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Moduli of analytic branches

Introduction

The purpose of this paper is to compute the underlying set of the moduli space for irreducible analytic branches embedded in affine-space over an algebraically closed field $k$ of arbitrary characteristic.

Let:

$$
C \hookrightarrow X = \text{Spec}(k[[T_0, \ldots, T_m]])
$$

be an analytic branch and $\{r_i\}_{i=0}^{\infty}$ the sequence of multiplicities of $C$ and its successive quadratic transforms. We will denote by $r$ the following:

$$
r = h + 2 \sum_{i=0}^{\infty} r_i (r_i - 1)^2
$$

where $h$ is the number of quadratic transformations needed to desingularise $C$.

Let $C_r$ be the $r$-th quadratic transform of $C$ let $X_r^F \rightarrow X$ be the $r$-th quadratic transform of $X$ in the direction of $C$ and

$$
\pi_r : X_r^F \rightarrow X
$$

the canonical map. If $X_0^C$ is the reduced exceptional fibre of $\pi_r$ we will denote by $I$ the sheaf of ideals defining $X_0$ and let $X_n^C \hookrightarrow X_r^F$ be the closed subscheme which is defined by $I^{n+1}$.

Let $L = I/I^2$ be the conormal sheaf to $X_0$ in $X_r^F$.

Definition: Two embedded branches $C \hookrightarrow X'$, $C' \hookrightarrow X$ are equisingular when $X_1^C$ and $X_1^{C'}$ are isomorphic schemes. That is, one defines the equisingularity of $C \hookrightarrow X$ to be the scheme $X_1^C$.

Let $X_1$ be an equisingularity. Also let $M(X_1)$ be the set of analytic equivalence classes of branches with equisingularity $X_1$.

Let $\mathcal{O}_{X_1}^K$ denote the sheaf of $\mathcal{O}_{X_0}$-modules.
Similarly let \( G_L \) denote the sheaf
\[
G_L = L^2 \otimes L^3 \otimes L^4 \otimes \cdots \otimes L^K.
\]

Let \( H^1 \) denote the \( k \)-vector space
\[
H^1 = H^1(X_0, \text{Der}_k(\mathcal{O}_{X_1}, \mathcal{O}_{X_0}) \otimes \mathcal{O}_{X_0}^* G_L^K).
\]

**Main theorem:** There exists a natural integer \( K \) and a certain quotient set \( M \) of \( H^1 \)
\[
\pi : H^1 \longrightarrow M
\]
such that \( M(X_1) \) is the subset of \( M \) defined by the vanishing of a 2-cycle obstruction class belonging to
\[
H^2(X_0, \text{Der}_k(\mathcal{O}_{X_1}, \mathcal{O}_{X_0}) \otimes \mathcal{O}_{X_0}^* G_L^K)
\]
and associated to each point of \( M \).

(More precision and details will be given below).

As a particular case; when \( k = \mathbb{C} \), the complex field, and \( \dim X = 2 \) one gets a result of O. Zariski [5] stating that the moduli space for plane analytic branches over \( \mathbb{C} \) is a quotient of a vector space.

In the present paper a sort of description of the fibres of \( \pi \) is given. I hope to come back to this problem in a future paper.

**0. Notations**

The ground field \( k \) will be algebraically closed and of arbitrary characteristic. We will denote the ring of formal power series in \( m+1 \) variables, with coefficients in \( k \), by
\[
A = k[[T_0, \ldots, T_m]].
\]

We will also denote by \( X \) the spectrum of \( A \) and by \( \hat{X} \) the formal spectrum of \( A \) (in the sense of Grothendieck [2]).

Given a natural integer \( n \) and an analytic branch (always irreducible) \( 1 : C \hookrightarrow X \), one defines the "\( n \)-th blowing up" of \( \hat{X} \) in the direction of \( C \) as the sequence
\[
\begin{array}{c}
{\hat{X}}(C) \\
\end{array}
\]

where \( {\hat{X}}(C) \) is the formal completion of the strict transform of \( C \) in \( \hat{X} \), the formalisation of the blow-up of \( X \) about \( C \).

is is algebraisable, that is, it is the formalisation of the blow-up of \( X \) about \( C \).

is exactly:

1. **Equivalence theorem**

One starts with \( 1 : h \) denotes the minimum

then if

is the \( r \)-th blowing up of \( X \) over \( C \),

\[ [r_n]_{n=0} \] then \( \hat{X}(C) \) is precisely:

**Theorem 1.1:** If \( [r_n]_{n=0} \) then \( \hat{X}(C) \) and \( \hat{X}(C) \) are isomorphic schemes; i.e.
sequence
\[ \hat{x}(C_n) \rightarrow \hat{x}(C_{n-1}) \rightarrow \ldots \rightarrow \hat{x}(C_0) = \text{Sp } f(A) \]

where \( \hat{x}(C_1) \) is the formal blowing up of \( \hat{x}(C_{i-1}) \) along the closed point of \( C_{i-1} \) (i.e.: the formalisation of the local blowing up of \( \hat{x}(C_{i-1}) \)) and \( C_1 \) is the strict transform of \( C_{i-1} \) starting with \( C_0 = C \).

The morphism:
\[ \pi_n: \hat{x}(C_n) \rightarrow \text{Sp } f(A) \]

is algebraisable, that is: there exists an ideal \( I \) of \( A \) such that the formalisation of the blowing up \( \hat{x}(C_n) \) of \( X \) along \( I \):
\[ \hat{x}(C_n) \rightarrow \text{Spec}(A) \]

is exactly:
\[ \pi_n: \hat{x}(C_n) \rightarrow \text{Sp } f(A) \].

1. Equivalence theorem. Upper bound for the conductor

One starts with the following: for every multiplicity sequence \( \{ r_n \}_{n=0}^{\infty} \), if \( h \) denotes the minimum integer such that \( r_h = 1 \), one defines \( r \) to be the integer:
\[ r = h + 2 \sum_{i=0}^{\infty} r_i (r_i - 1)^2 \]

then if
\[ \hat{x}(C_r) \rightarrow \hat{x} \]

is the \( r \)-th blowing up of \( \hat{x} \) directed by the branch \( C \) with multiplicity sequence \( \{ r_n \}_{n=0}^{\infty} \) then \( \hat{x}(C_r) \) determines the branch \( C \) up to analytic equivalence. More precisely:

**Theorem 1.1:** If \( C, C' \) are two branches with the same multiplicity sequence \( \{ r_n \}_{n=0}^{\infty} \) then \( C, C' \) are analytically equivalent if and only if the schemes \( \hat{x}(C_r) \) and \( \hat{x}(C'_r) \) are isomorphic.

To prove this theorem one needs some lemmas.

**Lemma 1.2:** \( C, C' \) are analytically equivalent if and only if they are isomorphic schemes; i.e.,
\[ C = \text{Spec}(\mathcal{O}_C) = \text{Spec}(\mathcal{O}_{C'}) = C' \].

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if one denotes by

$$C = \text{Spec}(k[[T]])$$

the desingularization of $C$ and by $c$ the conductor, then $\mathcal{C} = T^c k[[T]]$, where

$$c = \ell(O_C/C)$$

is the length of the conductor and one has

$$T^c k[[T]] \cong O_C.$$

Lemma 1.3: If $\ell \geq c$, then $O_C$ and $O_C$ are isomorphic if and only if the respective subalgebras of $k[[T]]/(t^c)$ they induce are isomorphic, i.e.: if and only if there exists an automorphism of $(T^c) k[[T]]$ that maps one subalgebra onto the other.

We will now prove that the length $c$ of the conductor of a branch $C$ is bounded by the multiplicity sequence $(r_i)_{i=0}^{\infty}$. More precisely, there exists a positive integer $K (= 2 \sum_{i=0}^{\infty} r_i (r_i - 1)/2)$, depending only on the given multiplicity sequence, such that $c \leq K$.

Let $O_C = k[[T]]$ be the ring of the desingularization of $C$ and, let $v_C$ be the valuation of the field of fractions of $O_C$ induced by $O_C$. If $m_C$ is the maximal ideal of $O_C$ and $t$ is an element of $m_C$ with minimum value for $v_C$ then

$$v_C(t) = \text{multiplicity of } O_C = r_0.$$

Moreover, one has

$$\dim_k (m_C/m^2_C) \leq r_0 = \dim_k (m_C/tm_C).$$

We will denote by $d_C$ the embedding dimension of $C$:

$$d_C = \dim_k (m_C/m^2_C);$$

it is clear that $d_C$ is a formal analogue of elements which generate the $k$-algebra $O_C$.

Lemma 1.4: For a plane branch $C$, i.e.: $d_C = 2$, then

$$m_C^n = tm_C^{n-1}$$

for all $n > r_0 - 1$.

As a corollary on $m^n_C$ algebra contains all the subalgebras and with multiplicity $m^n_C$.

Lemma 1.5: $m^n_C = t^n C_1$

If $O_C$ is the $C_1$

and also $d_C \leq r_0$. We shall prove:

Corollary 1.6: $m^n_C = t^n C_1$

Proof: The second

(recall that $q = (r_0 > \ldots )$

because $r_0 = \ell(O_C/t)$

Corollary 1.7: $d_C = 2$

If one uses the

one concludes that

Corollary 1.8: The multiplicity sequence
As a corollary one gets the following general result in the case \( d_C \geq 1 \). As the algebra

\[
\mathcal{O}_C = k[[t, t_1, \ldots, t_{d_C - 1}]]
\]

contains all the subalgebras \( k[[t, t_i]] \) for \( 1 \leq i \leq d_C - 1 \), which are plane branches and with multiplicities \( \leq r_0 = v_C (t) \), one can apply lemma 1.4 to get

Lemma 1.5: \( m^n_C = t m_{C}^{n-1} \) for all \( n > (r_0 - 1)(d_C - 1) = q \) and so one has

\[
m^n_C = t^{n-q} m^q_C .
\]

If \( \mathcal{Q}_{C_1} \) is the first quadratic transform of \( \mathcal{O}_C \), then

\[
\mathcal{Q}_{C_1} = \bigcup_{i=0}^{\infty} t^i \mathcal{O}_C = \mathcal{O}_C (t, t^2, \ldots, t^q, \ldots) \quad \text{(field of fractions of } \mathcal{O}_C),
\]

and also \( d_{C} \leq r_0 \). The lemma applies and gives

Corollary 1.6: \( m^{n-1}_{C} \mathcal{Q}_{C} = m^{n-1}_{C} \) for all \( n > (r_0 - 1)^2 \) and so

\[
\ell (\mathcal{O}_{C_1} / \mathcal{O}_C) \leq r_0 (r_0 - 1)^2 .
\]

**Proof:** The second part results from the fact that

\[
m^q_{C_1} = m^q_C
\]

(recall that \( q = (r_0 - 1)(d_C - 1) > (r_0 - 1)^2 \)). So one has

\[
\ell (\mathcal{O}_{C_1} / \mathcal{O}_C) \leq \ell (\mathcal{O}_{C_1} / m^q_{C_1} \mathcal{O}_C) = \ell (\mathcal{O}_{C_1} / t^q \mathcal{O}_{C_1}) = q \cdot r_0
\]

because \( r_0 = \ell (\mathcal{O}_{C_1} / t \mathcal{O}_{C_1}) \).

Corollary 1.7: \( \ell (\mathcal{O}_{C_1} / \mathcal{O}_C) \leq \sum_{i=0}^{\infty} r_1 (r_1 - 1)^2 .
\]

If one uses the inequality

\[
\ell (\mathcal{O}_{C_1} / \mathcal{O}_C) \leq \ell (\mathcal{O}_{C_1} / \mathcal{O}_C)
\]

one concludes that

\[
c = \ell (\mathcal{O}_{C_1} / \mathcal{O}_C) \leq \ell (\mathcal{O}_{C_1} / \mathcal{O}_C) + \ell (\mathcal{O}_{C_1} / \mathcal{O}_C) \leq 2 \sum_{i=0}^{\infty} r_1 (r_1 - 1)^2 .
\]

Corollary 1.8: The length \( c \) of the conductor of a branch \( C \) with multiplicity sequence \( \{ r_n \}_{n=0}^\infty \) is bounded as follows:
Let \( h \) be the number of quadratic transformations necessary to desingularise \( C \), that is \( h \) is the least integer such that \( r_1 = 1 \).

Let

\[ \pi_h : \hat{X}(C_h) \to Spec f(A) \]

be the desingularisation map of \( C \). Suppose \( C' \to X \) is another branch and \( \hat{C'} \to \hat{X}(C_h) \) its strict transform for \( \pi_h \) (\( C' \) is supposed to have the same multiplicity sequence as \( C \)). Then one has

**Lemma 1.9:** If \( C_h \) and \( \hat{C}' \) have a contact of order \( m \geq 1 \), then \( O_C \) and \( O_{\hat{C}'} \) induce isomorphic subalgebras in \( k[[T]]/(T^m) \).

**Proof:** As \( C, C' \) have the same multiplicity sequence, if \( C_h \) and \( \hat{C}' \) intersect, then \( \hat{C}' = C_h' \) is simple and \( C, C' \) are direct identical \( h \)-blowing ups of \( \hat{X} \). Let \( T \) be a function on \( \hat{X}(C_h) \) that is a parameter for \( C_h \) and \( C_h' \). If \( A = k[[T_0, \ldots, T_m]] \), then by the contact condition \( T_0, \ldots, T_m \) will have the same expansion up to order \( m \) along both branches \( C, C' \).

This means that the following diagram of natural maps is commutative:

\[
\begin{array}{ccc}
O_C & \to & O_{\hat{C}'} \\
\downarrow & & \downarrow \\
A & \to & k[[T]]/(T^m) \\
\end{array}
\]

**Lemma 1.10:** For the formal schemes \( \hat{X}(C_{h,m}) \) and \( \hat{X}(C_{h,m}') \) to be isomorphic it is necessary and sufficient that there exists an automorphism \( \tau \) of \( X \) such that the desingularisation \( \tau(C)_h \) of \( \tau(C) \) has a contact of order \( m \) with \( C'_h \).

**Proof:** Given an isomorphism

\[ \Phi : \hat{X}(C_{h,m}) \to X(C_{h,m}) \]

by taking global sections one gets an automorphism \( \tau \) of \( X \). Conversely, given a \( \tau \) with the properties of the lemma, \( \tau \) induces an isomorphism

\[ \Phi : \hat{X}(C_{h,m}) \to \hat{X}(C_{h,m}) = \hat{X}(C_{h,m}'). \]

**Proof (of theorem 1.1):** Taking \( m = 2 \sum_{i=0}^{\infty} r_i (r_i - 1)^2 \) and applying lemmas 1.10, 1.9, then corollary 1.8 and lemmas 1.3, 1.2 (in this order) one concludes that if the formal schemes \( \hat{X}_h \) in \( X \) are analytically equivalent, then

2. Characterization of \( \hat{X}_h \)

Let \( \{r_n\}_{n=0}^\infty \) be the \( r \)-th formal blowing up in \( X \), with multiplicity \( \hat{C}' \) the \( r \)-th blowing up

in the direction of \( \hat{X}_h \).

Let \( \hat{X}' \) be a formal scheme in \( X \) starting with \( \hat{X}_h \) and \( \hat{X}' \) is locally isomorphic to \( \hat{X} \).

**Proof:** Let \( \hat{X}_h \) be an \( r \)-blowing up of \( X \), \( \hat{X}_h \) are isomorphic. Suppose \( \hat{X}_h \) is a closed point, then the

\[ O_{\hat{X}_h} \]

where \( \cdot \) is the number of points through \( x \). This proves...
the formal schemes $\hat{X}(\mathcal{C}_r)$ and $\hat{X}(\mathcal{C}'_r)$ are isomorphic, then the branches $\mathcal{C}$, $\mathcal{C}'$ in $X$ are analytically equivalent. The converse is immediate.

2. Characterization of the "blowing ups" which are directed by a branch

Let $[r_n]_{n=0}^\infty$ be a multiplicity sequence of a branch $\mathcal{C}$ embedded in $X$. Let

$$r = h + 2 \sum_{i=0}^{\infty} r_i(r_i - 1)^2$$

where $h$ is minimal with the condition that $r_h = 1$. For every embedded branch $\mathcal{C}'$ in $X$, with multiplicity sequence $[r_n]_{n=0}^\infty$, one denotes by

$$\pi'_r: \hat{X}(\mathcal{C}'_r) \rightarrow \hat{X}$$

the $r$-th formal blowing up in the direction of $\mathcal{C}'$; $X'_o$ will denote the exceptional reduced fibre of $\pi'_r$ and $L'$ the conormal sheaf to $X'_o$ in $\hat{X}(\mathcal{C}'_r)$.

We now fix an embedded branch $\mathcal{C}$ in $X$ with multiplicity sequence $[r_n]_{n=0}^\infty$. Let $(X'_o, L)$ be the exceptional fiber and the conormal sheaf in the $r$-th blowing up

$$\hat{X}(\mathcal{C}_r) \rightarrow \hat{X}$$

in the direction of $\mathcal{C}$.

Let $\hat{\mathcal{C}}'$ be a formal scheme along a closed subscheme isomorphic to $X'_o$. Suppose that the conormal sheaf to $X'_o$ in $\hat{\mathcal{C}}'$ is isomorphic to $L$. The main result characterizing the $r$-th blowing up is the following.

**Theorem:** $\hat{\mathcal{C}}'$ is isomorphic to the composition of $r$ formal blowing ups starting with $\hat{\mathcal{C}}'$ and with centers at closed points if and only if the sheaf $\mathcal{O}_{\hat{\mathcal{C}}'}$ is locally isomorphic to $\mathcal{O}_{\hat{X}(\mathcal{C}_r)}$ along $X'_o$.

**Proof:** Let

$$\pi'_r: \hat{X}(\mathcal{C}'_r) \rightarrow \hat{X}$$

be an $r$-blowing up of $\hat{X}$ in the direction of $\mathcal{C}' \subset X$. Suppose also that $X'_o$ and $X'_o$ are isomorphic. One can prove easily by induction on $r$ that if $x \in X'_o$ is a closed point, then the local ring at $x$ is:

$$\mathcal{O}_{\hat{X}(\mathcal{C}'_r), x} = \left[ k[Y_0, \ldots, Y_t][[Y_0^{-1}, \ldots, Y_t^{-1}]] \right]$$

where $t$ is the number of irreducible connected components of $X'_o$ which pass through $x$. This proves the only if part.
Conversely, let \( X_0 \) be embedded in \( \tilde{X}' \) with the conditions of the theorem. Suppose

\[
X_0 = X_0^{1} U \cdots U X_0^{r}
\]

is the decomposition of \( X_0 \) into irreducible components, and

\[
\tilde{x}(C_r) \longleftarrow X_0 \longleftarrow \tilde{x}'
\]

are the given embeddings. Then there are positive integers \( n_1, \ldots, n_r \) such that \( \mathcal{O}(-n_1 X_0^{1} + \cdots + n_r X_0^{r}) \) is an ample line sheaf for

\[
\pi_r: X(C_r) \rightarrow \tilde{x}.
\]

So its inverse image \( i^* \mathcal{O}(-n_1 X_0^{1} + \cdots + n_r X_0^{r}) \) is ample on \( X_0 \). This implies that the other inverse images \( i^* \mathcal{O}(-n_1 X_0^{1} + \cdots + n_r X_0^{r}) \) are also ample on \( X_0 \) (because by the hypothesis \( i^* \mathcal{O}(-X_0) = \mathcal{L} \equiv \mathcal{L}^r = i^* \mathcal{O}(-X_0) \)). One can then apply H. Artin’s theorem on contractions ([4], Corollary (6.10)) to conclude that there exists a modification (see [4] for definition):

\[
\tilde{x}' : \tilde{x} \rightarrow \tilde{x}
\]

where \( \tilde{x} \) is the formal spectrum of a complete local ring. By Grothendieck’s algebraization theorem [3], there is a scheme \( \tilde{X}' \) containing \( X_0 \) as a closed subscheme and such that \( \tilde{X}' \) is the formalization of \( X' \) along \( X_0 \); corollary (6.11) of [4] applied to the subscheme \( \tilde{X}' = \mathbb{P}_m(k) \) allows us to contract \( X_0 \) to a point. That is, there exists a contraction

\[
f: X' \rightarrow X''
\]

such that the formalization of \( f \) along \( f(X_0) = f(X_0') U \cdots U f(X_0'^{r-1}) \) is a formal blowing up with center at the closed point \( f(X_0'^{r}) \) of \( X'' \). One concludes by induction on the number \( r \) of irreducible components of \( X_0 \).

3. The scheme of r-blowing ups and a theorem of boundedness

Let

\[
\tilde{x}(C_r) \longleftarrow X(C_r) \longrightarrow \tilde{x}(C_r')
\]

be two r-blowing ups, and \( I, I' \) be the respective sheaves of ideals defined by the reduced exceptional fibres. Denote by \( X_k, X'_k \), the subschemes defined respectively by \( I^{k+1}, I'^{k+1} \).

We want to prove the

Theorem 3.1. (Boundedness) The formal schemes \( \tilde{x}(C_r) \) and \( X'_k \) are isomorphic.

Proof: The conclusion is clear if we prove (1) for every r-blowing up

\[
l_k = k[T_0, \ldots, T_n]
\]

where \( m \) is the maximal ideal of \( k \).

It is clear that there exists an isomorphism \( \mathbb{P}_m \rightarrow \mathbb{P}_n \) defined by the sheaf automorphism \( \tilde{\tau} \) that induces \( \tau \). Let \( \tau \) be the blowing up at \( \tau(a) \) is another locally principal.

So the morphism

factors through the
Theorem 3.1 (Boundedness): For every \( r \) there exists a \( K \) such that the formal schemes \( \hat{X}(C_r) \) and \( \hat{X}(C'_r) \) are isomorphic if and only if the schemes \( X_K \) and \( X'_K \) are isomorphic.

Proof: The condition is obviously necessary for every \( K \). To see the sufficiency suppose the following \((\ast)\) is true:

\((\ast)\) for every positive integer \( r \) there exists a \( \lambda \) such that the \( r \)-th blowing up

\[
\hat{X}(C_r) \longrightarrow \hat{X} = \text{Sp } f(A),
\]

with \( A=k[[T_0, \ldots, T_m]] \), is the blowing up of an ideal \( \alpha \) of \( A \) such that

\[
m_\alpha \subseteq \alpha \subseteq A,
\]

where \( m \) is the maximal ideal of \( A \) and \( \pi_\alpha(\alpha_{\hat{X}(C_r)}) = \alpha'' \).

It is clear that there exists a positive integer \( K_1 \) such that

\[
m_\alpha \subseteq \alpha_{\hat{X}} \subseteq K_1
\]

for every \( r \)-blowing up \( \hat{X} \) of \( \hat{X} \) (It is enough to take \( K_1 = r! \) and to prove it by induction on \( r \)). Let \( K = (\lambda + 1)K_1 \). Suppose

\[
\phi: X_K \cong X'_K
\]

is an isomorphism. This induces another isomorphism between the subschemes defined by the sheaves \( m^{\lambda+1}\alpha_{\hat{X}} \) and \( m^{\lambda+1}\alpha_{\hat{X}}' \). By taking global sections one gets an automorphism \( \tau \) of \( \text{Spec}(A/m^{\lambda+1}) \). Let \( \tau \) be an automorphism of \( \text{Spec}(A) \) that induces \( \tau \). Let \( \alpha \) be an ideal of \( A \) such that

\[
\pi_\alpha(\alpha_{\hat{X}(C_r)}) \longrightarrow \hat{X}
\]

is the blowing up along \( \alpha \), and which satisfies the condition \((\ast)\) above, then \( \pi_\alpha(\alpha_{\hat{X}(C_r)}) \) is another ideal in \( \hat{X} \) such that

\[
\pi_\alpha(\alpha_{\hat{X}(C_r)}) = \phi(\alpha_{\hat{X}'(C_r)}).
\]

So the morphism

\[
\hat{X}(C_r) \longrightarrow \text{Sp } f(A)
\]

factors through the \( r \)-blowing up

\[
\hat{X}(C'_r) \longrightarrow \text{Sp } f(A) \longrightarrow \text{Sp } f(A),
\]

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that is, there exists a morphism

\[ f : \hat{X}(C_r) \longrightarrow \hat{X}(C'_r) \]

such that

\[ \pi_r = \tau \circ \pi'_r \circ f \]

One concludes that \( f \) is an isomorphism, because \( X(C_r) \) and \( X(C'_r) \) are \( r \)-blowing ups.

To finish one has only to prove (*). Let \( X \) be the spectrum of a local \( k \)-algebra of finite type. One has:

**Theorem 3.2 (Representability):** There exists a noetherian scheme of finite type over \( k \) and a blowing up

\[ \hat{\pi}_n : \hat{\pi} \longrightarrow X \times \Delta_n \]

such that for every closed point \( x_n \in \Delta_n \) the blowing up of \( X \) induced by \( \pi_n \) on the closed subscheme \( X = X \times x_n \) of \( X \times \Delta_n \) is an \( n \)-blowing up of \( X \) and every \( n \)-blowing up is obtained in this way. Moreover, if \( \mathcal{P}_{x \times \Delta_n}^X \) is the sheaf of ideals of the subscheme \( x \times \Delta_n \) of \( X \times \Delta_n \), then the blowing up \( \hat{\pi}_n \) is defined by a sheaf of ideals \( \mathcal{P} \) such that

\[ \mathcal{P}_{x \times \Delta_n}^X \subseteq \mathcal{P} \subseteq \mathcal{P}_{x \times \Delta_n} \]

**Proof:** We will only give the construction of \( \Delta_n \). The properties of \( \Delta_n \) follow from the general properties of a blowing up and the definition of \( \Delta_n \).

Firstly it is easy to see that if

\[ \pi : \hat{\pi}' \longrightarrow X' \]

is a blowing up of schemes over \( Y \), and \( Z \) is a flat scheme over \( Y \), and

\[ X'_Z = X' \times_Y Z \longrightarrow Z \]

the map obtained by base change, then the blowing up that \( \pi n \) induces on \( X'_Z \) is precisely

Construction of \( \Delta_n \)

be the blowing up of

be the diagonal, and

be the blowing up of

by blowing up the closed subscheme \( \pi_n(\Delta_n) \) over \( \pi_n(\Delta_n) \) of \( \pi_n(\Delta_n) \)

is the diagonal. It

(with \( \Delta_1 = x \)) by a

is an \( n \)-immersion of

\( (1 \leq n) \). Let

be the closed immersion

where \( \pi_X, \pi_n \) are
\[ \bar{\mathcal{I}}_{i} \times \mathcal{V}_{j} \xrightarrow{\bar{\pi}_{i} \times \bar{\pi}_{j}} \bar{\mathcal{N}} \times \mathcal{N} \]

Construction of \( \Delta_{n} \): let

\[ \pi_{1} : \bar{\mathcal{N}} \rightarrow \mathcal{N} \]

be the blowing up of \( \mathcal{N} \) at its closed point and \( \Delta_{2} \) the exceptional fibre. Let

\[ \iota_{2} : \Delta_{2} \hookrightarrow \bar{\mathcal{N}} \times \Delta_{2} \]

be the diagonal, and

\[ \pi_{2} : \bar{\mathcal{N}} \rightarrow \bar{\mathcal{N}} \times \Delta_{2} \]

be the blowing up of \( \bar{\mathcal{N}} \) along \( \iota_{2}(\Delta_{2}) \). Inductively one defines

\[ \pi_{n} : \bar{\mathcal{N}} \rightarrow \bar{\mathcal{N}} \times \Delta_{n} \]

by blowing up the closed subscheme \( \iota_{n} : \bar{\mathcal{N}}_{n-1} \hookrightarrow \bar{\mathcal{N}}_{n-1} \times \Delta_{n} \), and \( \Delta_{n+1} \) as the fibre over \( \iota_{n}(\Delta_{n}) \) of \( \pi_{n} \) and where

\[ \iota_{n+1} : \Delta_{n+1} \hookrightarrow \bar{\mathcal{N}} \times \Delta_{n+1} \]

is the diagonal. It is clear that \( \bar{\mathcal{N}}_{n} \) is obtained from

\[ R = \mathcal{N} \times \Delta_{1} \times \ldots \times \Delta_{n} \]

(with \( \Delta_{1} = \mathcal{N} \)) by a sequence of blowing ups. One also has projections

\[ f_{i} : \Delta_{n} \rightarrow \Delta_{1} \]

(\( 1 \leq n \)). Let

\[ f : \mathcal{N} \times \Delta_{n} \rightarrow \mathcal{N} \times \Delta_{1} \times \ldots \times \Delta_{n} \]

be the closed immersion defined by

\[ f = (\pi_{\mathcal{N}} \circ \Delta_{n}, \ldots, f_{n} \circ \Delta_{n}) \]

where \( \pi_{\mathcal{N}}, \pi_{\Delta_{n}} \) are the projections on the factors (with \( f_{n} = \text{Id}_{\Delta_{n}} \)). Consider

\[ \xymatrix{ \bar{\mathcal{N}} \ar[r] \ar[d]^{\bar{\pi}_{n}} & \bar{\mathcal{N}} \ar[d]^{\bar{\pi}} \\ \mathcal{N} \times \Delta_{n} \ar[r] & R } \]
where \( \pi \) is the blowing up constructed above and \( \pi_n \) the one induced on \( X \times \Delta_n \) by \( \pi \).

(*) Follows from this, because if

\[
\pi: \tilde{X} \longrightarrow X
\]
is a blowing up, there exists a closed point \( x_n \in \Delta_n \) such that

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\pi} & X \\
\downarrow & & \downarrow \\
\tilde{x} & \xrightarrow{\tilde{\pi}} & x
\end{array}
\]

where \( \tilde{\pi} \) is the blowing up induced by \( \tilde{\pi} \). But \( \tilde{\pi} \) is defined by the blowing up of the sheaf of ideals \( \mathcal{A} \) such that

\[
p_{\pi_n}^\mathcal{A} \subseteq \mathcal{A} \subseteq p_{\pi_n}^X.
\]

Restricting everything to \( x = x \times x_n \) one has that \( \pi \) is defined by the blowing up of a sheaf of ideals \( \mathcal{A} \) such that

\[
\mathcal{A}^\mathcal{O}_X \subseteq \mathcal{A} \subseteq \mathcal{O}_X.
\]

Moreover, one can easily see that \( \beta = \pi_*(\mathcal{O}_\tilde{X}) \) defines the same blowing up as \( \alpha \), and it verifies

\[
\pi_*(\beta \mathcal{O}_\tilde{X}) = \pi_*(\alpha \mathcal{O}_\tilde{X}) = \beta,
\]

with which one concludes the proof of condition (*).

**Observation:** The scheme \( \Delta_n \) parametrizes the analytic branches in \( \tilde{X} \) up to order \( n \) modulo the relation:

\( C \equiv C' \) if and only if the both direct the same \( n \)-blowing up.

4. Classification theorem

Let \( \{r_n\}_{n=0}^\infty \) be the multiplicity sequence of an embedded branch in \( X \). We will suppose all branches have multiplicity sequence equal to \( \{r_n\}_{n=0}^\infty \). Let

\[
\pi_r: \tilde{X}(C_r) \longrightarrow \text{Sp} f(A),
\]

with \( A = k[[T_0, \ldots, T_m]] \), be the \( r \)-blowing up of \( X \) in the direction of a branch \( C \).
embedded in $X$. We will denote by $X_0$, the exceptional reduced fibre of $\pi$, by $I$ the sheaf of ideals defining $X_0$ in $\tilde{X}(C_r)$ and by $I^{n-1}$ the conormal sheaf. Let $X_n$ be the closed subscheme of $\tilde{X}(C_r)$ defined by $I^{n-1}$.

To a given branch $C$ embedded in $X$, we associate the scheme $X_1$. Note that the pair $(X_0, L)$ is part of the information $X_1$ carries which we have defined as the equisingularity of $C$ as embedded in $X$. In this paragraph, we want to classify embedded branches with the same, up to isomorphism, associated scheme $X_1$. So we will fix the scheme $X_1$. The pair $(X_0, L)$ is also fixed and we will also fix, as a reference to classify, the formal scheme $\tilde{X}(C_r)$.

By theorem (1.4), one knows that to classify, up to analytic equivalence, the branches embedded in $X$ with multiplicity sequence $\{r_n\}_{n=0}^{\infty}$ and given associated pair $(X_0, L)$ amounts to classifying isomorphism classes of formal schemes obtained by $r$-blowing ups of $X$ ($r = h + 2 \sum_{i=0}^{\infty} r_i(r_i - 1)^2$) such that their associated pair is $(X_0, L)$. But by the theorem of equivalence (section 2) to classify these classes of formal schemes is the same as to classify the isomorphism classes of formal schemes $\tilde{X}'$ which contain $X_0$ as the closed subscheme, which are complete along $X_0$ and such that the conormal sheaf to $X_0$ in $\tilde{X}'$ is isomorphic to $L$ and $\mathcal{O}_{\tilde{X}'}$ is locally isomorphic to $\mathcal{O}_{\tilde{X}}$ on $X_0$. Besides, by the boundedness theorem (3.1), a formal scheme such as $X'$ determined, up to isomorphism, by the closed subscheme $X'_K$ of $\tilde{X}'$, where $X'_K$ is the subscheme defined by the sheaf of ideals $I'$, $I'$ being the defining sheaf of $X_0$ in $\tilde{X}'$ (and $K$ is a positive integer whose existence is guaranteed by theorem 3.1).

So, it is enough to classify the schemes $X'_K$ so obtained.

Consider now, for each pair of positive integers $n \geq m$, the sheaf $\mathcal{Aut}_{X'_{m,n}}$ of groups over $X_0$ consisting of local automorphisms of the scheme $X_n$ which restrict to the identity on the subscheme $X_m$.

Denote by

$$\rho_{X, n}^m: \mathcal{Aut}_{X, n} \rightarrow \mathcal{Aut}_{X, m}$$

(for $n \geq m \geq 1$) the natural restriction maps.

**Lemma 4.1.** For each $r > 1$, the sheaf of groups $\mathcal{Aut}_{X, n}^r$ is canonically isomorphic to the sheaf of groups

$$\mathcal{Der}_k(\mathcal{O}_{X, n}, \mathcal{I}^n)$$

In particular

$$\mathcal{Aut}_{X, n}^r \rightarrow \mathcal{Aut}_{X, m}^r$$

is a subsheaf contained in the center of this last sheaf of groups.

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Proof: The map
\[ \phi: \text{Aut}_{X_{n-1}}(X_n) \to \text{Der}_k(\mathcal{O}_{X_1}, L^n) \]
defined by
\[ \phi(\tau) = \tau - \text{Id} = \tau_0 \]
is a morphism of sheaves of groups. The image of \( \phi \) is contained in
\[ j_* \text{Der}_k(\mathcal{O}_{X_1}, L^n) \]
where
\[ j: X_1 \to X_n \]
is the canonical injection. Conversely, given a derivation \( D \in \text{Der}_k(\mathcal{O}_{X_1}, L^n) \) one defines \( \phi^{-1}(D) \) to be
\[ \phi^{-1}(D) = \text{Id} + j_* D = \tau_0 D . \]

It is easily seen that \( \tau_0 \) is an automorphism of \( X_n \) giving the identity on \( X_1 \). The rest follows from this.

Corollary 4.2. The sheaves of groups \( \text{Aut}_{X_1}(X_n) \) for \( n > 1 \) have resolutions by sheaves of coherent \( \mathcal{O}_{X_0} \)-modules of the form
\[ \text{Der}_k(\mathcal{O}_{X_1}, L^n) \]
for \( 2 \leq h \leq n \). More precisely, the sequences
\[ (n) \quad 0 \to \text{Der}_k(\mathcal{O}_{X_1}, L^n) \overset{i_n}{\to} \text{Aut}_{X_1}(X_n) \to \text{Aut}_{X_1}(X_{n-1}) \to 0 \]
are exact, where \( i_n \) is defined as in the above lemma, identifying
\[ \text{Der}_k(\mathcal{O}_{X_1}, L^n) = \text{Aut}_{X_{n-1}}(X_n) . \]

To simplify the notation, let us write
\[ D^n = \text{Der}_k(\mathcal{O}_{X_1}, \mathcal{O}_{X_0}) \otimes L^n \]
\[ A^n = \text{Aut}_{X_1}(X_n) \]
\[ G^n = L \otimes L^3 \otimes \ldots \otimes L^n . \]

Corollary 4.3: For every \( n \) there exists a quotient \( H_1 \) of the abelian group
\[ H_1(X_0, D^n \otimes X_0 G^n) \]
and a map
\[ \chi \]
which identifies with the subset of

Proof: By induction

and \( f_1 = 0 \) by applying \( \text{Der}_k(\mathcal{O}_{X_1}, L^n) \) gives an exact sequence
\[ B = H_1 \]
(see [1]). The first equality holds on \( H_1(A^n) \). The rest follows from this.

So

We define

where

is the natural projection

where \( g : M_{n-1} \to \text{complement} \).

As the sheaves of groups, the set
and a map

\[ f_n : M_n \rightarrow H^2(X_0, D^0 \otimes \mathcal{O}_n) \]

which identifies

\[ H^1(X_0, \mathcal{A}^n) \]

with the subset of \( M_n \) defined the elements \( c \in M_n \) such that \( f_n(c) = 0 \).

**Proof:** By induction on \( n \). For \( n = 2 \) one has

\[ M_1 = H^1(X_0, D^0) \]

and \( f_1 = 0 \) by applying lemma 5.1. If \( n > 2 \) the cohomology sequence associated with \((n)\) gives an exact sequence

\[ B = H^1(D^n)/\partial H^0(A^{n-1}) \rightarrow H^1(A^n) \rightarrow H^1(A^{n-1}) \otimes \delta^n \rightarrow H^2(D^n) \]

(see [1]). The first term on the left is an abelian group and acts freely on the left on \( H^1(A^n) \). The orbits of this action are the fibres of

\[ \text{Im}(\tau^n) = (\delta^n)^{-1}(0). \]

So

\[ H^1(A^n) = \text{Im}(\tau^n) \times B \subset H^1(A^{n-1}) \times B \subset M_{n-1} \times B. \]

We define

\[ f_{n-1} : M_n \rightarrow H^2(D^n) \otimes H^2(D^0 \otimes \mathcal{O}_{n-1}) = H^2(D^0 \otimes \mathcal{O}_n) \]

to be

\[ f_{n-1} = f_{n-1} \circ \pi, \]

where

\[ \pi : M_{n-1} \times B \rightarrow M_{n-1} \]

is the natural projection, and

\[ f_{n-1} = f_{n-1} + g \]

where \( g : M_{n-1} \rightarrow B \) is equal to \( \delta^n \) on \( H^1(A^{n-1}) \) and is zero on the complement.

As the sheaves \( \mathcal{O}_{X_K} \) are locally isomorphic to \( \mathcal{O}_{X_K} \), they are classified by the set

\[ H^1(X_1, \text{Aut}_{X_1}(X_K)) \]
(see [1]). The quotient of this $H^1$ by the action of the group $\text{Aut}(X_1)$ classified the schemes $X'_K$ which contain $X_1$ as a closed subscheme, and whose structure sheaf $\mathcal{O}_{X'_K}$ is locally isomorphic to $\mathcal{O}_{X_K}$.

From Corollary 4.3 and the considerations at the beginning of this paragraph the main theorem of this introduction follows easily.

REFERENCES