet a given line. The number of osculating planes that pass through a given point is called the class of the curve. In general, for a curve $X_0 \subset P = \mathbb{P}(V)$ in $n$-space, we define its $m$-rank to be the number $r_m$ of its osculating $m$-spaces that meet a given (general) sub-$\mathbb{P}(n - m - 1)$-space of $P$. As in §2, let $X \to X_0$ denote the normalization and $f : X \to P$ the induced map. We define $r_0$ to be the number of osculating $0$-spaces of $X_0$ as the image of the map $f_0 : \mathbb{P}(W^{l_0}) \to P$ obtained by composing $f^0(X_0) : \mathbb{P}(W^{l_0}) \to X \to P$ with the projection to $P$. Moreover, the $1$-quotient $\lambda^+X_0 \to \lambda^+X_0'$ defines the $\phi$ associated curve $\phi^+X_0 \to \mathbb{P}(\lambda^{n+1}V)$. The corresponding interpretations of the ranks are given in the next proposition.

**Proposition (3.1).** The following numbers are equal (for $0 \leq m \leq n - 1$):

(i) the $m$-rank $r_m$.

(ii) the degree $d(\phi^{m+1})$ of the osculating bundle of order $m$.

(iii) the degree of the $\phi$ osculating developable $X_0 = \mathbb{P}(\phi^{m+1}) \subset P$.

(iv) the degree of the $\phi$ associated curve $\phi^{m+1}X_0 \to \mathbb{P}(\lambda^{n+1}V)$.

PROOF. Since the quotient $\phi^+X_0 \to \mathbb{P}(W^{l_0})$ represents the osculating $m$-spaces of the curve, we see that the $m$-rank $r_m$ is equal to the degree of the pullback $X_0 \to \mathbb{P}(V)$ of the Schubert cycle $\sigma \subset \mathbb{Gr}(n+1, V)$ of $P = \mathbb{P}(V)$ which meets a given $(n - m - 1)$-space, via the map $G(\phi^{m+1})X_0 \to \mathbb{Gr}(n+1, V)$ defined by $\phi^m$. Consider the Plücker embedding, $\mathbb{Gr}(n+1, V) \subset \mathbb{P}(\Lambda^{n+1}V)$; then $\phi$ is a hyperplane section, and the $\phi$ associated map $\phi^{m+1}$ is the composition of $G(\phi^{m+1})$ with $\phi$. Hence we get $r_m$ by taking $\deg(\phi^{m+1}) = \deg(\mathbb{Gr}(n+1, V)) - \deg(\phi^{m+1})$. This proves (i) = (iv).

We have $\deg(Y) = (c_1(G(\phi^{m+1})))^m - \deg(\phi^{m+1})$ and, since $X_0 \to X_0$ is birational, so is $\phi^{m+1} X_0 \to X_0$, therefore we have $\deg(Y) = (c_1(G(\phi^{m+1})))^m - \deg(\phi^{m+1})$. By Chern class theory this is equal to $(c_1(\phi^{m+1}X_0), \deg(\phi^{m+1})) \phi^{m+1}$, which, by the projection formula, equals $(c_1(\phi^{m+1}X_0) \phi^{m+1}, \deg(\phi^{m+1}))$.

EXAMPLE 1. ($n = 3$). The twisted cubic $X_0 \subset \mathbb{P}^3$. We can write $X_0$ as the intersection of three quadrics $Q_1, Q_2, Q_3$, with its tangent developable defined by $Q_1 \cap Q_2 \cap Q_3 = 0$. Hence $X_0$ has 1-rank $r_1 = 4$.

Let $x \in X$ and consider, as in §2, a parametrization $\phi^m = (x_1, x_2, x_3, x_4, x_5)$, $x_4 \neq 0(i), i = 0, \ldots, n, 0 = l_0 \leq l_1 \leq \cdots \leq l_n$. The integer $l_{n-1} - l_n$ is called the $\phi$ stationary index of the curve $f : X \to P$ at $x$, and we define the $\phi$ stationary index of $f$ to be $m = \sum_{i=0}^{n-1} (l_i - l_{i+1})$. Hence, in the case that $f$ is non-singular of order $m = 1$, $k_m$ is the number (counted properly) of hyperosculating (or stationary) m-spaces.

**Theorem (3.2).** There are formulas

$$ r_m = (n + 1)k_m (m + 1) = m \sum_{i=0}^{m} k_i, $$

for $m = 0, 1, \ldots, n - 1$, and

$$ m \sum_{i=0}^{m} k_i = (n + 1)(m + 1)k_m, $$

for $m = 0, 1, \ldots, n - 1$, and

$$ m \sum_{i=0}^{m} k_i - (n + 1)(m + 1)k_m = -g \sum_{i=0}^{m} k_i, $$

where $g$ denotes the genus of $X$ (see [2], p. 200).

PROOF. Put $\mathcal{D}_m = \text{Coker}(\phi^m)$. The exact sequences $0 \to \mathbb{P}(\phi^{m+1}) \to X_0 \to 0$ gives $r_m = \deg(\phi^{m+1}) = \deg(\mathbb{P}(\phi^{m+1})) - \deg(f_x)$, where $f_x$ is the 0th Fitting ideal $f^0(\mathcal{D}_m)$ of $\mathcal{D}_m$, i.e., $f_x$ is the image of the homomorphism $(\Lambda^{n+1}W)^* \otimes (\Lambda^{n+1}W)^{m+1} \to X_0$. Locally at a point $x \in X$, in terms of a parametrization $\phi^m$, $f_x$ is generated by an element of the form $M_n(\phi(\psi))^{n+1}$. Hence we get $\deg(f_x) = \sum_{x \in X} \deg(f_x) = \sum_{i=0}^{m} k_i (m - i)k_m$. The
exact sequences (82)
\[ 0 \to \mathcal{O}_X \otimes \mathcal{E} \to \mathcal{O}_{P(V)}(\mathcal{E}) \to \mathcal{O}_{P(V)}(\mathcal{E}^{-1}) \to 0 \]
give
\[ \deg \mathcal{O}_{P(V)}(\mathcal{E}) = \sum_{i=1}^{\infty} \deg (\mathcal{O}_X \otimes \mathcal{E}) + \deg \mathcal{E}. \]

Hence we get:
\[ r_n = \frac{m(m+1)}{2} \deg \mathcal{O}_X + (m+1) \deg \mathcal{E} - \sum_{i=1}^{n} (m - i)k_i.
\]
\[ = (m+1)(c_0 + m(m-1)) - \sum_{i=1}^{n} (m - i)k_i. \]

(Since \( c_0 = \deg \mathcal{E} \) and \( \deg \mathcal{O}_X = 2(g - 1) \).) The last equation follows from the fact that \( r_n \geq 0 \) holds; the surjection \( \mathcal{E} \to \mathcal{E}^n \) is necessarily an isomorphism, since the bundles have the same rank.

We note that the map \( a^*: \mathcal{E} \to \mathcal{O}_{P(V)}(\mathcal{E}) \) is canonically isomorphic to the map \( f^*\mathcal{O}_{P(V)}(\mathcal{E}) \to \mathcal{O}_{P(V)}(\mathcal{E}) \) (6.4). Hence we get \( \text{Coker}(a^*) = \mathcal{O}_{P(V)}(\mathcal{E}) \) (7), 16.4.18), and \( k_0 \) is the degree of the ramification divisor; we have \( k_0 \) the number of cusps of \( f(X) \), if \( \text{char} \neq 2 \) (if \( \text{char} = 2 \), then \( k_0 = 2\# \text{cusps} \)).

**Corollary (3.3).** There are equalities:
1. \( (m(m-1) - (m - n) = 2(g - 1) - k_0 \)
2. \( (m(m+1) - (m - n) = 3(m - n - 1) \)
3. \( (m(m+1) - (m - n) = 2(m - n - 1) \).

The formulas (1) are usually referred to as the generalized Plücker formulas for curves (see [13], p. 207; [8], 4.26; [15], p. 43). In the case that the curve is plane \( (n = 2) \), it is the classical formula
\[ r_n = m^2 + 2m - 2 - k_0 \]
giving the class in terms of the degree, genus, and number of cusps.
The other Plücker formulas for the class of a plane curve can be written
\[ r_n = r_0 - 1 - e \]
where \( e = \deg (f^* \mathcal{E}) \) and \( f \) denotes the pullback to \( X \) of the Jacobian ideal of \( f(X) \). (This formula follows from the fact that, for a plane curve, \( \text{Ker}(a^*) \) is equal to \( f^* \mathcal{E} \otimes f^* \mathcal{E} \).

One sees \( e = 2m + k_0 \), where \( 2m \) denotes the degree of the conductor of \( X \) in \( f(X) \). In particular, if \( f(X) \) has no other singularities than \( D \) (ordinary) double points and \( K \) (simple) cusps, we get \( e = 2D + 3K \), if \( \text{char} \neq 2, 3 \).

**Example 1.** \( R_n \) has no hypercubingulating points. We get
\[ r_n = (m+1)(n-m) = r_{n-m-1}. \]

hence the degree of the \( n \)th associated curve \( f^*(n) : X \to \mathbb{P}^{n+1}(V) \to \mathbb{P}(\mathcal{O}^{n+1}) \) is equal to the dimension of \( \mathbb{P}(\mathcal{O}^{n+1}) \).

**Example 2.** We deduce, from the Riemann-Roch theorem, \( h_m = 0, m = 0, \ldots, n - 2 \). Hence we get \( r_n = (n+1)d, m = 0, \ldots, n-1 \) and \( h_{n-1} = (n+1)d \), so that the curve \( E_i \) has exactly \( (n+1)! \) hypercubingulating hyperplanes.

**Example 3.** For the curve \( G \), we have \( k_0 = 0, k_1 = (n+1)^{d-1} \), \( i = (d - 1) \) for \( i \geq 1 \), from which we can compute the formulas for \( r_n \).

Let us consider the case \( d = 2, n = 4 \). Then \( r_2 = 8, g = 5, k_2 = 0, k_1 = 0, k_2 = 40, k_3 = 40 \). Hence we get \( r_2 = 84, r_3 = 48, r_4 = 40 \). The number of hypercubingulating points on \( \Gamma \) is \( 40 \), but their 'weighted number' \( \sum_{i=0}^{n} (4 - i)k_i \) is 120. In fact the curve \( G \) is \( P \)-canonical, i.e. the embedding is given by the canonical bundle \( \mathcal{O}_{\mathbb{P}^3} \). So the \( 40 \) hypercubingulating points are the Weierstrass points of \( \Gamma \); the above remark about their number shows that these points are not ordinary Weierstrass points.

§4. Projections and sections

If we project a curve in \( 3 \)-space from a (general) point, we obtain a plane curve, of the same degree, whose class is equal to the rank of the space curve. Another way of obtaining a plane curve from the space curve is to take the intersection of its tangent developable with a (general) plane. We expect this plane curve to have the same class as the space curve, while its degree is that of the tangent developable, hence equal to the rank of the space curve. Veronese ([13]) used the principle of projections and sections to obtain \( n - 1 \) plane curves from a given curve in \( n \)-space; he then applied the formulas (1), (II) (and their duals) of §3 to these curves, deducing relations among the numerical characters of the given curve (interpreting these in terms of the characters of the plane curves).

Assume from now on that \( X \) is a smooth curve, given with a finite map \( f: X \to \mathbb{P}^{n-1}(V) \) such that \( a^*: \mathcal{E} \to \mathcal{O}_{P(V)}(\mathcal{E}) \) is generically surjective ([1], 22). A \((q+1)\)-dimensional subspace \( W \subseteq V \) such that \( f(X) \cap \mathbb{P}(W(V)) = \emptyset \) (i.e., such that the induced map \( \mathcal{E} \to f^* \mathcal{O}_{P(V)}(1) \) is surjective) defines a projection \( X \to \mathbb{P}^{n-1} \). If, in addition, the induced map \( \mathcal{O}_{P(V)}(\mathcal{E}) \to \mathcal{O}_{P(V)}(\mathcal{E}) \) is surjective (here \( \mathcal{E}^{q+1} \) denotes the osculating bundle of order \( q - 1 \) of \( f \), as in [22]), we say that \( p(f) \) is a \( q \)-projection of \( f \). In fact, a general subspace \( W \subseteq V \) has this property, as the following lemma shows.

**Lemma (4.1).** If \( Y \) is a \( d \)-dimensional variety and \( V \to \mathcal{E} \) is an \( r \)-quotient on \( Y \), then a general \((r-d)\)-dimensional subspace \( V \) of \( V \) has the property that the induced map \( V \to \mathcal{E} \) is surjective.
exact sequences (82)

\[ 0 \to S^m \Omega_X \otimes \mathcal{O}(X) \to \mathcal{O}_{\mathbb{P}^m}(m+1) \to \mathcal{O}_{\mathbb{P}^m}(m) \to 0 \]

give

\[ \deg(\mathcal{O}_{\mathbb{P}^m}(m+1)) = \sum_{i=0}^{n-1} \deg(S^i \Omega_X \otimes \mathcal{O}(X)) \cdot \deg(X) \]

Hence we get

\[ r_m = \frac{m(m+1)}{2} \deg \Omega_X + (m+1) \deg X - \sum_{i=0}^{m-1} \deg(S^i \Omega_X \otimes \mathcal{O}(X)) \cdot \deg(X) \]

\[ = (m+1)(\deg \Omega_X + (m+1) \deg X) - \sum_{i=0}^{m-1} \deg(S^i \Omega_X \otimes \mathcal{O}(X)) \cdot \deg(X) \]

(Since \( r_m = \deg X \) and \( \deg \Omega_X = 2\deg(X) \).) The last equation follows from the fact that \( r_m = 0 \) holds: the surjection \( A^* \to \mathcal{K}^* \) is necessarily an isomorphism, since the bundles have the same rank.

We note that the map \( A^* \to \mathcal{K}^* \) is canonically isomorphic to the map \( f^* \mathcal{K}(1) \to \mathcal{K}(2) \) (6.4). Hence we get \( \text{Coker}(A^*) = \text{Coker}(\mathcal{K}(2)) \) (7), 16.4.18), and \( k_i \) is the degree of the ramification divisor; we have \( k_0 = \text{the number of cusps of } f(X) \), if \( \text{char} \neq 2 \) (if \( \text{char} = 2 \), then \( k_4 = 2\# \text{cusps} \)).

**Corollary (3.3). There are equalities**

(1) \( (r_m + r_{m-1}) - (r_{m-1} - r_m) = 2(\deg X) - \deg \Omega_X \)

(2) \( (r_{m+1} + k_m) - (r_m + k_{m+1}) = 3(\deg \Omega_X - r_m) \)

(3) \( (r_{m+2} + k_{m+1}) - (r_m + k_2) = 2(r_m - r_0) \)

The formulas (i) are usually referred to as the generalized Plücker formulas for curves (see [13], p. 207; [8], 4.26; [15], p. 43). In the case that the curve is plane (\( n = 2 \)), it is the classical formula

\[ r_1 = n^2 - 2(n-1) - k_0 \]

giving the class in terms of the degree, genus, and number of cusps. The other Plücker formula for the class of a plane curve can be written

\[ r_3 = r_0 r_1 - 1 - \varepsilon \]

\( \text{where } \varepsilon = \deg(X) \) and \( \varepsilon \) denotes the pullback to \( X \) of the Jacobian ideal of \( f(X) \). This formula follows from the fact that, for a plane curve, \( \text{Ker}(A^*) \) is equal to \( f^* X \otimes \mathcal{O}(X) \), where \( X \) is the conormal of \( f(X) \) in \( P \) (12.1).)

One shows \( \varepsilon = 28 - k_0 \), where 28 denotes the degree of the conductor of \( X \) in \( f(X) \). In particular, if \( f(X) \) has no other singularities than \( D \) (ordinary) double points and \( K \) (simple) cusps, we get \( \varepsilon = 2D + 3K \), if \( \text{char} \neq 2, 3 \).

**Example 1.** \( R_n \) has no hyperosculating points. We get

\[ r_m = (m+1)(n-m) = r_{m-1} \]

hence the degree of the \( n \)-th associated curve \( f^{(n)} X \to P^{(n+1)} X \) is equal to the dimension of \( G^{(n+1)} X \).

**Example 2.** We deduce, from the Riemann-Roch theorem, \( k_m = 0, m = 0, 1, \ldots, n-2 \). Hence we get \( r_m = (m+1)d, m = 0, 1, \ldots, n-1 \) and \( k_m = (n+1)d \), so that the curve \( D_0 \) has exactly \( (n+1)d \) hyperosculating hyperplanes.

**Example 3.** For the curve \( D_0 \) we have \( k_0 = 0, k_i = (n+1)(d-1)(i-1) \) for \( i \geq 1 \), from which we can compute the formulas for \( r_m \).

Let us consider the case \( d = 2 \), \( n = 4 \). Then \( r_0 = 8, r_1 = 5, \) and \( k_1 = 0, k_2 = 40, k_3 = 40 \). Hence we get \( r_1 = 24, r_2 = 48, \) and \( r_3 = 40 \). The number of hyperosculating points on \( \Gamma = \Gamma_2 \) is 40, but their "weighted number" \( \sum_0^{4} (4-i)k_i = 120 \). In fact the curve \( \Gamma \to \mathbb{P}^d \) is canonical, i.e. the embedding is given by the canonical bundle \( \Omega_\Gamma \). So the 40 hyperosculating points are the Weierstrass points of \( \Gamma \); the above remark about their number shows that these points are not ordinary Weierstrass points.

**§4. Projections and sections**

If we project a curve in 3-space from a (general) point, we obtain a plane curve, of the same degree, whose class is equal to the rank of the space curve. Another way of identifying a plane curve from the space curve is to take the intersection of its tangent developable with a (general) plane. We expect this plane curve to have the same class as the space curve, while its degree is that of the tangent developable, hence equal to the rank of the space curve. Veronese ([4], [5]) used the principles of projections and sections to obtain an -1 plane curves from a given curve in \( n \)-space; he then applied the formulas (I), (II) (and their duals) of §3 to these curves, deducing relations among the numerical characters of the given curve (interpreting these in terms of the characters of the plane curves).

Assume from now on that \( X \) is a smooth curve, given with a finite map \( f: X \to P(V) \) such that \( A^* \to \mathcal{K}^* \) is generically surjective ([1], [2]). A \( (q+1) \)-dimensional subspace \( V' \subseteq V \) such that \( f(X) \cap (V'V) = \emptyset \) (i.e., such that the induced map \( V' \to \mathcal{K}^* \) is generically surjective) defines a projection \( X \to P(V) \) unique. If, in addition, the induced map \( V' \to \mathcal{K}^* \) is surjective (here \( \mathcal{K}^* \) denotes the osculating bundle of order \( q-1 \) of \( f \), as in §2), we say that \( p(f) = p(V); X \to P(V) \) is a projection of \( f \). In fact, a general subspace \( V' \subseteq V \) has this property, as the following lemma shows.

**Lemma (4.1).** If \( Y \) is a \( d \)-dimensional variety and \( Y \to \mathcal{S} \) is an \( r \)-quotient on \( Y \), then a general \((r+d) \)-dimensional subspace \( V' \subseteq V \) has the property that the induced map \( Y \to \mathcal{S} \) is surjective.
PROOF. Let \( \phi : Y \rightarrow \text{Grass}(V) \) be the morphism defined by the quotient.

We want to show (10), (2.6) that for a general \( V \in \mathcal{V} \), we have
\[
\phi^{-1}(\alpha_{g_\lambda}(V)) = \emptyset,
\]
where \( \alpha_{g_\lambda}(V) \in \text{Grass}(V) \) denotes the \((d+1)\)th special Schubert cycle defined by \( V \). Note that \( \alpha_{g_\lambda}(V) \) has codimension \( d+1 \).

Let \( G \) denote the general linear group acting on \( V \). Given \( V \in \mathcal{V} \), there is an open dense subset \( U \) of \( G \) such that \( \phi^{-1}(\alpha_{g_\lambda}(gV)) = \emptyset \) for all \( g \in U \). (11), Cor. 4. There is a surjective map \( \psi : G \rightarrow \text{Grass}(\alpha_{g_\lambda}(V)) \) and (by the theorem of generic flatness) there is an open dense subset \( U' \) of \( G' \) such that \( \psi^{-1}(U') \subseteq U \). Hence \( \psi \) satisfies the condition of the lemma, for all \( g \in U' \).

PROPOSITION (4.2). The osculating bundles of \( p_0 : X \rightarrow P \) are equal to the osculating bundles of \((\text{order } \le q-1)\) of \( f : X \rightarrow P \), and a \( q' \)-projection of \( p_0 \) is also a \( q' \)-projection of \( f \).

Proof. From the general properties of the maps \( a^m \) (6.1) it follows that \( a^m(p_0) : V_0 \rightarrow \mathcal{P}(\mathcal{L}(\mathcal{L})) \) is equal to the composition of \( a^m(f) : V_k \rightarrow \mathcal{P}(\mathcal{L}(\mathcal{L})) \) with the inclusion \( V_k \hookrightarrow V_k \). Therefore the image of \( a^m(p_0) \) is equal to \( \mathcal{P}(\mathcal{L}(\mathcal{L})) \), since this true for \( m = q - 1 \) and since the canonical surjections \( \mathcal{P}(\mathcal{L}(\mathcal{L})) \rightarrow \mathcal{P}(\mathcal{L}(\mathcal{L})) \) induce surjections \( \mathcal{P}(\mathcal{L}(\mathcal{L})) \rightarrow \mathcal{P}(\mathcal{L}(\mathcal{L})) \). To prove the last assertion, we observe that if \( W_0 \rightarrow V \) defines a \( q' \)-projection of \( p_0 \), then \( W_0 \rightarrow V \) defines a \( q' \)-projection of \( f \).

Assume now that \( V \subseteq \mathcal{V} \) is an \( m \)-dimensional subspace. Using lemma (4.1) to the \((m-1)\)-quotient \( V_0 \rightarrow (\text{Ker}(a^m(p_0)))' \) we see that for a general such subspace \( V' \), the induced map \( V_0 \rightarrow \mathcal{P}(\mathcal{L}(\mathcal{L})) \) is locally split. As in (2.2), let \( f = (\mathcal{P}(\mathcal{L}(\mathcal{L})) \rightarrow \mathcal{P}(\mathcal{L}(\mathcal{L})) \) and call \( f : X_0 \rightarrow P(W) \) an \( m \)-section of \( f \). We let \( X_0 \) denote the line bundle \( (\mathcal{P}(\mathcal{L}(\mathcal{L})) \rightarrow \mathcal{P}(\mathcal{L}(\mathcal{L}))' \).

PROPOSITION (4.3). With the above notations, the map \( q : X_0 \rightarrow X \) induced by \( \mathcal{P}(\mathcal{L}(\mathcal{L})) \rightarrow X \) is equal to the canonical isomorphism \( \mathcal{P}(\mathcal{L}(\mathcal{L})) \rightarrow X \).

Proof. Put \( X_0 = \text{Ker}(a^m) \). From the commutative diagram of bundles on \( X \)

\[
\begin{array}{c}
0 & \rightarrow & V_0 & \rightarrow & X_0 & \rightarrow & X & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & V_k & \rightarrow & X & \rightarrow & X & \rightarrow & 0 \\
0 & \rightarrow & V_{k'} & \rightarrow & X_0 & \rightarrow & X_0 & \rightarrow & 0 \\
0 & \rightarrow & V & \rightarrow & X & \rightarrow & X & \rightarrow & 0
\end{array}
\]

it follows that the curves \( P(X_0) \subseteq \mathcal{P}(\mathcal{L}(\mathcal{L})) \) and \( X_0 = (\mathcal{P}(\mathcal{L}(\mathcal{L})) \rightarrow \mathcal{P}(\mathcal{L}(\mathcal{L}))' \) are equal.

The next result we will prove is that an \( m \)-section of an \( m \)-section of \( f \) is an \((m+m)\)-section of \( f \).

THEOREM (4.4). Suppose \( f : X_0 \rightarrow P(W) \) is an \( m \)-section of \( f \) such that \( a^m(f) \) is generically surjective, let \( \mathcal{P}(\mathcal{L}) \) denote the osculating bundle of order \( m' \) of \( f \), and let \( f : X \rightarrow P(W) \) be an \( m \)-section of \( f \). Suppose
\[
f : X_0 \rightarrow X \quad \text{is an } m \text{-section of } f.
\]

Then
\[
f : X_0 \rightarrow X_0 \quad \text{is an } (m+m) \text{-section of } f.
\]

Proof. From the above notations, we have a canonical isomorphism \( \mathcal{P}(\mathcal{L}(\mathcal{L})) \rightarrow X \). Hence we get a commutative diagram of bundles on \( X \), where all arrows are surjective and the three vertical arrows all have kernel equal to \( \mathcal{P}(\mathcal{L}(\mathcal{L})) \).

\[
\begin{array}{c}
V_0 & \rightarrow & W_0 & \rightarrow & W_0 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
V_k & \rightarrow & W_k & \rightarrow & W_k & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
V_{k'} & \rightarrow & W_{k'} & \rightarrow & W_{k'} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
V & \rightarrow & W & \rightarrow & W & \rightarrow & 0
\end{array}
\]

Put \( V = \text{Ker}(V \rightarrow W) \). Then \( V \) is equal to \( \text{Ker}(\mathcal{P}(\mathcal{L}(\mathcal{L})) \rightarrow \mathcal{P}(\mathcal{L}(\mathcal{L}))' \), hence is a \( (m+m) \)-bundle of \( \mathcal{P}(\mathcal{L}(\mathcal{L}))' \). This means that \( V \) defines an \((m+m)\)-section \( f : X_0 \rightarrow X \). The existence of the cartesian diagram follows from the existence of the above commutative diagram of bundles.

It remains to establish (4.5).
it follows that the curves $\mathcal{P}(\mathcal{L}_\infty) \subseteq \mathcal{P}(\mathcal{P}^n)$ and $X_m = \mathcal{F}(\mathcal{P}(\mathcal{W})) \subseteq \mathcal{P}(\mathcal{P}^n)$ are equal.

The next result we will prove is that an $m'$-section of an $m$-section of $f$ is an $(m + m')$-section of $f$.

**Theorem (4.4).** Suppose $f_m : X_m \to \mathcal{P}(\mathcal{W})$ is an $m$-section of $f$ such that $a_m^*(f_m)$ is generically surjective, let $\mathcal{P}^n_m$ denote the osculating bundle of order $m'$ of $f_m$, and let $f_{m,m'} : \mathcal{P}(\mathcal{P}^n_m) \to \mathcal{P}(\mathcal{W})$ denote the $m'$th osculating developable of $f_m$. Suppose

$$f_{m,m'} : X_m \to \mathcal{P}(\mathcal{P}^n_m) \to \mathcal{P}(\mathcal{W})$$

is an $m'$-section of $f_m$. Then

$$f_{m,m'} : X_m \hookrightarrow \mathcal{P}(\mathcal{P}^n_m) \to \mathcal{P}(\mathcal{W})$$

is an $(m + m')$-section of $f$ and there is a cartesian diagram

$$\begin{array}{ccc}
X_{m,m'} & \longrightarrow & X_m \times_{X_{m'}} X_m \\
\mathcal{P}(\mathcal{P}^n_m) & \longrightarrow & \mathcal{P}(\mathcal{P}^n_m) \times_X X_m
\end{array}$$

where $u$ is an isomorphism and $v$ is an embedding.

**Proof.** We will show that there is a canonical exact sequence

$$0 \to V_{m} \to \gamma^m \mathcal{P}^n_{m'} \to \mathcal{P}^n_m \to 0,$$

where $V = \text{Ker}(\mathcal{W} \to \mathcal{P}(\mathcal{W}))$, and $\gamma = \gamma_m : X_m \to X$ is the isomorphism of (4.3). Once (4.5) is established, the theorem follows. Put $X_{m,m'} = \text{Ker}(\mathcal{P}^n_{m+m'})$. We have $\text{Ker}(\mathcal{W}_{m'} \to \mathcal{P}^n_m) = \text{Ker}(\mathcal{W}_{m'} \to \gamma^m \mathcal{P}^n_m) = \gamma^m X_{m,m'}$. Put $W' = \text{Ker}(\mathcal{W} \to \mathcal{P}(\mathcal{W}))$ and $Z' = \text{Ker}(\mathcal{W}_{m'} \to \mathcal{P}^n_m)$. Then $Z'$ is invertible (since $W'$ defines an $m'$-section of $f_m$ and there is an induced surjection $W_{m'} \to Z'$ with kernel $\gamma^m X_{m,m'}$. Hence we get a commutative diagram of bundles on $X$, where all arrows are surjective and the three vertical arrows all have kernel equal to $\gamma^m X_{m,m'}$.

$$\begin{array}{ccc}
V_m & \longrightarrow & W_m \longrightarrow \mathcal{W}_m \\
\mathcal{P}^n_{m} & \longrightarrow & \gamma^m \mathcal{P}^n_{m'} \longrightarrow \mathcal{P}^n_m
\end{array}$$

Put $V' = \text{Ker}(\mathcal{W} \to \mathcal{P}(\mathcal{W}))$. Then $V'$ is equal to $\text{Ker}(\mathcal{P}^n_{m+m'} \to \gamma_m Z')$, hence it is a sub-$(m + m')$-bundle of $\mathcal{P}^n_{m+m'}$. This means that $V'$ defines an $(m + m')$-section $f_{m,m'} : X_{m,m'} \to \mathcal{P}(\mathcal{P}^n_{m'} \times_X X_m)$ of $f$. The existence of the cartesian diagram follows from the existence of the above commutative diagram of bundles.

It remains to establish (4.5).
THEOREM (4.6). A $q$-projection of an $m$-section of $f: X \to P$ is the same as an $m$-section of a $(q+m)$-projection of $f$.

PROOF. Let $f_*: X_* \to P(V^*)$ be an $m$-section of $f$, defined by $0 \to V^* \to V \to V^* \to 0$. Let $p^{\mathfrak{n}}_{m}: X_* \to P(W^*)$ be a $q$-projection of $f_*$, defined by $0 \to W^* \to V^* \to W^* \to 0$. Put $W = \ker (V \to W)$ (note that $W$ has dimension $m+q+1$). We want to show that $0 \to W \to V \to W^* \to 0$ defines a $(q+m)$-projection of $f$, i.e., that the induced map $c: X_* \to \mathfrak{P}^m_\ast$ is surjective.

Since $p^{\mathfrak{n}}_{m}$ is a $q$-projection of $f_*$, we have a commutative diagram

\[
\begin{array}{cccccc}
W & \to & V^* & \to & V^* & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
W & \to & V^* & \to & V^* & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
W^* & \to & V^* & \to & V^* & \to & 0 \\
\end{array}
\]

which fits into the following commutative diagram (with exact rows)

\[
\begin{array}{cccccc}
0 & \to & V_* & \to & V^* & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & V_* & \to & V^* & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & V_* & \to & V^* & \to & 0 \\
\end{array}
\]

An easy diagram chase shows that $c$ is surjective.

The exact sequence $0 \to V^* \to W \to W^* \to 0$ defines an $m$-section of the $(q+m)$-projection given above; in fact, we get a commutative diagram

\[
\begin{array}{cccccc}
0 & \to & V_* & \to & W_* & \to & W^* & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & V_* & \to & W_* & \to & W^* & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & V_* & \to & W_* & \to & W^* & \to 0 \\
\end{array}
\]

which also shows that this $m$-section is equal to $p^{\mathfrak{n}}_{m}$.

If we suppose, on the other hand, that we are given an $m$-section of a $(q+m)$-projection of $f$, a similar argument shows that this curve is also a $q$-projection of an $m$-section of $f$.

If we put this last result together with (4.2) and (4.4), we see that any curve $f: X \to P$ which is obtained from $f$ by projections and sections can be viewed as a $q$-projection of an $m$-section of $f$, $p^{\mathfrak{n}}_{m}$, where $q = \dim P$ and $m$ is uniquely determined. In particular, from the given curve $f: X \to P$ in $n$-space, we obtain by these methods $n-1$ (and not more) plane curves $p^{\mathfrak{n}}_{m}$, $m = 0, 1, \ldots, n-2$.

PROPOSITION (4.7). Let $(t_0, t_1, \ldots, t_n)$ and $(k_0, \ldots, k_{n-1})$ denote the ranks and stationary indices of the curve $f: X \to P$. The ranks and stationary
Stru 1. Put $\varepsilon = \text{Coker} (\alpha_{x_{m+1}} W_0).$ Then $\varepsilon$ is an $(m+1)$-bundle.

This follows since $c$ is equal to the composition of the locally split map $X_{m+1} \rightarrow X_0$ (induced by the surjection $\beta_{m+1} \rightarrow \beta_0$) with the map $X_0 \rightarrow W_0$ (which is locally split because $W$ defines an $m$-section of $f$).

Stru 2. The map $\alpha : W_0 \rightarrow P_0^2 (L_m)$, obtained by composing $\beta_{m+1} : W_0 \rightarrow P_0^2 (L_m)$ with $P_0^2 (\alpha) = P_0^2 (\alpha) = P_0^2 (L_m)$ (66) factors through $\beta : W_0 \rightarrow L_m$.

Consider the following commutative diagram of exact rows (since $X$ is smooth, $P_0^{2n} (-)$ is an exact functor (7), 16.7.3)).

$$
0 \rightarrow V_1 \rightarrow V_2 \rightarrow W_1 \rightarrow 0
$$

$$
0 \rightarrow P_0^2 (V_1) \rightarrow P_0^2 (V_2) \rightarrow P_0^2 (W_1) \rightarrow 0
$$

$$
0 \rightarrow P_0^2 (V_1) \rightarrow P_0^2 (W_1) \rightarrow 0.
$$

The injection $P_0^2 (\alpha) \rightarrow P_0^2 (\alpha)$ gives an injection $\beta : P_0^2 (\alpha) \rightarrow P_0^2 (\alpha)$.

The map $\alpha^* : V_1 \rightarrow P_0^2 (\alpha)$, obtained by composing $\alpha$ with $j$, factors through $\alpha^* : V_1 \rightarrow P_0^2 (\alpha)$ via $\alpha^* : P_0^2 (\alpha) \rightarrow P_0^2 (\alpha)$ (6.2(ii)). Thus we get an injection $\alpha^* : V_1 \rightarrow P_0^2 (\alpha)$ (since $\beta$ is injective), also we see that $\alpha^* : V_1 \rightarrow P_0^2 (\alpha)$. Therefore the map $\delta : \alpha^* \rightarrow \alpha^* \alpha$ induces a map $\delta : \alpha^* \alpha \rightarrow \alpha^* \alpha$. It follows that there is a commutative diagram with exact rows

$$
0 \rightarrow V_1 \rightarrow V_2 \rightarrow W_1 \rightarrow 0
$$

$$
0 \rightarrow P_0^2 (V_1) \rightarrow P_0^2 (V_2) \rightarrow P_0^2 (W_1) \rightarrow 0
$$

$$
0 \rightarrow P_0^2 (V_1) \rightarrow P_0^2 (W_1) \rightarrow 0.
$$

Stru 3. The map $\gamma : \alpha_1 : W_0 \rightarrow \gamma P_0^2 (L_m)$ is equal to the map $\alpha_1 (\alpha_1)$.

Observe first that proposition (4.3) is a special case ($m=0$) of the lemma. Therefore we know that the canonical 1-quotient $\alpha_1 (\alpha_1)$ of $X_m$ is equal to $\gamma P_0^2 (W_0) \rightarrow \gamma P_0^2 (X_m)$. Applying (6.2(ii)) to $\gamma : X_m \rightarrow X$ then gives a commutative diagram

$$
W_0 \mapright{\alpha_1 \gamma} P_0^2 (\gamma P_0^2 (X_m))
$$

This shows that (4.5) exists: Since $\alpha_1 (\alpha_1)$ is generically surjective, the quotients $W_0 \rightarrow \gamma P_0^2 (W_0) = \gamma P_0^2 (X_m)$ are equal. This completes the proof of theorem (4.4).

Theorem (4.6). A $q$-projection of an $m$-section of $f : X \rightarrow P$ is the same as an $m$-section of a $(q+m)$-projection of $f$.

Proof. Let $f_0 : X_0 \rightarrow P_0$ be an $m$-section of $f$, defined by $0 \rightarrow V_0 \rightarrow W_0 \rightarrow 0$. Let $p_0^2 : X_0 \rightarrow P_0$ be a $q$-projection of $f_0$, defined by $0 \rightarrow W_0 \rightarrow V_0 \rightarrow 0$. Put $W = \text{Ker} (V_0 \rightarrow W_0)$ (note that $W$ has dimension $m+q+1$). We want to show that $0 \rightarrow W_0 \rightarrow V_0 \rightarrow W_0 \rightarrow 0$ defines a $(q+m)$-projection of $f$, i.e., that the induced map $\varepsilon : W_0 \rightarrow \gamma P_0^2 (W_0)$ is surjective.

Since $p_0^2$ is a $q$-projection of $f_0$, we have a commutative diagram

$$
\begin{array}{ccc}
W_0 & \xrightarrow{\varepsilon} & V_0 \\
\downarrow & & \downarrow \\
\gamma P_0^2 (W_0) & \xrightarrow{\gamma P_0^2 (\varepsilon)} & \gamma P_0^2 (V_0)
\end{array}
$$

which fits into the following commutative diagram (with exact rows)

$$
\begin{array}{ccc}
0 & \xrightarrow{\varepsilon} & 0 \\
\downarrow & & \downarrow \\
W_0 & \xrightarrow{\gamma P_0^2 (\varepsilon)} & V_0
\end{array}
$$

An easy diagram chase shows that $\varepsilon$ is surjective.

The exact sequence $0 \rightarrow V_0 \rightarrow W_0 \rightarrow W_0 \rightarrow 0$ defines an $m$-section of the $(q+m)$-projection above; in fact, we get a commutative diagram

$$
\begin{array}{ccc}
0 & \xrightarrow{\varepsilon} & 0 \\
\downarrow & & \downarrow \\
W_0 & \xrightarrow{\gamma P_0^2 (\varepsilon)} & V_0
\end{array}
$$

which also shows that this $m$-section is equal to $p_0^2$.

If we suppose, on the other hand, that we are given an $m$-section of a $(q+m)$-projection of $f$, a similar argument shows that this curve is also a $q$-projection of an $m$-section of $f$.

If we put this last result together with (4.2) and (4.4), we see that any curve $f : X \rightarrow P$ which is obtained from $f$ by projections and sections can be viewed as a $q$-projection of an $m$-section of $f$, $p_0^2$, where $q = \dim P$ and $m$ is uniquely determined. In particular, from the given curve $f : X \rightarrow P$, we obtain by these methods $n-1$ (and not more) plane curves $p_0^2$, $m = 0, 1, \ldots, n-2$.

Proposition (4.7). Let $(p_0, p_1, \ldots, p_{n-1})$ and $(k_0, \ldots, k_{n-1})$ denote the ranks and stationary indices of the curve $f : X \rightarrow P$. The ranks and stationary
indices of a q-projection of an m-section $p_i^p: X_n \to P^p$ of $f$ are given by:

$$
n_i(p_i^p) = n_i + \beta_i, \quad 0 \leq i < q - 1,
$$

$$
\beta_i(p_i^p) = \beta_i + \kappa_i, \quad \text{if } q > 1,
$$

$$
\kappa_i(p_i^p) = \kappa_{i+1}, \quad 1 \leq i < q - 1,
$$

$$
\kappa_{q-1}(p_i^p) = \kappa_{n+1} + \kappa_{n+1} + \cdots.
$$

PROOF. The osculating bundle of order $i, \mathcal{P}_{n,i}$, of an m-section $f_m$ fits into an exact sequence (see (4.5))

$$
0 \to \mathcal{V}_{m,i} \to \gamma^{(m+1)} \to \mathcal{P}_{n,i} \to 0.
$$

Hence we get $r_i(f_m) = \deg(\mathcal{P}_{n,i}) = \deg(\gamma^{(m+1)}) = \kappa_{n+1}$. From (4.2) it follows that the osculating bundle of $p_i^p$ is equal to that of $f_m$. Hence we get also $r_i(p_i^p) = \kappa_{n+1}$.

The formulas for $k_i(p_i^p)$ are straightforward applications of the formula (3.3) to the curves $p_i^p$ and $f$.

EXAMPLE 1. The $i$th rank of an m-section of a q-projection of the rational $n$-ic $R_i$ is $(m + i)(m - n)$. It has $m(m - n + 1)$ cusps if $n > 1$, $(m + q)(m - n + 1)d$ stationary hyperplanes, and no other points of hyperosculations.

EXAMPLE 2. For the curve $p_i^p(E_i)$ we get $r_i = (m + i + 1)d$, $\beta_i = md$ (if $q > 1$), $\kappa_i = 0$ for $0 < i < q - 1$ (if $m + i \leq n - 2$), and $\kappa_{q-1} = (m + q + 1)d$. In particular, if we take $q = 2$, $m = n - 2$, we get a plane curve of degree $(n - 1)d$, class $md$, with $(n - 2)d$ cusps and $(n - 1)d$ flexes.

EXAMPLE 3. Starting with the curve $V_2^X$, we obtain 3 plane curves $p_i^p$, $i = 0, 1, 2$, with the following numerical characters:

<table>
<thead>
<tr>
<th>$r_i$(degree)</th>
<th>$r_i$(class)</th>
<th>$\beta_i$(cusps)</th>
<th>$\kappa_i$(flexes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_0^p$</td>
<td>8</td>
<td>24</td>
<td>0</td>
</tr>
<tr>
<td>$p_1^p$</td>
<td>24</td>
<td>48</td>
<td>8</td>
</tr>
<tr>
<td>$p_2^p$</td>
<td>48</td>
<td>40</td>
<td>64</td>
</tr>
</tbody>
</table>

$\S 5$. Duality

Suppose $f: X \to P$ is finite and $a^i: V_X \to \mathcal{P}_{n,i}(x)$ is generically surjective.

Assume also that we have either $k = 0$ or char $k > n$. The dual curve $f^*: X \to P^* = \mathcal{P}(V^*)$ of $f: X \to P$ is defined by the 1-quotient $V_X^* \to \mathcal{X}_{n-1}^*$, where we have put $\mathcal{X}_{n-1}^* = \text{Ker}(a^{n-1}: V_X \to \mathcal{P}^{n-1})$. Hence a point $\bar{f}(x)$ of the dual curve is equal to the osculating hyperplane to $f$ at $x$, considered as a point in the dual projective space.

**THEOREM (5.1).** (i) The m-rank of $f$ is equal to the $(n - 1 - m)$-rank of $f$, i.e., $r_m(f) = r_{n-1-m}(f)$ for $m = 0, 1, \ldots, n - 1$, and similarly for the stationary indices: $\kappa_m(f) = \kappa_{n-1-m}(f)$ for $m = 0, 1, \ldots, n - 1$.

(ii) The dual curve of $f$ is equal to $f$ (we write $f^* = f$).

(iii) The dual of a q-projection of an m-section of $f$ is equal to a q-projection of an $(n - m - q)$-section of $f$ (we write $p_i^q(f^*) = p_{n-i}^{n-q}(f)$).

**PROOF.** Let $\mathcal{P}_{n,i}$ denote the nth osculating bundle of $f$ and $a_i^q: V_X \to \mathcal{P}_{n,i}$ the canonical quotient. Put $\mathcal{X}_{n,i}^* = \text{Ker}(a_i^q)$ and $\mathcal{X}_{n-1-i}^* = \text{Ker}(a^{n-1-i})$. The key result is the following:

**LEMMA (5.2).** There are canonical isomorphisms of exact sequences (for $m = 0, \ldots, n - 1$)

$$
0 \to \mathcal{P}_{m,i}^X \to \mathcal{V}_{m,i} \to \mathcal{P}_{n,i} \to 0
$$

$$
0 \to \mathcal{P}_{m+1,i}^X \to \mathcal{V}_{m+1,i} \to \mathcal{P}_{n,i} \to 0.
$$

Assuming for a moment that this holds, let us see how it implies the theorem. First of all, (i) follows:

$$
r_m(f) = \deg \mathcal{P}_{n,i}^X \text{ (by (3.1)) = deg(\mathcal{P}_{n-i-1}^X)}
$$

$$
= - \deg \mathcal{X}_{n-1-i} = \deg \mathcal{P}_{n-i-1}^X = r_{n-1-m}(f).
$$

The curve $f^*$ is defined by the 1-quotient $V_X \to \mathcal{X}_{n-1}^*$, i.e., by $V_X \to \mathcal{P}^{n-1}$, therefore (ii) holds.

Suppose $p_i^q(f)$ is defined by an $(n + 1 - m)$-quotient $V \to V^*$ and a $(q + 1)$-subspace $W \to V^*$. We obtain (as in the proof of (4.6)) a commutative diagram

$\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\end{array}$
indices of a $q$-projection of an $m$-section $p_\alpha^2: X_\alpha \to P$ of $f$ are given by:

$$
\begin{align*}
&\tau_i(p_\alpha^2) = \tau_{i+1}, \\
&\kappa_0(p_\alpha^2) + \kappa_1, \\
&\kappa_i(p_\alpha^2) = \kappa_{i+1}, \\
&\kappa_{q-1}(p_\alpha^2) = \kappa_n + \kappa_{n+1}
\end{align*}
$$

PROOF. The osculating bundle of order $i$, $\mathcal{O}_\alpha^i$, of an $m$-section $f_\alpha$ fits into an exact sequence (see (4.5))

$$
0 \to V_{\kappa_i} \to \gamma_{\mathcal{O}_\alpha^{i+1}} \to \mathcal{O}_\alpha^i \to 0.
$$

Hence we get $\tau_i(f_\alpha) = \deg(\mathcal{O}_\alpha^i) = \deg(\mathcal{O}_\alpha^{i+1}) = \tau_{i+1}$. From (4.2) it follows that the osculating bundle of $p_\alpha^2$ is equal to that of $f_\alpha$, hence we get also $\tau_i(p_\alpha^2) = \tau_{i+1}$.

The formulas for $\kappa_i(p_\alpha^2)$ are straightforward applications of the formula (3.3) (ii) to the curves $p_\alpha^2$ and $f$.

EXAMPLE 1. The $i$th rank of an $m$-section of a $q$-projection of the rational $n$-ic $R_\alpha$ is $(m+i+1)(n-m-i)$; it has $m(n-m+1)$ cusps if $q > 1$, $(m+q+1)(n-m-q)$ stationary hyperplanes, and no other points of hyperosculations.

EXAMPLE 2. For the curve $p_\alpha^2(f_\alpha)$ we get $\tau_i = (m+i+1)d$, $\kappa_0 = 0$ if $q > 1$, $\tau_i = 0$ for $0 < i < q-1$, $\tau_i = (m+i+1)d$ in particular, if we take $q = 2$, $m = n - 2$, we get a plane curve of degree $(n-1)d$ class $md$, with $(n-2)d$ cusps and $2(n+1)d$ flexes.

EXAMPLE 3. Starting with the curve $X_2 = P^3$, we obtain 3 plane curves $p_\alpha^i$, $i = 0, 1, 2$, with the following numerical characters:

| $\tau_0(\text{degree})$ | $\tau_i(\text{class})$ | $\kappa_0(\text{cusps})$ | $\kappa_1(\text{flexes})$
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_0^0$</td>
<td>8</td>
<td>24</td>
<td>0</td>
</tr>
<tr>
<td>$p_1^1$</td>
<td>24</td>
<td>48</td>
<td>8</td>
</tr>
<tr>
<td>$p_2^2$</td>
<td>48</td>
<td>40</td>
<td>64</td>
</tr>
</tbody>
</table>

\section{5. Duality}

Suppose $f: X \to P$ is finite and $\alpha^*: V_X \to \mathfrak{g}_0^{\alpha^*}(2)$ is generically surjective. Assume also that we have either $\vartheta = 0$ or char $\kappa > n$. The dual curve $f: X \to P^* = \mathfrak{g}(V^*)$ of $f: X \to P$ is defined by the 1-quotient $V_X \to \mathfrak{X}_{\alpha^*}$, where we put $\mathfrak{X}_{\alpha^*} = \text{Ker}(a^{\alpha^*}: V_X \to \mathfrak{g}^\alpha)$. Hence a point $f(x)$ of the dual curve is equal to the osculating hyperplane to $f$ at $x$, considered as a point in the dual projective space.

\textbf{THEOREM (5.1).} (i) The $m$-rank of $f$ is equal to the $(n-1-m)$-rank of $f$, i.e., $\tau_i(f) = \tau_{i-1-m}(f)$, for $m = 0, 1, \ldots, n-1$, and similarly for the stationary indices: $\kappa_i(f) = \kappa_{i-1-m}(f)$ for $m = 0, 1, \ldots, n-1$.

(ii) The dual curve of $f$ is equal to $f$ (we write $(\mathfrak{X})^\alpha = f$).

(iii) The dual of a $q$-projection of an $m$-section of $f$ is equal to a $q$-projection of an $(n-m-q)$-section of $f$ (we write $p_\alpha^2(f)^* = p_\alpha^{n-q-m}(f)$).

\textbf{Proof.} Let $\mathfrak{g}_\alpha^{\alpha^*}$ denote the $m$th osculating bundle of $f$ and $\mathfrak{g}_0^{\alpha^*}(2)$ the canonical quotient. Put $\mathfrak{g}_0^{\alpha^*} = \text{Ker}(a^{\alpha^*})$ and $\mathfrak{X}_{\alpha^*} = \text{Ker}(a^{\alpha^*})$. The key result is the following:

\textbf{LEMMA (5.2).} There are canonical isomorphisms of exact sequences (for $m = 0, \ldots, n-1$)

$$
\begin{align*}
0 &\to \mathfrak{g}_\alpha^{\alpha^*} \to V_X \to \mathfrak{g}_0^{\alpha^*}(2) \to 0, \\
&\to (\mathfrak{g}_\alpha^{\alpha^*})^\alpha \to V_X \to (\mathfrak{X}_{\alpha^*})^\alpha \to 0.
\end{align*}
$$

Assuming for a moment that this holds, let us see how it implies the theorem. First of all, (i) follows:

$$
\tau_i(f) = \deg(\mathfrak{g}_\alpha^{\alpha^*}) = \deg(\mathfrak{X}_{\alpha^*}) = \tau_{i-1-m}(f).
$$

The curve $(\mathfrak{X})^\alpha$ is defined by the 1-quotient $V_X \to \mathfrak{X}_{\alpha^*}^\alpha$, i.e., by $V_X \to \mathfrak{g}^\alpha$, therefore (ii) holds.

Suppose $p_\alpha^2(f)^*$ is defined by an $(n+1-m)$-quotient $V^* \to V^*$ and a $(q+1)$-subspace $W^* \to V^*$. We obtain (as in the proof of (4.6)) a commutative diagram

\begin{align*}
0 &\to f \to X_{\alpha^*-1-m} \to W_X^* \to 0, \\
&\to W_X^* \to V^* \to W^*_X \to 0, \\
\mathfrak{X}_{\alpha^*}^\alpha = \mathfrak{g}^\alpha &\to 0.
\end{align*}
The dual curve $p^{*}(t)$ is defined by the 1-quotient $(W^{n}x)^{n} \to F^{*}$. Let $W = \ker(V \to W)$, we have an induced surjection $W \to W'$, and we note $\dim W = m + 1 + q$. The quotient $V' \to W'$ defines an $(n - m - q)$-section of $f$, we get an exact sequence $0 \to \mathcal{X}^{n} \to W_{Q} \to W' \to 0$ defining this section.

The isomorphism (5.2) then shows that the 1-quotient $W' \to F'$ defining the projection $p_{m}^{n}(t)$ corresponding to the subspace $W'^{\circ} \subset W'$ is canonically isomorphic to the 1-quotient $W_{Q} \to F'$. This proves (iii).

Proof of (5.2). (The proof is essentially the one given by Weyl ([15], p. 47) for analytic curves.) We will first show that the map between the sequences exists, by showing that the composition $\mathcal{A} = (a^{*} - y)^{n} = 0$. Then we show that $\mathcal{A}$ has rank $m + 1$, for $m = 0, 1, \ldots, n$ (so that the dual curve spans $P$). From these two facts it then follows that the generically surjective map $a^{*} - y = 0$ holds locally on $X$. Let $x \in X$ and put $A = C_{x}$. With the notations of the proof of (2.1), the map $a(u - y)^{n} = 0$ holds locally on $x$. Let $y = A = A^{*} = 0$ be the $(n + 1, n)$-matrix corresponding to the (locally split) map $\mathcal{X}^{n} \to \mathcal{W}$ at $x$, so that the 1-quotient $V_{Q} \to \mathcal{W}_{Q} \subset \mathcal{W}$ defining $f$ is given, locally at $x$, by $y^{T}$. It follows (2.1) that the map $\mathcal{A}$ is given by the matrix $w$. The entries of the matrix $w^{*} = (w^{*})^{T}$ are $y_{i}$ for $0 \leq i \leq m$, $0 \leq j \leq n - 1 - m$, $0 \leq m \leq n - 1$.

By definition of $y$, $D_{i}^{*} = 0$ holds for $j = 0, 1, \ldots, n - 1$. Assume $D_{i}^{*} = 0$ holds for some $i$, and for $j = 0, i, \ldots, n - 1 - i$. Then we can apply the differential operator $d'$:

$0 = d'(D) = \sum_{i = 0}^{n} \left[ (i + 1)d^{*}y_{i}d_{i} + (i + 1)d^{*}y_{i}d_{i}^{*}s_{i} \right] = (i + 1)D_{i}^{*} + (i + 1)D_{i}^{*}$

Hence we get $D_{i}^{*} = 0$ for $j = 0, \cdots, n - 1 - i$, and we can prove (1) by induction.

It remains to prove that $a^{*}(t)$ is generically surjective, for $0 \leq m \leq n$; since we have surjections $\mathcal{A} \to \mathcal{A}_{Q}^{*}$, it is enough to show that $a^{*}(t)$ is generically surjective. Hence we may assume that $x \in X$ is an arbitrary point (see §2), so that $w(x) = \det(w(x))$ is a unit in $A = C_{x}$. Suppose we can prove the equality

$w(x)w(y) = (-1)^{n(m-1)}t(w(x))^{n} \prod_{i=1}^{n} \frac{\det(w(x))}{\det(w(y))}$

where we have put $w(y) = \det(w(x))$. It would follow that $w(y)$ is a unit in $A$, hence $a(t)$ is surjective at $x \in X$, as desired.

Consider the matrix $w^{*} = (w^{*})^{T}$. Since $D_{i}^{*} = 0$ holds for $0 \leq i \leq n - 1$, this matrix is lower right triangular; its anti-diagonal entries are $D_{i}^{*}$, In order to prove (1), it thus suffices to show

$D_{i}^{*} = (-1)^{n}w(z)$.

Using the definition of $y$ (and linear algebra) we see that (1) holds. Assume (1) holds and apply $a^{*}$ to $D_{i}^{*}$. From this we obtain $D_{i} = (-1)^{n}D_{i}^{*} = 0$ and the proof can be completed by induction.

Together with the results of §4, theorem (5.1) shows that from a curve $f : X \to P$ in $n$-space we obtain, by the method of projections, sections and dual curves, $n - 1$ (and not more) plane curves $p^{*}(t)$ and their duals. Applying formulas (5) and (13) to these $(n - 1)$ plane curves we obtain the formulas of Veronese ([14], §3), if we (slightly to what we did) interpret the character $c$ of (II) for each curve $p^{*}(t)$, $p_{2}(t)$ in terms of (actual and apparent) double osculating spaces of the curve $f$. We will not discuss this interpretation here; let us only, as an application of (5.1), give the Cayley-Plücker formulas for a curve in $3$-space ([2], p. 191).

**Corollary (5.3). Suppose $f : X \to P \to P$ is a curve such that $a^{*}(t)$ is generically surjective. Suppose char $k = 2, 3$. Let $\gamma_{i}$, $\gamma_{j}$, $\gamma_{k}$ denote the ranks and stationary indices of $f$, and let $E_{i}$ (resp. $E_{j}$) denote the degree of the Jacobian ideal of the plane curve $p_{i}(t)$ (resp. $p_{j}(t)$), for $i = 0, 1, 2, 3$.

**Example 1.** The rational $n$-ic is self-dual: its dual curve is also a rational $n$-ic.

**Example 2.** $E_{i} \subseteq P^{3}$. We have:

$a_{i} = d(d - 3)$, $b_{i} = d(4d - 3)$

$E_{i} = d(4d - 5)$, $b_{i} = d(9d - 5)$. 


The dual curve $p^*_n(f')$ is defined by the 1-quotient $(W^m)\rightarrow \mathcal{A}_n$. Put $W = \text{Ker}(V \rightarrow W)$. We have an induced surjection $W \rightarrow W$, and we note $\dim W = m + 1 \cdot k$. The quotient $V \rightarrow W$ defines an $(n-m-1)$-section of $f$, we get an exact sequence $0 \rightarrow \mathbb{Z} \rightarrow W \rightarrow \mathcal{A}_n \rightarrow 0$ defining this section. The isomorphism (5.2) then shows that the 1-quotient $W^m \rightarrow \mathcal{A}_n$ defining the projection $p^*_n(f')$ corresponding to the subspace $W^m \subseteq W$ is canonically isomorphic to the 1-quotient $W^m \rightarrow \mathcal{A}_n$. This proves (iii).

**Proof of (5.2).** (The proof is essentially the one given by Weyl ([15], p. 47) for analytic curves.) We will first show that the map between the sequences exists, by showing that the completion $\pi^*_n = (\pi^*_n)$ is 0. Then we show that $\pi^*_n$ has rank $m + 1$, for $m = 0, 1, \ldots, n$ (so that the dual curve spans $\mathcal{A}_n$). From these two facts it then follows that the generically surjective map $(\pi^*_n)$ factors through the quotient $V^m \rightarrow \mathcal{A}_n$, hence its image $\pi^*_n$ is isomorphic to this quotient.

It suffices to show that $a_{x}^* = (a_{x}^{*})$ holds locally on $X$. Let $x \in X$ and put $A = \mathcal{A}_X$. With the notations of the proof of (2.1), the map $(a_{x}^{*})$ is given by $(w_{x}^{\alpha^{*}}): A^* \rightarrow A^{*1}$. Let $(x): A \rightarrow A^{*1}$ be the $(n+1, 1, 0)$-matrix corresponding to the locally split map $Z_{x} \rightarrow V_{x}$ at $x$, so that the 1-quotient $V^m \rightarrow Z_{x} = (\mathbb{Z}^m)_x$ defining $f$ is given, locally at $x$, by $(x)^{\alpha}$. It follows (2.1) that the map $a_{x}^*$ is given by the matrix $w_{x}^{\alpha}$. The entries of the matrix $w_{x}^{\alpha} = (w_{x}^{\alpha_{j}^{*}})$ are $\sum_{s=0}^{n} a_{j}^{s} d_{x}^{s}$. Hence we want to show:

$$D^{\alpha} = \sum_{y=0}^{\alpha-n} d_{x}^{y} d_{y} = 0 \quad \text{holds for} \quad 0 \leq i \leq m, \quad 0 \leq j \leq n - 1 - 1, \quad (*)$$

$$0 \leq m \leq n - 1.$$

By definition of $(y)$, $D^{\alpha} = 0$ holds for each $n = 1, \ldots, n - 1$. Assume $D^{\alpha} = 0$ holds for some $i$, and for $j = 0, 1, \ldots, n - 1 - 1$. Then we can apply the differential operator $d^{*}$:

$$0 = d^{*}(D^{\alpha}) = \sum_{y=0}^{\alpha-n} \left[ (i+1) d_{x}^{y} d_{y} + (j+1) d_{x}^{y} d_{y} \right]$$

$$= (i+1)(D^{\alpha})^{(i+1)}$$

Hence we get $D^{\alpha} = 0$ for $i = 0, 1, \ldots, n - 1 - (i+1)$, and we can prove (*) by induction.

It remains to prove that $a_{x}^{*}$ is generically surjective, for $0 \leq m \leq n$; since we have surjections $\pi^*_n \rightarrow \pi^*_n$, it is enough to show that $a_{x}^{*}$ is generically surjective. Hence we may assume that $x \in X$ is an ordinary point (see §5), so that $w(x) = \text{det}(w_{x})$ is a unit in $A = \mathcal{A}_X$. Suppose we can prove the equality

$$w(x)w(y) = (-1)^{\delta(x,y)} w(x)^{-1} w(y)^{-1} \left( \begin{array}{c} n \end{array} \right)$$

where we have put $w(x) = \text{det}(w_{x})$. It would follow that $w(x)$ is a unit in $A$, hence $a_{x}^{*}$ is surjective at $x \in X$, as desired.

Consider the matrix $w_{x}^{\alpha}$. Since $D^{\alpha} = 0$ holds for $0 \leq i + 1 \leq n - 1$, this matrix is lower right triangular; its anti-diagonal entries are $D_{w_{x}^{\alpha}}$. In order to prove (**), it thus suffices to show

$$D_{w_{x}^{\alpha}} = (-1)^{\left( \begin{array}{c} n \end{array} \right)} w(x).$$

Using the definition of $(y)$ (and linear algebra) we see that (***)$_{0}$ holds.

Assume (***)$_{0}$ holds and apply $d^{*}$ to $D^{\alpha}$. From this we obtain $D^{\alpha+1} = 0$ and the proof can be completed by induction.

Together with the results of §4, theorem (5.1) shows that from a curve $f: X \rightarrow P$ in $n$-space we obtain, by the method of projections, sections and dual curves, $n - 1$ (and not more) plane curves $p_{i}(f')$ and their duals. Applying formulas (1) and (12) of the $2n$-plane curves we obtain the formulas of Veronese ([14], §2). If we (similarly to what he does) interpret the character $\epsilon$ of (II) for each curve $p_{i}(f')$, $p_{i}(f')$ in terms of (actual and apparent) double osculating spaces of the curve $f$. We will not discuss this interpretation here; let us only, as an application of (5.1), give the Cayley-Pliedler formulas for a curve in 3-space ([2], p. 191).

**Corollary (5.3).** Suppose $f: X \rightarrow P$ is a curve such that $a_{x}^{*}\pi_{x} = \partial^{*}(x)$ is generically surjective. Suppose char $k$ is $2, 3$. Let $(n, r, n)$ and $(n, s, k, x)$ denote the ranks and stationary indices of $f$, and let $\epsilon_{i}$ (resp. $\delta_{j}$) denote the degree of the Jacobian ideal of the plane curve $p_{i}(f')$ (resp. $p_{j}(f'')$), for $i = 0, 1$. There are formulas

$$\begin{array}{ll}
(1) & k_{1} = 2n_{1} - (n_{1} + 2) + 2k - 2 \\
(2) & k_{2} = 2n_{2} - n_{1} + 2k - 2 \\
(3) & k_{3} = 2n_{3} - n_{1} - 2k - 2 \\
(4) & \epsilon_{1} = -3k_{2} + k_{3} = (n_{1} - 1)(n_{1} - 6) + 2n_{1} - 6 \\
(5) & \epsilon_{2} = -3k_{2} + k_{3} = (n_{1} - 1)(n_{1} - 6) + 2n_{1} - 6 \\
(6) & \epsilon_{3} = 3k_{2} - (n_{1} - 1)(n_{1} - 6) + 2n_{1} - 6 \\
(7) & \epsilon_{3} = 3k_{2} - (n_{1} - 1)(n_{1} - 6) + 2n_{1} - 6 \\
\end{array}$$

**Example 1.** The rational $n$-ic is self-dual: its dual curve is also a rational $n$-ic.

**Example 2.** $E_{i} \subset \mathbb{P}^{3}$. We have:

$$\begin{array}{ll}
\epsilon_{0} = d(d - 3) \\
\epsilon_{1} = d(d - 3) \\
\epsilon_{2} = d(d - 5), \\
\end{array}$$
EXAMPLE 3. $\Gamma_2 \in P^3$ is an elliptic curve of degree 4, with no cusps and no flexes. It has rank 8 and class 12. The number of apparent double points is $|\alpha_4| = 2$, and the degree of the double curve of its tangent developable is $|\alpha_5 - 3\alpha_6 + \alpha_7| = 16$. The corresponding characters for the dual curve are $|\alpha_5 - 3\alpha_6 + \alpha_7| = 38$ and $\alpha_6 = \alpha_7 = 8$.

§6. Appendix. The functor of principal parts

Let $f: X \to S$ be a morphism of schemes. For each integer $m \geq 0$, consider the functor of relative principal parts (or jets) of order $m$

$$\mathcal{P}^m_{f,S} = \mathcal{P}^m(f): \mathcal{C}_X \to \mathcal{C}_X \text{-mod}.$$

Here $\mathcal{P}^m(f)$ denotes the composition of the $m$th projection $p_m: X \times_S X \to X$ with the $m$th infinitesimal neighborhood $h_m: X^{(m)} \to X \times_S X$ of the diagonal in $X^{(m)}$.

Let $\sigma(f): f^\ast f_\ast \to 1$ denote the canonical natural transformation. There is a natural transformation

$$p_m \circ h_m \circ \sigma(f) \circ p_m \circ h_m \circ \sigma(f) \circ p_m \circ h_m \circ \sigma(f) = \mathcal{P}^m_{f,S}.$$

Define $a^m = a^m(f): f^\ast f_\ast \to \mathcal{P}^m_{f,S}$ to be the natural transformation obtained by composing the one above with the base change transformation

$$(f_\ast \circ p_m)(p_m)^\ast f^\ast f_\ast \to p_m \circ h_m \circ \sigma(f) \circ p_m \circ h_m \circ \sigma(f) \circ p_m \circ h_m \circ \sigma(f)$$

(where $\#$ denotes the isomorphism $\text{HomS}(\# f_\# \#) \to \text{HomS}(f_\# \# - \#))$. Define also a natural transformation of functors from $\text{HomS}(\# f_\# \#)$ to $\mathcal{C}_X \text{-mod}$

$$\beta^m = \beta^m(f): f^\ast f_\ast \to \mathcal{P}^m_{f,S} \circ f^\ast f_\ast$$

by composing $p_m \circ h_m \circ \sigma(f) \circ p_m \circ h_m \circ \sigma(f) \circ p_m \circ h_m \circ \sigma(f)$ with the isomorphism

$p_m \circ h_m \circ \sigma(f) \circ p_m \circ h_m \circ \sigma(f) \circ p_m \circ h_m \circ \sigma(f)$

(whose existence follows since we have $f_\# \circ p_m \circ h_m \circ \sigma(f) \circ p_m \circ h_m \circ \sigma(f)$).

Note that

$$\beta^m(\mathcal{P}^m_{f,S}) \circ \sigma(f) \circ \beta^m(\mathcal{P}^m_{f,S}) = \mathcal{P}^m_{f,S} \circ \beta^m(\mathcal{P}^m_{f,S})$$

is the isomorphism which defines the (not $\mathcal{C}_X$-algebra structure on $\mathcal{P}^m_{f,S}(\mathcal{P}^m_{f,S})$.

Proposition 6.1. There is a factorization $a^m = (\mathcal{P}^m_{f,S}(a^m(f)) \circ \beta^m(\mathcal{P}^m_{f,S})$.

Proof. By the adjoint property (3.5.3.4) we know that $a^m$ factors through $\beta^m(\mathcal{P}^m_{f,S})$,

$$a^m = (p_m^\ast \circ a^m \circ p_m) \circ (\beta^m(\mathcal{P}^m_{f,S}))$$

(here $\#$ denotes the isomorphism $\text{HomS}(\# f_\# \#) \to \text{HomS}(f_\# \# - \#))$. Thus it suffices to show that we have $\mathcal{P}^m_{f,S}(a^m) = p_m^\ast a^m \circ p_m$, or, equivalently, that $a^m = p_m^\ast \circ a^m \circ f$. The proof of this equality is straightforward.

The following proposition lists various properties of $a^m$. The proofs are straightforward (using the naturality of the functors involved) and will therefore be omitted.

Proposition 6.2,

(i) We have $a^m = \sigma(f)^\ast f_\ast \to 1$.

(ii) If $m \leq m$, the canonical diagram

$$\begin{array}{ccc}
    f^\ast f_\ast & \to & \mathcal{P}^m_{f,S} \\
    \downarrow \sigma & & \downarrow \beta^m \\
    f^\ast f_\ast & \to & \mathcal{P}^m_{f,S} \\
\end{array}$$

commutes.

(iii) If $g: Y \to X$ is an $S$-morphism, there is a canonical commutative diagram

$$\begin{array}{ccc}
    g^\ast f^\ast f_\ast & \to & g^\ast \mathcal{P}^m_{f,S} \\
    \downarrow \sigma^m & & \downarrow \beta^m \\
    g^\ast f^\ast f_\ast & \to & g^\ast \mathcal{P}^m_{f,S} \\
\end{array}$$

(iv) Let $a^m$: $\mathcal{P}^m_{f,S} \to \mathcal{P}^m_{g,S}$. Define the natural transformation defined in (7) 16.8.8.2. The following diagram commutes

$$\begin{array}{ccc}
    f^\ast f_\ast & \to & g^\ast \mathcal{P}^m_{f,S} \\
    \downarrow a^m & & \downarrow \beta^m \\
    f^\ast f_\ast & \to & g^\ast \mathcal{P}^m_{f,S} \\
\end{array}$$

Proposition 6.3. Let $\mathcal{B}$ be a bundle on $S$. Put $P = \mathcal{P}(\mathcal{B})$ and $\mathcal{E} = \mathcal{O}(1)$. The map $a^m(\mathcal{B}): \mathcal{B} \to \mathcal{P}^m(\mathcal{B})$ is an isomorphism.

Proof. It suffices to show that $a^m(\mathcal{B})$ is surjective (since it is a map between bundles of the same rank). The question is local on $S$, so we may assume $S = \text{Spec}(A)$ and $\mathcal{B} = \bigoplus_{i \in \mathbb{A}} A$. Write $\text{Sym}(E) = A[T_0, \ldots, T_r]$ and let $U = \{ \mathcal{B} \in \mathcal{O} \cap \mathcal{B} \}$. Then $B = A[T_0, \ldots, T_r] \cap \mathcal{B}$ is the ring of $U$. The map $a^m(\mathcal{B}) \circ \mathcal{B} \to \mathcal{B}$ is the map which sends $\alpha$ to $(T_0, T_1, \ldots, T_r)$, and it follows from the factorization $a^m(\mathcal{B}) = p_m^\ast \mathcal{B}^m(f)(\mathcal{B}) \circ \beta^m(\mathcal{B})$. (6.1) that $a^m(\mathcal{B}): \mathcal{B} \to \mathcal{P}^m(\mathcal{B}) \mid U = (\mathcal{B} \cap \mathcal{B})^{(1)} \cap \mathcal{B} \eta \mathcal{B}^m(\mathcal{B})$ is the map which sends $\alpha$ to $1 \otimes (T_0, T_1, \ldots, T_r)$ (here we have
The following proposition lists various properties of \( \alpha^* \). The proofs are straightforward (using the naturality of the transformations involved) and will therefore be omitted.

**Proposition (6.2).**
(i) We have \( \alpha^* \circ f^\sharp \phi_{XY} \rightarrow 1. \)
(ii) If \( m \leq m \), the canonical diagram

\[
\begin{array}{ccc}
\phi_{XY} & \rightarrow & \phi_{XY} \\
\downarrow & & \downarrow \\
X & \rightarrow & X
\end{array}
\]

commutes.
(iii) If \( g: Y \rightarrow X \) is an \( S \)-morphism, there is a canonical commutative diagram

\[
\begin{array}{ccc}
g^\sharp \phi_{YX} & \rightarrow & g^\sharp \phi_{YX} \\
\downarrow & & \downarrow \\
g^\sharp & \rightarrow & g^\sharp
\end{array}
\]

(iv) Let \( \eta: \eta^\natural: \phi_{XY} \rightarrow \phi_{XY} \phi_{XY} \) denote the natural transformation defined in (6.8.2.2). The following diagram commutes

\[
\begin{array}{ccc}
f^\sharp \phi_{XY} & \rightarrow & f^\sharp \phi_{XY} \\
\downarrow & & \downarrow \\
f & \rightarrow & f
\end{array}
\]

**Proposition (6.3).** Let \( \mathcal{B} \) be a bundle on \( S \). Put \( P = P(\mathcal{B}) \) and \( \mathcal{E} = \mathcal{O}(1). \) The map \( \alpha^*(\mathcal{D}): \mathcal{E} \rightarrow \mathcal{P}(\mathcal{D}) \) is an isomorphism.

**Proof.** It suffices to show that \( \alpha^*(\mathcal{D}) \) is surjective (since it is a map between bundles of the same rank). The question is local on \( S \), so we may assume \( S = \text{Spec } (A) \) and \( E = \bigoplus \mathcal{O}_S \). Write \( \text{Sym } (E) = A[T_0, \ldots , T_r] \) and let \( U \subseteq P \) denote the open affine scheme defined by \( T_r \neq 0 \). Then

\[
\begin{align*}
B & = A[T_0, \ldots , T_r, T_1] \\
U & = (0 \cap B) / \bigoplus B T_1.
\end{align*}
\]

It follows from the factorization \( \alpha^*(\mathcal{D}) = \phi_{XY} \circ \phi_{XY}^\sharp \circ \eta \circ f^\sharp \phi_{XY} \) (6.1) that \( \alpha^*(\mathcal{D}): \eta^\natural \circ f^\sharp \phi_{XY} | U = (0 \cap B) / \bigoplus B T_1 \) is the map which sends \( \eta \) to \( 1 \otimes (T_r T_1) T_1 \) (here we have
put \( I = (b \otimes 1 - 1 \otimes b) B \otimes B \). Since these elements generate \((B \otimes B)/I \otimes B T_{an}\), we have shown that \(a(\mathcal{X})\) is surjective.

**Remark (6.4).** Put \( \mathcal{X} = \text{Ker}(a(\mathcal{X}))\). The lemma shows that there is a canonical isomorphism of exact sequences

\[
0 \rightarrow \mathcal{X} \rightarrow L \rightarrow \xi \rightarrow 0
\]

\[
0 \rightarrow \Omega^1(\mathcal{X}) \rightarrow \Omega^1_\mathcal{X}(\mathcal{X}) \rightarrow \mathcal{X} \rightarrow 0.
\]

The isomorphism \( a \) is, of course, the canonical isomorphism called the second fundamental form in \((1), I, 3.1\). (The isomorphism \( a \) can be described as follows: We see that \( \mathcal{X} \mid U \) is the free \( B \)-module generated by \( \{ e_i \}_{i \in I} \), where \( e_i = (T/T_i)e_i - e_i \). Hence

\[
a(e_i) = a(\xi)((T/T_i)e_i - e_i) = (T/T_i \otimes 1)(1 \otimes T_i) - (1 \otimes (T/T_i)T_i) = (T/T_i \otimes 1 - 1 \otimes T/T_i)(1 \otimes T_i)
\]

\((1), I, 3.1\).

Note that the map \( U : (B \otimes B)/I \otimes B T_{an} \rightarrow B T_{an} \) is given by

\[
e(1 \otimes (T/T_i)(1 \otimes T_i)) = e(1 \otimes (T/T_i)T_i) = (T/T_i)T_i
\]

We shall use \( a(\mathcal{X}) \) to identify \( \Omega^1_\mathcal{X}(\mathcal{X}) \) (as a left \( \Omega^1_{\mathcal{X}} \)-module) with \( \mathcal{Y}_{\mathcal{X}} \), and we will denote \( \Omega^1_{\mathcal{X}}(\mathcal{X}) \) with \( \mathcal{X} \) via \( a \).

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put $I = (B \otimes 1 - 1 \otimes B)B \otimes B).$ Since these elements generate $(B \otimes B)/I^0 \otimes B I,$ we have shown that $a'(2)$ is surjective.

**Remark (6.4).** Put $X = \text{ker}(a'(2)).$ The lemma shows that there is a canonical isomorphism of exact sequences

$$0 \longrightarrow X \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$  

The isomorphism $a$ is, of course, the canonical isomorphism called the second fundamental form in ([1], I, 3.1). (The isomorphism $a$ can be described as follows: We see that $X | U$ is the free $B$-module generated by $[e_i]_{i \in \mathbb{N}}$, where $e_i = (T/T_i) \epsilon_i - \epsilon_i$.

$$a(e_i) = a'(2)((T/T_i) \epsilon_i - \epsilon_i)$$

$$= (T/T_i \otimes 1)(1 \otimes T_i) - (1 \otimes (T/T_i) T_i)$$

$$= (T/T_i \otimes 1 - 1 \otimes T_i)(1 \otimes T_i).$$

Note that the map $\sigma : U : (B \otimes B)/I^0 \otimes B I \longrightarrow B I$ is given by

$$\sigma((1 \otimes (T/T_i))(1 \otimes T_i)) = \epsilon(1 \otimes (T/T_i) T_i) = (T/T_i) \epsilon_i.$$

We shall use $a'(2)$ to identify $H^0(\mathcal{O}_Z)$ (as a left $\mathcal{O}_Z$-module) with $\mathcal{E}_n$, and we will identify $\Omega^1(\mathcal{O}_Z)$ with $X$ via $a$.

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