ON THE EXPONENTS
AND THE GEOMETRIC GENUS
OF AN ISOLATED HYPERSURFACE
SINGULARITY

MORIHKO SAITO

1. Introduction. Let \( f \in \mathcal{O}_{\mathbb{C}^n, 0} \) be a germ of a holomorphic function such that \( f(0) = 0 \) and \( f \) has an isolated singularity at 0. Using the limit mixed Hodge structure J. H. M. Steenbrink defined some rational numbers associated with \( f \) (cf. [St2]), which we call the exponents of \( f \) (cf. Definition (3.2)).

The geometric genus \( p_g \) of the hypersurface \( V := \{ f = 0 \} \) is defined via a resolution of singularity \( \bar{p} \rightarrow V \):

\[
p_g := \dim_{\mathbb{C}} \left( R^{n-1} \mathcal{O}_{\bar{p}} / \partial \mathcal{O}_{\bar{p}} \right)_{\mathcal{O}} \quad \text{for } n \geq 2,
\]

\[
p_g := \delta = \dim_{\mathbb{C}} \left( \partial \mathcal{O}_{\bar{p}} / \partial \mathcal{O}_{\bar{p}} \right)_{\mathcal{O}} \quad \text{for } n = 1.
\]

The main result of this paper is the following.

**Theorem 1.** The geometric genus \( p_g \) equals the number of the exponents not greater than 1.

**Remarks.** (1) By the definition of exponents this theorem is equivalent to the formula \( p_g = \dim_{\mathbb{C}} \mathcal{O}_{\bar{p}} H^q(X_0, \mathcal{O}) \) (cf. §4).

(2) For the quasihomogeneous case, this theorem was proved by Kimio Watanabe (cf. [W, Theorem 1.13, p. 71]).

Combining the theorem with the result of Steenbrink [St2, Theorem (4.11)], we obtain the following.

**Corollary 1.** \( 2p_g = \mu_+ + \mu_0 \) for \( n = 2 \), \( 2\delta = \mu + \mu_0 \) for \( n = 1 \).

Here \( \mu \) is the Milnor number of \( f \) and \( \mu_+ \) (resp. \( \mu_0 \)) is the number of positive (resp. zero) eigenvalues of the intersection form (cf. [St2]).
The formula for $u = 1$ is the so-called Milnor relation because $\mu_0 + 1$ equals the number of the irreducible components of $\{V, 0\}$. The case $n = 2$ was proved by A. Durfee, I. Deligne and M. Reid (cf. [Dur]).

As an application of our argument, we obtain the following result.

Let $V$ be a hypersurface of degree $d$ in $\mathbb{P}^{n+1}$. Assume that $\Sigma(V)$, the singular locus of $V$, is discrete. Let $\nu: \tilde{V} \rightarrow V$ be a resolution. Put $h^r(\tilde{V}) := \dim_{\mathbb{C}} H^r(\tilde{V}, \mathbb{R})$ and $p_\nu(V) := h^0(\tilde{V})$. Let $\rho_\nu(V, x)$ be the geometric genus of $x \in \Sigma(V)$. Then we have

$$\sum_{x \in \Sigma(V)} \rho_\nu(V, x) = h^{n-1}(\tilde{V}) + \binom{d-1}{n+1} - p_\nu(\tilde{V}) \quad \text{for } n \geq 2.$$ 

Suppose $V$ is rational. In particular, this is the case when there exists $x \in \Sigma(V)$ such that $\mu_\nu(V) = d-1$. Then the above formula turns out to be

$$\sum_{x \in \Sigma(V)} \rho_\nu(V, x) = \binom{d-1}{n+1} \quad \text{for } n \geq 2.$$ 

We remark that the number on the right-hand side is the geometric genus of a nonsingular hypersurface of degree $d$ in $\mathbb{P}^{n+1}$.

I thank K. Saito, I. Naruki, A. Fujiishi, K. Watanabe and T. Yano for helpful discussions.

2. Limit mixed Hodge structure. We review the theory of the limit mixed Hodge structure of Deligne-Saint-Field (cf. [St2]).

(2.1) Compactification. Let $f \in \mathcal{E}_{n+1, 0}$ be a holomorphic function with $f(0) = 0$. We assume that $f$ has an isolated critical point at $0$. Then we may assume that $f$ is a polynomial and $Y_0 := \{x \in \mathbb{C}^{n+1} : f(x) = 0\} \subset \mathbb{P}^{n+1}$ is nonsingular away from $0$. Define

$$Y := \{ (x, t) \in \mathbb{C}^{n+1} \times S : f(x) = t \} \subset \mathbb{P}^{n+1} \times S,$$

and

$$Y_t := Y \cap \{ \mathbb{P}^{n+1} \times \{ t \} \} \quad \text{for } t \in S,$$

where $S := \{ (t \in \mathbb{C} : |t| < \eta \}$ and the number $\eta$ is sufficiently small such that the projection $Y \rightarrow S$ is smooth away from $0$.

(2.2) Milnor’s fibration. Let $B := \{ x \in \mathbb{C}^{n+1} : \| x \| < \varepsilon \}$ be an open ball with radius $\varepsilon$. If $\varepsilon$ and $\eta$ are sufficiently small, the projection

$$X := Y \times (B \times S) \rightarrow S$$

is smooth away from $0$ and its restriction over $S^* := S - \{ 0 \}$ is a topological fibration and the topological type of a general fiber $X_t$ does not depend on $t$. If $x \neq 0$, $H^1(X_t, \mathbb{C}) = 0$ and $\dim_{\mathbb{C}} H^1(X_t, \mathbb{C})$ is called the Milnor number and is denoted by $\mu$. Since $\bigcup_{t \in \mathbb{C}^*} H^1(X_t, \mathbb{C})$ is a flat vector bundle, the monodromy transformation $\gamma$ acts on $H^1(X_t, \mathbb{C})$, and we call it the local monodromy.

In the same way $\gamma$ acts on $H^i(Y_t, \mathbb{C})$ and the restriction morphism $H^i(Y_t, \mathbb{C}) \rightarrow H^i(X_t, \mathbb{C})$ is equivariant with respect to $\gamma$.

(2.3) Resolution. Let $\rho: \tilde{Y} \rightarrow Y$ be an embedded resolution of $Y_t$, i.e., $\rho$ is a proper holomorphic map of a manifold $\tilde{Y}$ to $Y$ such that $\rho_\rho^{-1}(0) \rightarrow Y$ is biholomorphic and $E := \rho^{-1}(Y_t)$ is a divisor with normal crossings in $\tilde{Y}$. Let $E = \bigcup_{i \in \mathbb{N}} E_i$, be the decomposition into irreducible components. We may assume that $E_i$ is nonsingular and $E_0$ is the proper transform of $Y_t$. We put

$$m_i = \text{ord}_x f(x) \quad \text{and} \quad m = \text{LCM}(m_i).$$

(2.4) Base changes. Let $S_0$ be a universal covering space of $S^*$ and set $X_0 := X \times_{S_0} S_0$ and $Y_0 := Y \times_{S_0} S_0$.

Put $\pi: S \rightarrow S^*$ be an $m$-fold covering and $\tilde{B}$ be the normalization of $\tilde{B} \times S$. We remark that $\tilde{B}$ is a $\nu$-manifold and $D := \pi^{-1}(E)$ is a divisor with $\nu$-normal crossings in $\tilde{B}$, where $\nu$ is the natural map $\tilde{B} \rightarrow \tilde{B}$ (cf. [St2]). We put $D_i := \pi^{-1}(E_i)$ and $C_i := D_i \cap D$. We have $\pi|_{\tilde{B}_0} : \tilde{B}_0 \rightarrow E_0$ and $\pi|_{\tilde{B}_0} : C_0 \rightarrow E_0$, since $m_0 = 1$.

(2.5) The exact sequence. According to Deligne-Saint-Field (cf. [St2]), we can put mixed Hodge structures on $\tilde{H}^i(Y_0)$, $\tilde{H}^i(Y_t)$ and $\tilde{H}^i(X_0)$ such that

$$\tilde{H}^i(Y_0) \rightarrow \tilde{H}^i(Y_t) \rightarrow \tilde{H}^i(X_0) \rightarrow \tilde{H}^{i+1}(Y_0) \rightarrow \cdots$$

is an exact sequence of mixed Hodge structures, where $\tilde{H}^i$ is the reduced cohomology and the coefficient field is assumed to be $\mathbb{C}$ unless explicitly specified.

(2.6) Spectral sequences. We have the following spectral sequences with respect to the weight and Hodge filtrations (cf. [St1, St2]):

$$(u_I) \quad \tilde{E}^{p,q}_r = \tilde{H}^p(\tilde{X}^{(p)}) \rightarrow \tilde{H}^p(D, \tilde{\Omega}^q_{\tilde{X}^{(p)}}(\log D)) \rightarrow \tilde{H}^p(\tilde{Y}_t).$$

$$(u_{II}) \quad \tilde{E}^{p,q}_r = \bigoplus_{k=0}^{\infty} H^{p+k-2k}(\tilde{B}^{(2k+1)})(-r-k) \rightarrow \tilde{H}^p(\tilde{Y}_t).$$

$$(w_I) \quad \tilde{E}^{p,q}_r = \tilde{H}^p(\tilde{X}^{(p)}, \tilde{\Omega}^q_{\tilde{X}^{(p)}}) \rightarrow \tilde{H}^p(\tilde{Y}_t).$$

$$(w_{II}) \quad \tilde{E}^{p,q}_r = \bigoplus_{k=0}^{\infty} H^{p+k-2k}(\tilde{X}^{(p+1)})(-r-k) \rightarrow \tilde{H}^p(\tilde{Y}_t).$$

where we define $\tilde{X}^{(p)} := \bigcup_{\nu=1}^{\infty} C_{\nu} \cap \cdots \cap C_0$ (and $\tilde{D}^{(p)}$, $\tilde{E}^{(p)}$ similarly) and $\tilde{H}^p(\tilde{Y}_t) := \{ u \in \tilde{H}^p(\tilde{Y}_t) : \gamma^{-1}u = 0, \exists k \in \mathbb{Z}_+ \}$. We remark that $(u_{II})$ is a direct factor of $(u_I)$ on which $\gamma$ acts unipotently, that $\text{Im}(\tilde{H}^p(\tilde{Y}_0) \rightarrow \tilde{H}^p(\tilde{Y}_t))$ is contained in $\tilde{H}^p(\tilde{Y}_0)$, and it induces a morphism of $(u_I)$ to $(w_{II})$.

3. Exponents.

(3.1) For $\lambda \in \mathbb{C}$ we put

$$H^p(\tilde{X}_t) := \{ u \in H^p(\tilde{X}_t) : (\gamma - \lambda)^n u = 0, \exists k \in \mathbb{Z}_+ \}.$$
The formula for \( n = 1 \) is the so-called Milnor relation because \( p_{n+1} \) equals the number of the irreducible components of \((Y, 0)\). The case \( n = 2 \) was proved by A. Durfee, I. Dolgachev, and M. Reid (cf. [Dur], [Dol]).

As an application of our argument, we obtain the following result.

Let \( Y \) be a hypersurface of degree of \( d \) in \( \mathbb{P}^{n+1} \). We assume that \( S(Y) \), the singular locus of \( Y \), is discrete. Let \( \rho: \bar{Y} \to Y \) be a resolution. Put \( h^n(\bar{Y}) := \dim_{\mathbb{C}} H^n(\bar{Y}, \mathbb{R}) \) and \( p_n(\bar{Y}) := h^n(\bar{Y}) \). Let \( p_n(V, x) \) be the geometric genus of \( x \in \Sigma(V) \). Then we have

\[
\sum_{x \in \Sigma(V)} p_n(V, x) = h^{n-1} - \binom{d}{n+1} - \binom{d-1}{n+1} d \quad \text{for } n \geq 2.
\]

Suppose \( V \) is rational. In particular, this is the case when there exists \( x \in \Sigma(V) \) such that \( \mu V(V) = d - 1 \). Then the above formula turns out to be

\[
\sum_{x \in \Sigma(V)} p_n(V, x) = \binom{d-1}{n+1} \quad \text{for } n \geq 2.
\]

We remark that the number on the right-hand side is the geometric genus of a nonsingular hypersurface of degree of \( d \) in \( \mathbb{P}^{n+1} \).

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2. Limit mixed Hodge structure. We review the theory of the limit mixed Hodge structure of Deligne-Schnellbrink ([S2]).

(2.1) Compactification. Let \( f \in \mathbb{C}^{n+1} \) be a holomorphic function with \( f(0) = 0 \). We assume that \( f \) has an isolated critical point at \( 0 \). Then we may assume that \( f \) is a polynomial and \( Y_0 := \{ x \in \mathbb{C}^{n+1} : f(x) = 0 \} \subset \mathbb{P}^{n+1} \) is nonsingular away from 0. Define

\[
Y := \left\{ (x, t) \in \mathbb{C}^{n+1} \times S : f(x, t) = 0 \right\} \subset \mathbb{P}^{n+1} \times S,
\]

and

\[
Y_t := Y \cap \{ \mathbb{P}^{n+1} \times \{ t \} \} \quad \text{for } t \in S,
\]

where \( S := \{ t \in \mathbb{C} : |t| < \eta \} \) and the number \( \eta \) is sufficiently small such that the projection \( Y \to S \) is smooth away from 0.

(2.2) Milnor's fibration. Let \( B := \{ x \in \mathbb{C}^{n+1} : |x| < \epsilon \} \) be an open ball with radius \( \epsilon \). If \( \epsilon \) and \( \eta \) are sufficiently small, the projection

\[
X := Y \cap (B \times S) \to S
\]
is smooth away from 0 and its restriction over \( S^* := S - \{ 0 \} \) is a topological fibration and the topological type of a general fiber \( X_t \) does not depend on \( t \). If \( i \neq 0, n, \) then \( H_i(X, \mathbb{C}) = 0 \) and \( \dim_{\mathbb{C}} H_i(X, \mathbb{C}) \) is called the Milnor number and is denoted by \( \mu_i \). Since \( \bigcup_{x \in X^*} H^*(X, \mathbb{C}) \) is a flat vector bundle, the monodromy transformation \( \gamma \) acts on \( H^*(X, \mathbb{C}) \), and we call it the local monodromy.

In the same way \( \gamma \) acts on \( H^*(Y, \mathbb{C}) \) and the restriction morphism \( H^*(Y, \mathbb{C}) \to H^*(X, \mathbb{C}) \) is equivariant with respect to \( \gamma \).

(2.3) Resolution. Let \( \rho: \bar{Y} \to Y \) be an embedded resolution of \( Y \), i.e., \( \rho \) is a proper holomorphic map of a manifold \( \bar{Y} \) to \( Y \) such that \( \rho \circ \rho^{-1} : \bar{Y} \to \bar{Y} \) is biholomorphic and \( E := \rho^*(\bar{Y}) \) is a divisor with normal crossings in \( \bar{Y} \). Let \( E = \bigcup_{i \geq 0} E_i \) be the decomposition into irreducible components. We may assume that \( E_i \) is nonsingular and \( E_0 \) is the proper transform of \( Y \). We put \( m_i := \partial \rho^{-1}(E_i) \geq 0 \) and \( m := \text{LCM}(m_i) \).

(2.4) Base changes. Let \( S_0 \) be a universal covering space of \( S^* \) and set \( X_0 := X \times_{S_0} S \) and \( Y_0 := Y \times_{S_0} S \).

Let \( s \in S \). We put \( \pi_s : S \to S \) as an \( m \)-fold covering and \( \bar{S} \) be the normalization of \( \pi \bar{S}/S \). We remark that \( \bar{S} \) is a \( \nu \)-manifold and \( \bar{D} := \pi^* E \) is a divisor with \( \nu \)-normal crossings in \( \bar{S} \), where \( \nu \) is the natural map \( \bar{S} \to \bar{S} \) (cf. [S2]). We put \( D := \pi^{-1}(E) \) and \( C := D_0 \cap D_n \). We have \( \pi^{-1}(E) = D_0 \sim E_0 \) and \( \pi^{-1}(E) = D_0 \cap E_n \), since \( m_n = 1 \).

(2.5) The exact sequence. According to Deligne-Schnellbrink ([S2]), we can put mixed Hodge structures on \( \bar{H}^*(Y) \), \( H^*(Y) \) and \( \bar{H}^*(X) \) such that

\[
\cdots \to \bar{H}^*(Y_{\nu}) \to H^*(Y) \to \bar{H}^*(X) \to \cdots
\]

is an exact sequence of mixed Hodge structures, where \( \bar{H}^* \) is the reduced cohomology and the coefficient field is assumed to be \( \mathbb{C} \) unless explicitly specified.

(2.6) Spectral sequences. We have the following spectral sequences with respect to the weight and Hodge filtrations (cf. [St1, St2]):

\[
\begin{align*}
&\mathbb{E}^{p,q}_{\alpha} = H^q(\bar{Y}, H^p_{\alpha}(Y_0)), \quad \text{s.t.} \mathbb{E}^{p,0}_{\alpha} \cong Gr^{\alpha}_p H^p(Y_0), \\
&\mathbb{E}^{p,q}_{\alpha} = \bigoplus_{k=0}^\infty H^{p-k}(\bar{Y}, H^k(\bar{E}^{k+1})) \quad \text{for } \mathbb{E}^{p,q}_{\alpha} = \bigoplus_{k=0}^\infty H^{p-k}(\bar{Y}, H^k(\bar{E}^{k+1})), \\
&\mathbb{E}^{p,q}_{\alpha} \cong Gr^{\alpha}_p H^p(Y_0), \quad \text{s.t.} \mathbb{E}^{p,0}_{\alpha} \cong Gr^{\alpha}_p H^p(Y_0), \\
&\mathbb{E}^{p,q}_{\alpha} \cong Gr^{\alpha}_p H^p(Y_0), \quad \text{s.t.} \mathbb{E}^{p,0}_{\alpha} \cong Gr^{\alpha}_p H^p(Y_0), \\
&\mathbb{E}^{p,q}_{\alpha} \cong Gr^{\alpha}_p H^p(Y_0), \quad \text{s.t.} \mathbb{E}^{p,0}_{\alpha} \cong Gr^{\alpha}_p H^p(Y_0),
\end{align*}
\]

where we define \( \bar{E}^{k+1} := \bigcup_{i=0}^{k+1} C_i \) (and \( \bar{E}^{k+1} = \bigcup_{i=0}^{k+1} C_i \)) and \( \mathbb{E}^{p,q} \) (\( \mathbb{E}^{p,q} \)) similarly and \( H^*(\bar{Y}) := \{ u \in H^*(\bar{Y}) : (\gamma - 1)^u = 0, \exists k \in \mathbb{Z} \} \).

We remark that \( \mathbb{E}^{p,q}_{0} \) is a direct factor of \( \mathbb{E}^{p,q}_{\nu} \) on which \( \gamma \) acts unipotently, that \( \text{Im}(\bar{H}^*(Y) \to H^*(X)) \) is contained in \( H^*(X) \), and it induces a morphism of \( \mathbb{E}^{p,q}_{0} \) to \( \mathbb{E}^{p,q}_{\nu} \).

3. Exponents.

(3.1) For \( \lambda \in \mathbb{C} \) we put

\[
\mathbb{H}^p_{\lambda} := \{ u \in H^p(Y) : (\gamma - \lambda)^u = 0, \exists k \in \mathbb{Z} \}.
\]

\( \mathbb{H}^p \) := \( \sum_{\lambda} \mathbb{H}^p_{\lambda} \).
Since the direct decomposition \( H^*(X_0) = \bigoplus \lambda H^*(X_0) \), it is compatible with both filtrations, we have \( \sum_{\lambda, \rho, p} h^{p, q} = \mu = \dim_c H^*(X_0) \). Due to the monodromy theorem, if \( h^{p, q} \neq 0 \), \( \lambda \) is a root of unity.

(3.2) **Definition.** We define \( \mu \) rational numbers \( \{a_1, \ldots, a_\mu\} \) as follows, and we call them the *exponents* of \( f \).

(a) \( a_1 \leq a_2 \leq \cdots \leq a_\mu \),
(b) \( \forall \lambda \in \mathbb{C}, \forall p \in \mathbb{Z}, \lambda = 1 \Rightarrow \{ j : \exp 2\pi i a_j = \lambda^{-1} \} = \sum_{q} h^{p, q} \).

\[ \lambda = 1 \Rightarrow \{ j : a_j = n - p + 1 \} = \sum_{q} h^{p, q} \]

where \( \{a_j\} \) is the Gauss symbol, i.e., \( a_j = \max\{k \in \mathbb{Z} : k < a_j\} \). This is well defined because of \( \sum_{\lambda, \rho, p} h^{p, q} = \mu \).

(3.3) **Proposition (duality).** \( a_1 + a_{\mu+1-n} = n + 1 \).

**Proof.** Let \( N := \log \gamma_0 \), where \( \gamma_0 \) is the unipotent part of the local monodromy \( \gamma \). Due to Steenbrink [SI2], we have

\[ N^{p+e-i} : H^{p, q} \to H^{p, q+e-i} \] for \( \lambda = 1 \),

and

\[ N^{p+e-i-1} : H^{p, q} \to H^{p+1, q+e-i-1} \].

Since \( \gamma \) is defined over \( \mathbb{Z} \), we have \( H^{p, q}_\mathbb{Z} = H^{p, q}_\mathbb{C} \), which gives the desired result. Q.E.D.

4. **Proof of Theorem 1.** By the definition of the exponents, it suffices to prove the following.

(4.1) **Proposition.** \( p_\mu = \dim_c \operatorname{Gr}^{\mu} H^*(X_0) \), for \( n \geq 1 \).

First we reduce (4.1) to the following proposition.

(4.2) **Proposition.**

\[ \dim_c \lim_{\mu} \left( H^{p, q-1}(D_0, \mathfrak{c}, D_0) - \bigoplus_{i} H^{p, q-1}(C_i, \mathfrak{c}, C_i) \right) \]

\[ = \dim_c \operatorname{Coker}(\operatorname{Gr}^{\mu} H^*(Y_0) - \operatorname{Gr}^{\mu} H^*(X_0)) + \delta_{1, \mu} \] for \( n \geq 1 \).

Where \( \delta_{1, \mu} \) is the Kronecker \( \delta \).

We further reduce (4.2) to the following and prove (4.3).

(4.3) **Proposition.** Let \( \psi_{23} : \left( \bigoplus_{i} H^{p, q-1}(C_i, \mathfrak{c}, C_i) \to \operatorname{Gr}^{\mu} H^*(X_0) - \mathfrak{c}^\perp \right) \) be the Gysin morphism (cf. Lemma (4.5.4)). Then we have

\[ \dim_c \operatorname{Coker} \psi_{23} = \delta_{1, \mu} \] for \( n \geq 1 \).

(4.4) **Proposition (4.2) → Proposition (4.1).**

First we describe the geometric genus \( p_\mu \) as the difference of the Euler characteristics, i.e.,

(4.4.1)

\[ x(D_0, \mathfrak{c}, D_0) = x(D, \mathfrak{c}, D) \pm (-1)^{n-1} p_\mu. \]

**Proof.** By the invariance of Euler characteristics under a flat deformation, we have

\[ x(Y_0, \mathfrak{c}, Y_0) = x(Y, \mathfrak{c}, Y) = x(D, \mathfrak{c}, D). \]

Therefore it remains to show that

\[ x(D_0, \mathfrak{c}, D_0) = x(Y_0, \mathfrak{c}, Y_0) + (-1)^{n-1} p_\mu. \]

When \( n \geq 2 \), \( p_\mu \mathfrak{c}_0 = \mathfrak{c}_0 \) holds since \( Y_0 \) is normal. Using the Leray spectral sequence we obtain:

\[ x(D_0, \mathfrak{c}, D_0) = x(Y_0, \mathfrak{c}, Y_0) + \sum_{i=0}^{n-1} (-1)^i \dim_c H^i(D_0, \mathfrak{c}, D_0) \]

and

\[ R^i p_\mu \mathfrak{c}_0 = 0 \text{ for } 0 < i < n - 1 \text{ since } Y_0 \text{ is Cohen-Macaulay.} \]

The case \( n = 1 \) follows from

\[ x(D_0, \mathfrak{c}, D_0) = x(Y_0, p_\mu \mathfrak{c}_0) = x(Y_0, \mathfrak{c}, Y_0) + x(Y_0, p_\mu \mathfrak{c}_0 \mathfrak{c}, Y_0) \]

\[ = x(Y_0, \mathfrak{c}, Y_0) + \delta. \quad \text{Q.E.D.} \]

From the exact sequence (2.5.1) and the vanishing of \( H^*(X_0) \) for \( i \neq 0, n \), we obtain

(4.4.2) \( \dim_c \operatorname{Gr}^{\mu} H^i(Y_0) = \dim_c \operatorname{Gr}^{\mu} H^i(Y_0) \) for \( 0 < p < n \).

We calculate both sides of (4.4.2) using the spectral sequences in (2.6). First we have the following for \( \forall p \in \mathbb{Z} \) by (4.1).

(4.4.3) \( \dim_c \operatorname{Gr}^{\mu} H^i(Y_0) = \dim_c \operatorname{Gr}^{\mu} H^i(Y_0) = \dim_c H^i(D, \mathfrak{c}, D) \)

where \( \delta_0 \) is the zero map, and

\[ \dim_c \operatorname{Ker}(\operatorname{Gr}^{\mu} H^i(C_0) - \operatorname{Gr}^{\mu} H^i(C_0)) \]

\[ = \dim_c \operatorname{Ker}(H^i(D_0, \mathfrak{c}, D_0) - \bigoplus H^i(C_i, \mathfrak{c}, C_i)). \]

Secondly, we have by (\( \mathcal{H} \))

(4.4.4) \( \dim_c \operatorname{Gr}^{\mu} H^i(Y_0) = \dim_c \operatorname{Gr}^{\mu} H^i(Y_0) = \dim_c H^i(D, \mathfrak{c}, D) \) for \( \forall p \neq 0, n \).

As \( R^p p_\mu \mathfrak{c}_0 = 0 \) for \( 0 < p < n - 1 \), we obtain

\[ p^* : H^i(Y_0, \mathfrak{c}, Y_0) \to H^i(D_0, \mathfrak{c}, D_0) \] for \( 0 < p < n - 2 \).

Hence the restriction morphism \( H^i(D_0, \mathfrak{c}, D_0) \to H^i(C_i, \mathfrak{c}, C_i) \) is the zero map, and (4.4.3) turns out to be the following for \( p = 0, n - 1 \).

(4.4.5) \( \dim_c \operatorname{Gr}^{\mu} H^i(Y_0) = \dim_c H^i(D, \mathfrak{c}, D) \) for \( p = 0, n - 1 \).

Substituting (4.4.3)–(4.4.5) into (4.4.2) we have

(4.4.6) \( \dim_c H^i(D, \mathfrak{c}, D) = \dim_c H^i(D, \mathfrak{c}, D) \) for \( p = 0, n - 1 \).
Since the direct decomposition $H^*(X_0) = \bigoplus \lambda H^*(X_0)_\lambda$ is compatible with both filtrations, we have $\Sigma_{\lambda, r, s} h^s_\lambda = \mu = \dim_c H^*(X_0)_\mu$. Due to the monodromy theorem, if $h^s_\lambda \neq 0$, $\lambda$ is a root of unity.

3.2 DEFINITION. We define $\mu$ rational numbers $\{a_1, \ldots, a_n\}$ as follows, and we call them the exponents of $f$.

(a) $a_1 < a_2 < \cdots < a_n$.

(b) $\forall \lambda \in C$, $\forall \rho \in Z$, $\lambda = 1 \mapsto \{j: \exp 2\pi i a_j = \lambda^{-1}, [a_j] = n - p\} = \sum q h^s_q$,

$\lambda = 1 \mapsto \{j: a_j = n - p + 1\} = \sum q h^{s_0}_q$,

where $[a_j]$ is the Gauss symbol, i.e., $[a_j] = \max\{k \in Z: k < a_j\}$. This is well defined because of $\Sigma_{\lambda, r, s} h^s_\lambda = \mu$.

3.3 PROPOSITION (dual). $a_1 + a_{n+1-i} = n + 1$.

PROOF. Let $N := \log \gamma_0$, where $\gamma_0$ is the unipotent part of the local monodromy $\gamma$. Due to Steenbrink [S2], we have $N^{p+q-s}: H^p_\gamma \to H^q_\gamma$ for $\lambda \neq 1$,

and

$N^{p+q-s-1}: H^p_\gamma \to H^q_\gamma^{p+q-s-1} \cdot p$.

Since $\gamma$ is defined over $Z$, we have $H^p_\gamma = H^p_\gamma$, which gives the desired result.

Q.E.D.

4. Proof of Theorem 1. By the definition of the exponents, it suffices to prove the following.

4.1 PROPOSITION. $p_0 \equiv \dim_c \text{Gr}_p^f H^*(X_0)$, for $n > 1$.

First we reduce (4.1) to the following proposition.

4.2 PROPOSITION.

$\dim_c \text{Im}(H^{p-1}(D_0, E_{D_0}) - \bigoplus_t H^{p-1}(C_t, E_{C_t}))$

$= \dim_c \text{Coker} \left( \text{Gr}_p^f H^*(X_0) - \text{Gr}_p^f H^*(X_0) \right) + \delta_{1, \lambda}$ for $n > 1$.

where $\delta_{1, \lambda}$ is the Kronecker $\delta$.

We further reduce (4.2) to the following and prove (4.3).

4.3 PROPOSITION. Let $\psi_2: \text{Gr}_p^f H^{p-1}(E^{(1)} - E^{(2)}) - \text{Gr}_p^f H^{p-1}(E^{(1)} - E^{(2)})$ be the Gysin morphism (cf. Lemma (4.5.4)). Then we have $\dim_c \text{Coker} \psi_2 = \delta_{1, \lambda}$ for $n > 1$.

(4.4) Proposition (4.2) $\to$ Proposition (4.1).

First we describe the geometric genus $p_0$ as the difference of the Euler characteristics, i.e.,

(4.4.1) $X(D_0, E_{D_0}) = X(D, E_0) - (-1)^{n-1} p_0$

PROOF. By the invariance of Euler characteristics under a flat deformation, we have $X(D, E_{D_0}) = X(D, E_0)$. Therefore it remains to show that $X(D_0, E_{D_0}) = X(D_0, E_{D_0}) - (-1)^{n-1} p_0$.

When $n > 2$, $p_\lambda E_{D_0} = E_0$ holds since $Y_0$ is normal. Using the Leray spectral sequence we obtain: $X(D_0, E_{D_0}) = X(D_0, E_0) + \Sigma_{a_1}(-1)^{i} \dim_c(R^p\mu_* E_{D_0})_0$, and $R^p\mu_* E_{D_0} = 0$ for $0 < i < n - 1$ since $Y_0$ is Cohen-Macaulay.

The case $n = 1$ follows from

$X(D_0, E_{D_0}) = X(D_0, E_{D_0}) + X(D_0, E_{D_0}) + X(D_0, E_{D_0})$

$= X(D_0, E_{D_0}) + \delta$. Q.E.D.

From the exact sequence (2.5.1) and the vanishing of $H^*(X_0)$ for $\lambda \neq 1$, we obtain

(4.4.2) $\dim_c \text{Gr}_p^f H^p(Y_0) = \dim_c \text{Gr}_p^f H^p(Y_0)$, for $0 < p < n$.

We calculate both sides of (4.4.2) using the spectral sequences in (2.6). First we have the following for $\forall \rho \in Z$ by ($\rho = 1$).

(4.4.3) $\dim_c \text{Gr}_p^f H^p(Y_0) = \dim_c \text{Gr}_p^f H^p(Y_0)$

$= \dim_c \text{Ker} \left( \text{Gr}_p^f H^p(C^{(1)}) - \text{Gr}_p^f H^p(C^{(2)}) \right)$

Secondly, we have by ($\rho = 1$)

(4.4.4) $\dim_c \text{Gr}_p^f H^p(Y_0) = \dim_c \text{Gr}_p^f H^p(Y_0) = \dim_c H^p(D, E_0)$, for $0 < p < n - 1$.

As $R^p\mu_* E_{D_0} = 0$ for $0 < p < n - 1$, we obtain

$p^*: H^p(Y_0, E_{Y_0}) = H^p(D_0, E_{D_0})$ for $0 < n - 2$.

Hence the restriction morphism $H^p(D_0, E_{D_0}) \to H^p(C_t, E_{C_t})$ is the zero map, and (4.4.3) turns out to be the following for $p \neq 0$, $n - 1$.

(4.4.5) $\dim_c \text{Gr}_p^f H^p(Y_0) = \dim_c H^p(D_0, E_{D_0})$ for $p \neq 0$, $n - 1$.

Substituting (4.4.3)–(4.4.5) into (4.4.2) we have

(4.4.6) $\dim_c H^p(D, E_0) = \dim_c H^p(D_0, E_{D_0})$ for $p < n - 1$.
(4.4.7)\[\dim_c H^{n+1}(D, \partial D) = \dim_c \ker \left( H^{n+1}(D, \partial D) \to \bigoplus_i H^{n+1}(C_i, \partial C_i) \right) + \delta_{1,n}.\]

Substituting these into (4.4.1), we obtain
\[p_k = \begin{aligned}
&= \dim_c H^{n+1}(D, \partial D) - \dim_c H^{n+1}(D, \partial D) \\
&+ \dim_c \ker \left( H^{n+1}(D, \partial D) \to \bigoplus_i H^{n+1}(C_i, \partial C_i) \right) - \delta_{1,n} \\
&= \dim_c \ker \left( H^{n+1}(D, \partial D) \to \bigoplus_i H^{n+1}(C_i, \partial C_i) \right) - \delta_{1,n} \\
&= \dim_c \ker \left( H^{n+1}(D, \partial D) \to \bigoplus_i H^{n+1}(C_i, \partial C_i) \right).
\end{aligned}\]

The last equality follows from Proposition (4.2) and (2.5.1). Q.E.D.

(4.5) Proposition (4.3) ⇒ Proposition (4.2).

From the exact sequence (2.5.1), we have
\[
\begin{align*}
\dim_c \ker (H^n(Y^o) - H^n(X^o)) &= \dim_c \ker (H^n(Y^o) - H^n(X^o)).
\end{align*}
\]

Here we used the remark in (2.6).

Using the spectral sequences (4.1) and (4.2) in (2.6), we obtain
\[
\begin{align*}
Gr^p H^{n+1}(Y^o) &= Gr^p H^{n+1}(D), \\
Gr^p H^{n+1}(Y^o) &= Gr^p H^{n+1}(Y^o), \\
Gr^p H^{n+1}(Y^o) &= Gr^p H^{n+1}(Y^o).
\end{align*}
\]

We can verify that the morphism in the last expression is the Gysin morphism. We set
\[
\begin{align*}
A_1 &= Gr^{p-1} H^{n+1}(\hat{C}(1)), \\
B_1 &= Gr^{p+1} H^{n+1}(\hat{C}(0)), \\
A_2 &= Gr^{p-1} H^{n+1}(\hat{E}(3) - \hat{C}(0)), \\
B_2 &= Gr^{p+1} H^{n+1}(\hat{E}(1) - \hat{C}(0)),
\end{align*}
\]

and apply the following lemma to obtain (4.2).

(4.5.4) Lemma. Let \(\psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} \) be a linear map of finite dimensional linear spaces, satisfying
(a) \(\psi(A_2) \subset B_2, i.e., \psi_{12} = 0\).
(b) \((\psi_{21}, \psi_{22}): A_1 \otimes A_2 \to B_1 \otimes B_2\) is surjective.
(c) \(\ker(\psi_{11}; A_1 \to B_1) \subset \ker(\psi_{21}; A_1 \to \text{Coker } \psi_{22}).\)

Then we have
\[
\dim \ker (B_1 \to \text{Coker } \psi) = \dim \ker (B_1 \to \text{Coker } \psi) - \dim \ker (B_1 \to \text{Coker } \psi).
\]
(4.4.7) \[ \dim_c H^{n-1}(D, \partial D) = \dim_c \ker \left( H^{n-1}(D, \partial D) \rightarrow \bigoplus_i H^{n-1}(C_i, \partial C_i) \right) + \delta_{1,n}. \]

Substituting these into (4.4.1), we obtain
\[ p_2 = \left( \dim_c H^{n-1}(D, \partial D) - \dim_c H^{n-1}(D, \partial D) \right) + \left( \dim_c H^1(D, \partial D) - \dim_c H^1(D, \partial D) \right) \]
\[ = \dim_c \ker \left( H^{n-1}(D, \partial D) \rightarrow H^1(C_i, \partial C_i) \right) = \delta_{1,n} \]
\[ + \dim_c \ker \left( H^1(D, \partial D) \rightarrow H^1(C_i, \partial C_i) \right) \]
\[ = \dim_c \ker \left( H^1(D, \partial D) \rightarrow H^1(D, \partial D) \right) - \delta_{1,n} \]
\[ = \dim_c H^1(D, \partial D) - \delta_{1,n}. \]

The last equality follows from Proposition (4.2) and (2.5.1). Q.E.D.

(4.5) Proposition (4.3) Proposition (4.2).

From the exact sequence (2.5.1), we have
\[ (4.5.1) \quad \dim_c \ker \left( H^*(Y_0) - H^*(X_0) \right) = \dim_c \ker \left( H^*(Y_0) - H^*(X_0) \right). \]

Here we used the remark in (2.6).

Using the spectral sequences $p_1$ and $p_2$ in (2.6), we obtain
\[ (4.5.2) \quad \ker H^{n-1}(Y_0) = \ker H^{n-1}(D_0). \]
\[ (4.5.3) \quad \ker H^{n-1}(X_0) = \ker H^{n-1}(D_0). \]

We can verify that the morphism in the last expression is the Gysin morphism. We set
\[ A_1 := \ker H^{n-1}(\tilde{C}(0)), \quad B_1 := \ker H^{n-1}(\tilde{C}(0)), \]
\[ A_2 := \ker H^{n-1}(\tilde{C}(0)), \quad B_2 := \ker H^{n-1}(\tilde{C}(0)). \]

and apply the following lemma to obtain (4.2).

(4.5.4) Lemma. Let
\[ \psi = \left( \begin{array}{c} \psi_{11} \\ \psi_{21} \\ \psi_{22} \end{array} \right), \quad A_1 \oplus A_2 - B_1 \oplus B_2 \]
be a linear map of finite dimensional linear spaces, satisfying
\begin{itemize}
  \item[(a)] $\psi(A_2) \subseteq B_2 \Leftrightarrow \psi_{12} = 0$,
  \item[(b)] $(\psi_{21}, \psi_{22}) \colon A_1 \otimes A_2 \rightarrow B_1 \otimes B_2$ is surjective,
  \item[(c)] $\ker(\psi_{11} : A_1 \rightarrow B_1) \subseteq \ker(\psi_{21} : A_1 \rightarrow \operatorname{Coker} \psi_{22}).$
\end{itemize}
Then we have
\[ \dim \ker(\psi_1 \rightarrow \operatorname{Coker} \psi) = \dim \ker(\psi_{11}) - \dim \ker(\psi_{22}). \]
A NOTE ON TWO LOCAL HODGE FILTRATIONS

JOHN SCHERK

In [2], J. H. M. Steenbrink constructed a mixed Hodge structure on the cohomology of the generic fibre of the Milnor fibration of an isolated hypersurface singularity. Recently, A. N. Varchenko introduced the "asymptotic Hodge filtration" which, together with Steenbrink's weight filtration, also gives a mixed Hodge structure on $H^*(X_\nu)$. In [3] Varchenko proved that the two agree for curve singularities and quasihomogeneous singularities. In the general case, he showed that the two Hodge filtrations agree on $Gr^*\cdot H^*(X_\nu, C)$ (cf. [4]). The purpose of this note is to give a simple example of a function for which the two filtrations are not the same.

Suppose $f = f(x_1, \ldots, x_n)$ is an analytic function defined in some open ball $B$ about 0 in $\mathbb{C}^{n+1}$. Assume that $f(0) = 0$, and that 0 is the only critical point of $f$. As is well known, one can find an open neighbourhood $X$ of 0 in $\mathbb{C}^{n+1}$, and a disc $T$ about 0 in $\mathbb{C}$ such that $f(X) \subset T$ and

$$f : X - f^{-1}(0) \rightarrow T - \{0\}$$

is a $\mathbb{C}^n$ fibre bundle. Such neighbourhoods $X$ form a neighbourhood basis of 0. Let $T' = T - \{0\}$, $X' = X - f^{-1}(0)$.

Pick a path in $T'$ which travels once about 0 in a counter-clockwise direction as a generator of $\pi_1(T', 0)$. Then this determines an automorphism $\sigma$ of $H^k(X', C)$, the monodromy of $f$. For different values of $t$, the corresponding automorphisms will be conjugate. Write $\sigma = e^{\omega_t}$, where $\omega_t$ is semisimple and $\omega_0$ is unipotent.

$H^* = \bigcup_{k \geq 0} H^k(X', C)$ is a flat vector bundle with a canonical connection $\partial_t$. Let $\mathcal{E}_T(H^*)$ denote the sheaf of germs of holomorphic sections of $H^*$. $\mathcal{E}_T(H^*)$