PERIODICITIES IN ARNOLD'S LISTS
OF SINGULARITIES

Dirk Siersma

Abstract

V. I. Arnold discovered experimentally periodicities in the classification of singularities. These periodicities are explained for functions from $\mathbb{C}^n$ to $\mathbb{C}$, using the blowing up construction. Moreover the singularities of multiplicity 5 are classified.

In his paper 'Local forms of functions' Arnold [2] gives a list of normal forms of functions in the neighborhood of critical points (the classification of all singularities with number of modules $m=0, 1, 2$ or with multiplicity $\mu \leq 16$ included). In his introduction he mentions a periodicity in the decomposition of singularities into $\mu$-equivalence classes. According to Arnold the phenomenon of periodicity is only partially explained and for quasi-homogeneous singularities only. The explanation is based upon some root technique for the quasi-homogeneous Lie algebra, related to work of Enriques and Demazure [6].

The aim of this paper is to explain the periodicity for all isolated singularities of corank $\leq 2$ and some singularities of corank 3 (including all singularities in Arnold’s lists), using the theory of resolutions. We also give a list of singularities of corank 2 with multiplicity equal to 5.

This paper is an elaborated version of the lecture I gave at the Institut des Hautes Études Scientifiques (Bures-sur-Yvette, France) in May 1975 about Arnold’s paper: ‘Local forms of functions’.

I thank the I.H.E.S. for their hospitality.

Added in proof: Arnold communicated to me that:
1° Some of his students have also classified the singularities of corank 2 with multiplicity 5 (unpublished).
2° His remark about periodicities also concerns the actual computations that occur.

AMS(MOS) subject classification scheme (1970): 33 C 40, 58 C 23
1. Introduction

For results and definitions, mentioned in this introduction see Arnold [2] and Siersma [10].

1.1 The group $\mathfrak{G}$ of germs (or jets) of holomorphic mappings $(\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ acts on $\mathfrak{G}$, the set of germs at 0 (or jets) of holomorphic functions $\mathbb{C}^n \to \mathbb{C}$ by right-multiplication. A singularity class is a subset of $\mathfrak{G}$ invariant under this action. Each orbit is a singularity class. Two germs (or jets) belonging to the same orbit are called right-equivalent.

1.2 Let $m(f) = \dim \ker (\Delta(f)) = \dim \ker (\mu(f))$ and the Milnor number of $f$ is defined by $\mu(f) = \dim \ker (\Delta(f))$. These are related by:

$$\mu(f) = 1 + \dim \ker (\Delta(f)) = 1 + \dim \ker (\Delta(f)).$$

1.3 A germ $f \in \mathfrak{G}$ is called $k$-determined (or $k$-sufficient) if for any $g \in \mathfrak{G}$, $\mu(f) = \mu(g) \Rightarrow f$ in right-equivalent with $g$. It is well-known that the following are equivalent:

1. $\mu(f) < m$.
2. $0$ is an isolated critical point of $f$.
3. there exists $k \in \mathbb{N}$ such that $f$ is $k$-determined.

1.4 Two germs $f$ and $g$ are called:

(1) Right-left-equivalent if there is a $\phi \in \mathfrak{G}$, and $\psi \in \mathfrak{G}$ such that $\phi g = fb$.

(2) Contact-equivalent if there is a $\phi \in \mathfrak{G}$, and for every $x$ near 0 $\phi x \in \mathfrak{G}$ (analytically depending on $x$) such that $\phi g = fb$.

1.5 The modality $m(f)$ of $f \in \mathfrak{G}$ is the smallest number $k$ such that some neighborhood of zero is covered by a finite number of more than $m$-parametrized families of orbits of $\mathfrak{G}$ on $m$.

Another characterization of modality is as follows:

Let $(\phi_1, \ldots, \phi_{n-1})$ be a basis of $\text{det} (A(f))$ and let $F(x) = f(x) + \sum_{i=0} F_i(x)$ (versal deformation).

Define $S = \{|\mu(f) + \mu(g) = \mu(f)| f \in \mathfrak{G} \}$.

Gabrielov [7] proved that the modality of $f$ is equal to the dimension of $S$. So $m(f) = \mu(f) - \dim(c(f)) = \mu(f)$-1.

1.6 Splitting lemma. Let $f \in \mathfrak{G}$ then:

$$f(x_0, \ldots, x_n) \sim g(x_0, \ldots, x_0) + Q(x_0, \ldots, x_0)$$

where $Q \in \mathbb{R}$ and $Q$ a non-degenerate quadratic form.

(Here $\sim$ means Right-equivalent.)

The number $\rho$ is called the corank of $f$.

1.7 Arnold [2] classified in 105 theorems:

1* all singularities with Milnor number $\mu(f) \leq 16$.
2* all singularities with modality $m(f) \leq 2$.

The classification of singularities follows:

1* increasing corank.

2* increasing multiplicity $m(f)$.

3* different factorizations of $f$ (decreasing the number of factors).

In this way one finds (see also Siersma [10], [11]).

Corollary 0.1. type $A_n: f = x^{n+1} (n$ $\geq 0)$. 

(3) Topological-equivalent if there exist homeomorphisms $\phi: \mathbb{C}^n \to \mathbb{C}^n$ and $\psi: \mathbb{C}^n \to \mathbb{C}^n$ such that $\phi \psi = f$.

(4) $\mu$-equivalent if there exists a family $f$ with $\mu(f) = \mu$ (constant) and $f_0 = f$ and $f_0 = g$.

If $n \neq 3$: $\mu$-equivalexample: equivalence (cf. Lê Dũng Tráng and Ramanujam [8] and Teissier [12]).

A $\mu$-equivalence class will also be called $\mu$-class.

Remark. A $\mu$-class can consist of several orbits; examples are:

$$(x + x^2 + y)^3 + z^4, (x + y + z)^3 + x^2 y^2 + x w^2 y^2, x^2 + y^2 + x w z, x^2 y^2 + x w z.$$
Corollary 2:
(a) multiplicity 3 \( \Rightarrow i'(f) = 0 \), \( i'(f) \neq 0 \). Factorizations of \( i'(f) \):

(a1) three linear factors
\[ f = x^2 + y^2 \]

(a2) two linear factors
\[ f = x^2 + y^2 \]

(a3) one linear factor
\[ f = x^2 \]

(b) multiplicity 4: 5 different factorizations, etc.

If \( \mu \) increases the classification becomes more complicated. To work more systematically and to reduce some of the computations we propose, in the case of corank 2, the use of the blowing up construction.

2. The blowing up construction

We consider \( f: C^2 \rightarrow C \), a holomorphic mapping.

(2.1) Replace \( O \in C^2 \) by the set of all its tangent-directions (isomorphic to \( P^1(C) \)). We get a manifold \( M \) that can be covered by two charts \( (x_1, y_1) \) and \( (x_2, y_2) \), together with a projection \( \pi: M \rightarrow C^2 \).

The charts and the projection are related by the formulae:
\[
\begin{align*}
\pi(x_1, y_1) &= (x_1, y_1), \\
\pi(x_2, y_2) &= (x_2, y_2).
\end{align*}
\]

\( C = \pi^{-1}(0) = P^1(C) \) is given by \( y_1 = 0 \) or \( x_2 = 0 \) and is called the exceptional divisor.

\( f: C^2 \rightarrow C \) extends in a natural way to a map \( \tilde{f}: M \rightarrow C \):
\[
\begin{array}{ccc}
M & \xrightarrow{\pi} & C \\
\downarrow & \searrow \tilde{f} & \downarrow \\
C^2 & \xrightarrow{f} & C
\end{array}
\]

If \( f \) has multiplicity \( k \) we have:
\[
\tilde{f}(x_1, y_1) = f(x_1, y_1) = y_1^k \tilde{g}(x_1, y_1)
\]
and
\[
\tilde{f}(x_2, y_2) = f(x_2, y_2) = x_2^k \tilde{g}(x_2, y_2)
\]
for certain \( \tilde{g}_1 \) and \( \tilde{g}_2 \) defined in a neighborhood of \( C \).

In the neighborhoods of the points \( w_1, \ldots, w_s \), we can choose local coordinates \( (\xi, \eta) \) such that \( g_i(\xi, \eta) = \xi + \eta \eta(\xi, \eta)(k = 1, 2) \) where \( C \) is given by \( \eta = 0 \).

If an intersection point is covered by 2 charts then \( g_1(x_1, y_1) = x_1^k \tilde{g}_1(x_1, y_1) \), so \( g_1 \) and \( g_2 \) are \( \mu \)-equivalent near that intersection point.

(2.3) In the following paragraphs we shall obtain the following results:
1. The classification of \( \mu \)-classes of \( f \) can be done by local investigations around every intersection point.
2. Different intersection points are treated independently.
3. In each intersection point there is a periodicity in the classification of \( f \).

First we compare the Milnor number of \( f \) with the Milnor numbers at the intersection points. The following proposition is due to Pfaff [9].

(2.4) Proposition. Let \( f \) have an isolated critical point at \( O \in C^2 \). Let \( f^{-1}(0) \) have \( s \) different tangent directions \( w_1, \ldots, w_s \), in \( O \in C^2 \). Let \( M \) be constructed from \( C^2 \) by blowing up \( O \) and let \( \tilde{f}: M \rightarrow C \) be the natural extension of \( f \). Denote by \( \mu_i (i = 1, \ldots, s) \) the restrictions of \( g_i \) at \( g_i \) (defined above) to neighborhoods of the points \( w_1, \ldots, w_s \in M \).

Then:
\[
\mu(f) = \sum_{i=1}^s \mu_i(f) - s + 1
\]
where \( s \) is the multiplicity of \( f \).

Proof. We consider the real monomorphism \( \tilde{f} \) of \( f \), as constructed by A'Campo [1]. Let \( C \) be the set of zeros of \( \tilde{f} \), intersected by a small real 2-disc \( D_r \). The curve \( C \) has as its only singularities multiple normal crossings of branches. Let \( p_1, \ldots, p_s \) be the points of crossing of \( C \), and \( n_i \) the number of branches of \( C \) around the points \( p_i \).

A’Campo proved:
\[
\mu = \sum_{i=1}^s n_i (n_i - 1) - q + 1
\]
where \( q \) is the number of branches of \( f \).

The monomorphisms of \( f_i (i = 1, \ldots, s) \) are constructed from \( f \) by restricting
3. Periodicity of $\mu$-classes in $x^* + m^{-1}x$

(3.1) Let $f(x, y) = x^* + p(x, y)$ with multiplicity $\nu(p) \geq n + 1$. We blow up $\mathbb{C}^2$ at $p$.

Then:

$$f(x, y) = f(x_1, y_1) = x_1^* + p(x_1, y_1) = y_1^2 + y_1 q_1(x_1, y_1)$$

where $q_1(x_1, y_1) = y_1^{n-1} p(x_1, y_1)$ and $q_2(x_2, y_2) = x_2^{n-1} p(x_2, y_2)$.

We consider $x_1^* + y_1 q_1(x_1, y_1) = 0$ and $1 + x_2 q_2(x_2, y_2) = 0$.

The solutions have only one intersection point with the exceptional divisor, namely $(x_1, y_1) = (0, 0)$. We next consider the map germ:

$$f(x_1, y_1) = x_1^* + y_1 q_1(x_1, y_1)$$

in a neighborhood of $(x_1, y_1) = (0, 0)$.

(3.2) In order to classify $f$ up to $R$-equivalence it now seems enough to classify $f_1$ up to $R$-equivalence. However, we have to take into account the following:

1. Diffeomorphisms of $\mathbb{C}^2$ lift only to diffeomorphisms of $\mathbb{M}$ with a special form.

2. The polynomial $q_1(x_1, y_1)$ is not arbitrary since $q_1(x_1, y_1) = y_1^{n-1} p(x_1, y_1)$.

Since $\mu(f) = m(n-1) + \mu(f_1)$, we see that $\mu(f)$ depends only on $\mu(f_1)$.

Moreover, the $\mu$-class of $f$ follows from the configuration of its resolution. If we apply diffeomorphisms to $f_1$ such that this resolution is still the same, then we stay in the same $\mu$-class.

The resolution of $f$ depends only on the different possibilities for tangencies of $f_1^*(\mathbb{C})$ to the exceptional divisor $C$. This can give a subdivision of every $\mu$-class of $f_1$, each subclass giving a $\mu$-class for $f$.

(3.3) We now return to $f_1(x_1, y_1) = x_1^* + y_1 q_1(x_1, y_1)$.

1. Let the multiplicity $\nu(y_1 q_1) \equiv n$.

In this case a detailed study is necessary to find the possibilities for $f_1$ and the corresponding classes for $f$. In some sense the singularity $f_1$ is less complicated than $f$ and is already treated in an earlier part of the classification.

As an example consider $n = 3: x_1^* + y_1 q_1(x_1, y_1)$.

This singularity must be of type $A$ or $D$ if $\nu(y_1 q_1) \equiv 3$.

A detailed study (cf. §6) gives just the following possibilities for $f_1$:

$$A_2 \times A_2 \times D_4 \times D_4 \times D_4 \times D_4 \times D_4$$

and the following corresponding classes for $f$:

$$E_8 \times E_7 \times E_6 \times F_4 \times F_4 \times F_4 \times F_4 \times F_4$$

(Here and in the following we use the same notations as in Arnold [2]).

2. Let the multiplicity $\nu(y_1 q_1) \equiv n - 1$.

We now have more or less the same situation as before with $f_1(x_1, y_1) = x_1^* + y_1 q_1(x_1, y_1)$. So we blow up a second time and get: $f_1^2(x_2, y_2) = (x_2)^* + y_2 q_1(x_2, y_2)$. Now we can omit the detailed study, mentioned above, because of the following lemma:

(3.4) Periodicity Lemma. There is a 1-1-correspondence between $\mu$-classes of $f$, defined by $f_1$ and $\mu$-classes of $f$ defined by $f$.

Proof. Let $(f(x, y) = x^* + p(x, y)$ with $\nu(p) \equiv 4$.

1. Arrange the normal form:

$$f(x, y) = x^* + x^{n-1} A_2(x, y) + \cdots + x y^{n-1} A_2(y) + y^{n-1} A_2(y)$$

$$= x^* + x^{n-2} \sum_{k=0}^{n-2} a_k x y^{k} + \cdots + x y^{n-1} \sum_{k=0}^{n-1} a_k y^{k} + y^{n-1} \sum_{k=0}^{n-1} a_k y^{k}.$$
2. After blowing up:

\[ f(x_1, y_1) = x_1^2 + x_1^{-2}y_1 \sum_{k=0}^{n} a_{-2-k}y_1^k + \cdots + x_1y_1 \sum_{k=1}^{r} a_{1-k}y_1^{k-1} + y_1 \sum_{k=0}^{s} a_{0-k}y_1^k. \]

So

\[ v(y_1q_1) = n + 1 \Rightarrow \begin{cases} a_{-2-n} = a_{-n-2} = 0 \\ a_{-3-n} = a_{-n-3} = 0 \\ \vdots \\ a_{n-1} = a_{-n-1} = 0 \end{cases} \]

If \( v(y_1q_1) \geq n + 1 \) then:

\[ f_1(x_1, y_1) = x_1^2 + x_1^{-2}y_1 \sum_{k=0}^{n} a_{-2-k}y_1^k + \cdots + x_1y_1 \sum_{k=0}^{r} a_{1-k}y_1^{k-1} + y_1 \sum_{k=0}^{s} a_{0-k}y_1^k. \]

The normal forms of \( f \) and \( f_1 \) differ only by a shift of indices

\[ \begin{align*}
&f_{a_{-2-k} \rightarrow a_{-2-k}} \\
&f_{a_{-3-n} \rightarrow a_{-n-3}} \\
&\vdots \\
&f_{a_{n-1} \rightarrow a_{-n-1}} \\
&f_{a_{n-k} \rightarrow a_{0-k}}
\end{align*} \]

So if we blow up our \( f \) a second time we have to consider \( f_1(\frac{1}{x_1}, y_1) \) the same set of germs as before with \( f_1(x_1, y_1) \). However the induced action of diffeomorphisms has become even more complicated. But also in this case the \( \mu \)-class of \( f \) depends only on the possibilities for tangencies of \((f_1)^{-1}(O)\) to the exceptional divisors \( C \) and \( C' \):

\[ \begin{array}{ccc}
C & \rightarrow & C' \\
\uparrow & & \downarrow \\
\mu & & \mu'
\end{array} \]

Since \((f_1)^{-1}(O)\) and \( C \) intersect \( C' \) in different points, every \( \mu \)-class of \( f_1 \) is subdivided in the same classes as before, when we blow up once.

(3.5) From this lemma follows the periodicity in the classification of \( \mu \)-classes in \( x^8 + y^{n+1} \). In the case \( n = 3 \) we get the following pattern:

\[ \begin{array}{ccc}
\infty & \rightarrow & \infty \\
\uparrow & & \downarrow \\
(1, 1) & \rightarrow & (1, 1)
\end{array} \]

(3.6) Following Zariski’s definition of equisingularity, all information about the topological type is contained in the ‘resolution-tree’, which can be constructed as follows:

Write down the multiplicities \((m_1, \ldots, m_n)\) of the irreducible components of \( f^{-1}(O) \), blow up once and do the same for the multiplicities of each \( f_i \). Write this as follows:

\[ (m_1, \ldots, m_n) \rightarrow (m'_1, \ldots, m'_n) \rightarrow \cdots \]

Repeat this until you get everywhere one branch of multiplicity one.

**Examples**

\[ \begin{array}{ccc}
x^8 - y^8 & \rightarrow & x^8 + y^8 \\
(1, 1) & \rightarrow & (1, 1) \uparrow \\
(1, 1) & \rightarrow & 1 \\
1 & \rightarrow & 1 \downarrow \\
\end{array} \]

This construction and the following alternative proof of the periodicity-lemma were communicated to me by P. Slodowy.

**Proof.** There is a bijective map between trees of class I and trees of class II:

\[ \begin{array}{ccc}
\phi & : & I \\
\downarrow & & \downarrow \\
\psi & : & II \\
\end{array} \]

This construction and the following alternative proof of the periodicity-lemma were communicated to me by P. Slodowy.
\( \phi \) is defined by blowing up and \( \psi \) by transversal blowing down. The multiplicities of the new branches are the same as before.

Transversal blowing down is always possible: use e.g. the normal form:

\[
f(x, y) - x^n + x^{n-2}A_{n-2}(y) + \cdots + xy^nA_1(y) + y^{n+m}A_0(y)
\]

(3.7) It is possible to start the classification of these resolution-trees and to get in that way the classification of \( \mu \)-classes. We prefer to use mappers and to get the classification as a chain of sets of semi-algebraic sets in some jet-space, since it is then possible to compare the modality of \( f \) and \( f_1 \) (cf. (2.5)).

4. Independence

(4.1) Let \( f \) have multiplicity \( k \) and let \( f_1 \) have a different tangent directions. The \( k \)-jet of \( f \) can be given by a homogeneous polynomial \( f'(f) \) of degree \( k \), which factors into linear forms as follows:

\[
f'(f) = (a_1x + b_1y)^k \cdots (a_nx + b_ny)^k.
\]

From Henset's lemma it follows that we can factor \( f \) (at least over \( \mathbb{C}[x_1, x_2] \)) as \( f = f_1 \cdot \cdots \cdot f_n \) such that each \( f_i \) is given by \( f_i(x, y) = (a_i x + b_i y)^k + p(x, y) \), where \( v(p) > k_i \).

So we can change coordinates such that \( f_i(x, y) = x^{k_i} + p(x, y) \) with \( v(p) > k_i \).

Definitions. Let \( \Pi^n \) be the set of \( \mu \)-classes with multiplicity \( k \). Let \( k_1 \geq k_2 \geq \cdots \geq k_n \geq 1 \) with \( \Sigma k_i = k \). We define \( \Pi^n[k_1, \cdots, k_n] \) to be the subset of \( \Pi^n \), consisting of those \( f \in \Pi^n \) such that \( f \) has \( k_1 \) different tangent directions \( w_1, \cdots, w_{k_1} \) such that the corresponding \( f_1, \cdots, f_n \) have multiplicities \( k_1, \cdots, k_n \).

(4.2) Independence-Lemma. The map \( \Psi: \Pi^n[k_1, \cdots, k_n] \rightarrow \Pi^n[k_1] \times \cdots \times \Pi^n[k_n] \) defined by \( \Psi(f) = (f_1, \cdots, f_n) \) is an isomorphism. Here:

\[
(f_1, \cdots, f_n)(x, y) = (f_1(x, y), \cdots, f_n(x, y)) \oplus k_i = k_i.
\]

Proof. The surjectivity of \( \Psi \) is clear. The injectivity follows from the fact that a \( \mu \)-class is determined by the homomorphism class of its resolution (cf. Zariski [13]).


(b) \( \Pi^n[2, 1] = \Pi^n[2] \times \Pi^n[1] \) : A-series.

(4.3) If we combine the periodicity lemma with the independence lemma we see that singular classes are repeated in several ways. Moreover in the detailed study of the 'period' in \( \Pi^n[n] \) we meet a 1-1-correspondence with earlier treated cases.

This happens e.g. if \( f_1 = x^1 + y^2 \) and \( v(y^2) = n \).

We first apply a coordinate transformation of the type \( (x_1 = x_1 + \alpha y_1) \) such that the coefficient of \( xy^2 \) is equal to zero. If the \( n \)-jet of \( f_1 \) was given by \( (x_1 + \alpha y_1) y_1 \), then, after the transformation, \( v(x^1 + \alpha y_1) > n \) and the corresponding \( \mu \)-classes can be studied in the usual way by the periodicity-lemma. In the remaining cases \( v(y^2) = n \) and \( f_1 \) has two or more tangent directions. The possible \( \mu \)-classes are already treated in an earlier part of the classification, since in each tangent direction the corresponding singularity has multiplicity less than \( n \). Moreover it is not difficult to show that all these possibilities can occur.

Since \( f_1 = x^1 + y^1 \) and \( v(y^1) \geq n \), all tangent directions are transversal to the exceptional divisor. Consequently there is no subdivision, but a 1-1-correspondence between \( \mu \)-classes of \( f_1 \) and \( \mu \)-classes of \( f \).

So the full pattern of singularities of multiplicity \( n \) and with two or more tangent directions is repeated with a jump in \( \mu \) of \( n(n - 1) \).

This pattern becomes a part of the 'period' of \( \Pi^n[n] \) and shall be repeated by the periodicity lemma.

(4.4) Example. \( n = 2: \Pi^n[2] \) : Period:

\[
(A_4) \alert{\text{has multiplicity 2 and 2 tangent directions.}}
\]

\( n = 3: \Pi^n[3] \) : Period:

\[
\Pi^\ell \times \text{diagonal} \rightsquigarrow \text{diagonal}
\]

The \( D \)-series has multiplicity 3 and 3 or 2 tangent directions.

\( n = 4: \Pi^n[4] \) see §5.

5. Classification of singularities of corank \( \equiv 2 \) and multiplicity \( v \equiv 4 \)

(5.1) As an application of the independence lemma and the periodicity lemma we treat the classification of the singularities, mentioned in the title of this paragraph.

We shall only indicate results, since the classification itself is well-known (cf. Arnold [2], [3], [4], Siersma [11], for \( v = 3 \) see also Briançon [5]). We have to consider the following cases:

\( v = 1 \): The function is regular at \( O \), and can be given by \( f(x, y) = x \).

\( v = 2 \): (a) \( \Pi^2[1, 1] \)

(b) \( \Pi^2[2] \)
If $b^2 - 4c = 0$, then $x^2 + bx + y = (b/2)^2 + \cdots - x^2 + \cdots$ (differs respects exc. divisor).

So the detailed study gives the 'period':

\begin{align*}
A_2 & \rightarrow A_3 \\
(A_2) & \rightarrow (A_3)
\end{align*}

The periodicity lemma gives the classes $A_4, A_4, \ldots, A_{3k}, A_{3k+1}, \ldots$.

So in the $\Pi[2]$ we get the following singularity classes:

\begin{align*}
A_2 & \rightarrow A_3 & A_4 & \rightarrow A_5 & A_6 & \rightarrow (A_7) & (A_8)
& \rightarrow (A_9)
A_9 & \rightarrow f(x, y) = t^2 + t^{k+1}.
\end{align*}

In the above pictures the transform of $f^{-1}(0)$ is given after one blowing up.

\begin{align*}
\Pi[1, 1] & \rightarrow \Pi[1, 1] \\
\Pi[2, 2] & \rightarrow \Pi[2, 2] \\
\Pi[3, 1] & \rightarrow \Pi[3, 1]
\end{align*}

Let $f(x, y) = x^2 + p(x, y)$ with multiplicity $\nu(p) \geq 3$.

We blow up once and get: $f_1(x_1, y_1) = x_1^2 + y_1q_1(x_1, y_1)$.

We study the case, that multiplicity $\nu(q_1) \notin 2$, so:

\begin{align*}
f_1(x_1, y_1) & = x_1^2 + y_1x_1 + by_1 + cy_1^2 + \cdots \\
f_1(x, y) & = x^2 + ax' + by' + cy' + \cdots
\end{align*}

The singularity of $f_1$ have been classified before, so they must be of type $A_0$ or $A_1$.

We get the following detailed study:

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>generic $f(x_1, y_1)$</th>
<th>$f(x, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$x_1^2 + y_1$</td>
<td>$x^2 + y^2$</td>
</tr>
<tr>
<td>3</td>
<td>$x_1^2 + y_1^2$</td>
<td>$x^2 + y^2$</td>
</tr>
</tbody>
</table>

A class of type $D_{4k+2}$ can be given by: $f(x, y) = y(x^2 + y^{k+2})$.

\begin{align*}
\Pi[3] & \rightarrow \Pi[3] \\
\Pi[2, 1] & \rightarrow \Pi[2, 1] \\
\Pi[1, 1] & \rightarrow \Pi[1, 1]
\end{align*}

Let $f(x, y) = x^2 + p(x, y)$ with multiplicity $\nu(p) \geq 4$.

We blow up once and get: $f_1(x_1, y_1) = x_1^2 + y_1q_1(x_1, y_1)$.

We study the case, that multiplicity $\nu(q_1) \notin 3$, so:

\begin{align*}
f_1(x_1, y_1) & = x_1^2 + a_1x_1 + b_1x_1y_1 + by_1^2 + c_1x_1^2y_1 + c_2y_1^2 + \cdots \\
f_1(x, y) & = x^2 + ax' + by' + cy'+ \cdots
\end{align*}

The singularities of $f_1$ have been classified earlier, so they must be of type $A_0$ or $A_1$. 

\begin{align*}
A_2 & \rightarrow A_3 \\
(A_2) & \rightarrow (A_3)
\end{align*}
A or D. We get the following detailed study:

<table>
<thead>
<tr>
<th>𝜇</th>
<th>0 Bij</th>
<th>generic f(𝐱ₘ, 𝐲ₙ)</th>
<th>generic f(𝐱, 𝐲)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>𝐲₂</td>
<td>(x_1^2 + y_1)</td>
<td>(A₆) (x^2 + y^2)</td>
</tr>
<tr>
<td>7</td>
<td>𝐲₂</td>
<td>(x_1^2 + xy_1)</td>
<td>(A₇) (x^2 + xy)</td>
</tr>
<tr>
<td>8</td>
<td>𝐲₂</td>
<td>(x_1^2)</td>
<td>(A₈) (x^2)</td>
</tr>
<tr>
<td>10</td>
<td>𝐲₂</td>
<td>(x_1^2 + y_2^2)</td>
<td>(D₂₀) (x_2^2 + y_2^2)</td>
</tr>
<tr>
<td>10+1</td>
<td>𝐲₂</td>
<td>(x_1^2 + x_2y_1 + y_2^2)</td>
<td>(D₂₀) (x_2^2 + x_2y_1 + y_2^2)</td>
</tr>
</tbody>
</table>

The A₄-singularities with \(k \geq 3\) don't occur.

So the detailed study gives the 'period':

\[
\begin{array}{c|c|c}
\mu & 0 & 1 \\
\hline
\text{bij} & \text{bij} & \text{bij} \\
\end{array}
\]

The periodicity lemma gives classes \(E_{10}, E_{13}, E_{14}, J_{20}, J_{21}, \ldots\).

Thus in \(Π^{[3]}\) we get the following pattern of singularity classes:

\[
\begin{array}{c|c|c|c}
\mu & 0 & 1 & 2 \\
\hline
\text{bij} & \text{bij} & \text{bij} & \text{bij} \\
\end{array}
\]

**Remark about modality. First we recall the formula**

\[m(f) = \mu(f) + c(f) - 1\]

**Consider again the case Π^{[3]}.** It is known that \(\mu(E_3) = 6\); \(c(E_3) = 5\) and \(\mu(E_4) = 7\); \(c(E_4) = 6\).

The semi-algebraic set \(A_5\) can be constructed from the semi-algebraic set \(A_3\) by defining inequality (of course one has also to change the defining inequalities). After blowing up we see that also \(E_k\) can be constructed from \(E_4\) by one defining equation.

Next we try to do the same for \(A_5 \to A_3\). Since the \(A_5\)-type doesn't occur we see, after blowing up, that the corresponding defining equation for \(A_5\) must imply the equation for \(D_k\).

So \(c(J_{10}) = c(E_3) + 1\) but \(\mu(J_{10}) = \mu(E_3) + 2\).

So \(m(J_{10}) = m(E_3) + 1\).

In the same way one shows that \(m(J_{11}) = 1, m(E_{12}) = m(E_{13}) = \cdots = 1\) but \(m(J_{20}) = 2\). etc.

In general: \(m(J_{10}) = k\)

\[m(E_k) = m(E_{2k+1}) = m(E_{2k+2}) = k - 1.\]

This way of computing the modality works not only in this example but can be used in general:

\[Π^{[1]} [1, 1, 1] = \frac{1}{Π^{[1]} [1, 1, 1]} \]

We get \(Π^{[2]} [1, 1, 1] = Π^{[2]} [1, 1, 1] \times Π^{[1]} [1, 1, 1] = Π^{[1]} [1, 1, 1] = Π^{[2]} = A\)-series where \(X_{10} = X_{13} \to (A_0, A_0, A_0) \to (A_0, A_0, A_0)

We get the types:

\[Π^{[2]} [2, 1, 1] = Π^{[2]} [2, 1, 1] \times Π^{[2]} [2, 1, 1] = Π^{[2]} = A\)-series where \(X_{10} = X_{11} \to (A_0, A_0, A_0) \to (A_0, A_0, A_0)

A class of type \(X_{12}\) can be given by \(f(x, y) = x^2 + y^2 + y^2 + y^2 + y^2\).

\[Π^{[2]} [2, 2] = Π^{[2]} [2, 2] \times Π^{[2]} [2, 2] \]

We get \(Π^{[2]} [2, 2] = Π^{[2]} [2, 2] \times Π^{[2]} [2, 2] \times \ldots\).

The type \(Y_{10}\) corresponds to \((A_{10}, A_{10}, A_{10})\).

We have a double-\(A\)-series:

\[Π^{[2]} [3, 1] = Π^{[2]} [3, 1] \times Π^{[2]} [3, 1] \]

We get \(Π^{[2]} [3, 1] = Π^{[2]} [3, 1] \times Π^{[2]} [3, 1] \times Π^{[2]} [3, 1] \times Π^{[2]} [3, 1] \times Π^{[2]} [3, 1] \times \ldots\).
The type $Z_{45}$ corresponds to $E_6$ and $Z_{24.6}$ corresponds to $A_9$.

\[
\begin{array}{c|c|c|c|c}
 n & f_1(x_1, y_1) & g_1(x_1, y_1) & h_1(x_1, y_1) & i_1(x_1, y_1) \\
\hline
12 & x_1^4 + y_1 & & & \\
13 & x_1^3 + x_1 y_1 & & & \\
15 & x_1^2 + y_1^3 & & & \\
16 & b_1 = 0 & (A_3) & W_{13} & x_1^2 + y_1^3 \\
17 & c_0 = 0 & (A_3) & W_{13} & x_1^2 + y_1^3 \\
2q & 14 & (A_{3,1}) & W_{24} & (x_1^2 + y_1^3) \\
2q & 15 & (A_{3,1}) & W_{24} & (x_1^2 + y_1^3) \\
\end{array}
\]

The periodicity lemma gives the classes $W_6$, $X_6$, $Y_6$, $Z_6$, etc.

6. Corank 3

(6.1) In the case of corank 3 we need 3 coordinates. The blowing up process now gives $P^2(C)$ as exceptional divisor. The intersections of the branches with the exceptional divisor are now not necessarily isolated but form a 1-dimensional algebraic variety $X$. So the induced functions $f_3(x \in X)$ can have non-isolated singularities.

If the 3-jet of $f$ has no multiple factor, then $f_3$ has only singularities in the multiple points of $X$; in the other points $f_3$ is of type $A_9$.

In the cases we consider most $f_3$ has a singularity-type that has been studied before.

We list the topological classes of $f_3$ that can occur. Next we assume that different intersection points can be treated independently and compare the results with Arnold's list. The lists are identical and this means that we have found, in this way, all the $\mu$-classes for $f$. 

---

Periodicities in Arnold's Lists of Singularities

\[
\begin{array}{c|c|c|c|c|c|c}
 n & f_1(x, y) & g_1(x, y) & h_1(x, y) & i_1(x, y) & j_1(x, y) & k_1(x, y) \\
\hline
16 & b_1 = 0 & (A_3) & W_{13} & x_1^2 + y_1^3 \\
17 & c_0 = 0 & (A_3) & W_{13} & x_1^2 + y_1^3 \\
2q & 14 & (A_{3,1}) & W_{24} & (x_1^2 + y_1^3) \\
2q & 15 & (A_{3,1}) & W_{24} & (x_1^2 + y_1^3) \\
\end{array}
\]
We cannot prove this directly since $\mu$ constant $\leftrightarrow$ resolution constant is not true if $n = 3$.

(6.2) We consider $f(x, y, z)$ with the multiplicity of $f$ equal to 3. The corresponding variety $X \in \mathbb{P}^3(C)$ is given by $f^3(f) = 0$.

We have the following possibilities:

(a) $f = x^3 + y^3 + z^3 + p(x, y, z)$

$X$ is given by $xyz + x^2 + y^2 + z^2 = 0$ and is an elliptic curve without multiple points.

The type of this singularity is called $P_3$.

(b) $f = x^2 + y^2 + z^2 + p(x, y, z)$

$X$ is given by $xyz + x^2 + y^2 = 0$ and has one double point $\sigma$, where

$f + x_0 = x_0y_0 = x_0^2 + y_0 = x_0q_0$.

We get a singularity of type $\Pi^2[2]$, which all occur.

(c) $f = x^2z + y^2 + z^2 + p(x, y, z)$

$X$ is given by $xy^2 + z^2 = 0$ and has one double point $\sigma$ and $\tau$ where

$f + x_0 = x_0y_0 = x_0^2 + y_0 = x_0q_0$.

This singularity is of type $E$ or $f$ and all singularities of $\Pi^2[3]$ occur.

(d) $f = x^2 + x^2p(x, y, z)$

$X$ is given by $xyz + x^2 + z^2 = 0$ and has two double points $\sigma$ and $\tau$, where:

$f + x_0 = x_0y_0 = x_0^2 + y_0 = x_0 q_0$.

Both are singularities of type $A$, and all possible combinations can be found.

\[
\begin{align*}
\{ f_0 \text{ of type } A_{1,3} \} & \leftrightarrow f_0 \text{ of type } R_{1,3,1} \\
\{ f_0 \text{ of type } A_{1,0} \} & \leftrightarrow f_0 \text{ of type } R_{1,3,1}.
\end{align*}
\]

This singularity class is isomorphic to $\Pi^2[2] \times \Pi^2[2] = \Pi^2[4, 2]$.

(e) $f = x^2 + y^2 + p(x, y, z)$

$X$ is given by $x^2 + y^2 + z^2 = 0$ and has one multiple point $\sigma$, where

$f + x_0 = x_0y_0 = x_0^2 + y_0 = x_0^2 + y_0 q_0$.

We get the singularities of $\Pi^2[4]$, which all occur.

\[
\begin{align*}
\{ f_0 \text{ of type } W_0 \} & \leftrightarrow f_0 \text{ of type } S_0 \\
\{ f_0 \text{ of type } W_0 \} & \leftrightarrow f_0 \text{ of type } S_0 \\
\{ f_0 \text{ of type } W_0 \} & \leftrightarrow f_0 \text{ of type } S_0.
\end{align*}
\]

etc., in general:

\[
\begin{align*}
\{ f_0 \text{ of type } W_0 \} & \leftrightarrow f_0 \text{ of type } S_0 \\
\{ f_0 \text{ of type } X_0 \} & \leftrightarrow f_0 \text{ of type } SP_0 \\
\{ f_0 \text{ of type } Y_0 \} & \leftrightarrow f_0 \text{ of type } SR_0 \\
\{ f_0 \text{ of type } Z_0 \} & \leftrightarrow f_0 \text{ of type } SO_0.
\end{align*}
\]

(f) $f = xz + p(x, y, z)$

$X$ is given by $xyz = 0$ and has three multiple points $\sigma$, $\tau$, and $\mu$, where

\[
\begin{align*}
\{ f_0 \} & \leftrightarrow \{ f_0 \} \\
\{ f_0 \} & \leftrightarrow \{ f_0 \} \\
\{ f_0 \} & \leftrightarrow \{ f_0 \}.
\end{align*}
\]

\[\text{case a)} \quad \text{case b)} \quad \text{case c)} \quad \text{case d)} \quad \text{case e)} \quad \text{case f)} \quad \text{case g)}\]
We get three singularities of type $A$ and all combinations can be obtained:

\[
\begin{cases}
\{ f_0 \text{ of type } A_{p-1} \} \\
\{ f_0 \text{ of type } A_{p-1} \} \rightarrow f \text{ of type } T_{p-1} \\
\{ f_0 \text{ of type } A_{p-1} \}
\end{cases}
\]

(g) \( f = x^3 + xz^2 + p(x, y, z) \)

\( X \) is given by \( x^2 + zz^2 = 0 \) and has one multiple point, where

\[
f_0 = x_1^2, x_2^2 + y_0.
\]

The following possibilities can occur:

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>( \text{generic } f(x, y, z) )</th>
<th>( \text{generic } f(x, y, z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>( x_2^3 + x_1^2 z + y_0 )</td>
<td>( (A_0) )</td>
</tr>
<tr>
<td>( \times )</td>
<td>The ( A_1 )-singularity cannot occur</td>
<td>( (A_1) )</td>
</tr>
<tr>
<td>2q+14</td>
<td>( x_2^3 + x_1^2 z_2 + x_1 x_2^3 + z_2^{q+1} )</td>
<td>( U_{2q} )</td>
</tr>
<tr>
<td>2q+15</td>
<td>( x_2^3 + x_1^2 z_3 + x_1 x_2^3 + z_3^{q+1} )</td>
<td>( U_{2q+1} )</td>
</tr>
<tr>
<td>16</td>
<td>( x_2^3 + x_1^2 z_2 + y_0 )</td>
<td>( (A_2) )</td>
</tr>
<tr>
<td>( \times )</td>
<td>The higher order corank 2 singularities don't occur.</td>
<td>( (A_2) )</td>
</tr>
</tbody>
</table>

The next case is: \( x_2^3 + x_1^2 z + y_0 \) with multiplicity \( v(y_0, q) \leq 3 \).

The 3-jet is

\[
x_2^3 + x_1^2 z + y_0 \text{ with } \mu = 2q+14.
\]

We have no restrictions on the type of the 3-jet and find all singularities of type \( P, R, T, O \) and \( S \).

We call the corresponding types of \( f : X \) \( U_{11}, U_{12}, U_{13}, U_{14}, \) and \( U_{15} \).

So the 'period' of \( x^2 + xz^2 \) is given by:

\[
\begin{align*}
& U_{11} \rightarrow U_{12} \rightarrow U_{13} \rightarrow U_{14} \rightarrow U_{15} \rightarrow U_{16} \\
& (A_1) \rightarrow (A_2) \rightarrow (A_3) \\
& (B_1) \rightarrow (B_2) \rightarrow (A_{11}) \\
& (B_3) \rightarrow (B_4) \rightarrow (A_{12}) \\
& (B_5) \rightarrow (B_6) \rightarrow (A_{13}) \\
& (B_7) \rightarrow (B_8) \rightarrow (A_{14}) \\
& \times \hspace{1cm} \text{The } A_1 \text{-singularity cannot occur, so we have:}
\end{align*}
\]

\[
\begin{align*}
& 23 \hspace{1cm} x_1^3 + y_1^2 + x_1 y_1 \\
& \rightarrow \hspace{1cm} (A_3) \\
& x^3 + y^2 \hspace{1cm} (A_3)
\end{align*}
\]
We next get two possibilities: \( c_1 = 0 \) and \( b_2 = 0 \)

<table>
<thead>
<tr>
<th>( 24 )</th>
<th>( c_1 = 0 )</th>
<th>( x^4 + y^4 )</th>
<th>((A_2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-)</td>
<td>Other singularities of type ( A_2 ) don't occur, because of ( x^4 ).</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 24 )</td>
<td>( b_2 = 0 )</td>
<td>( x_1^4 + x_1^2x_2 + y_1^4 )</td>
<td>((D_2))</td>
</tr>
<tr>
<td>( 25 )</td>
<td>( x_1^4 + x_1^2x_2 + y_1^4 )</td>
<td>((D_2))</td>
<td></td>
</tr>
<tr>
<td>( 26 )</td>
<td>( x_1^4 + x_1^2x_2 )</td>
<td>((D_2))</td>
<td></td>
</tr>
<tr>
<td>( 27 )</td>
<td>( x_1(y_1 - x_2^2)^2 + y_1x_1^2 )</td>
<td>((D_2))</td>
<td></td>
</tr>
<tr>
<td>( 20 + k )</td>
<td>( x_1(y_1 - x_2^2)^2 + y_1x_1^2 )</td>
<td>((D_2))</td>
<td></td>
</tr>
<tr>
<td>( 21 + 2\rho )</td>
<td>( x_1(y_1 - x_2^2)^2 + y_1x_1^2 )</td>
<td>((D_2))</td>
<td></td>
</tr>
<tr>
<td>( x )</td>
<td>The ( E_6 ) singularity cannot occur</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| \( 27 \) | \( x_1^4 + x_1^2y_1 + y_1^4 \) | \((E_6)\) |
| \( 28 \) | \( x_1^4 + y_1^4 \) | \((E_6)\) |

| \(-\) | Other singularities of type \( J \) or \( E \) cannot occur |

| \( 29 \) | \( x_1^4 + x_1^2y_1 + y_1^4 \) | \((X_6)\) |
| \( 30 \) | \( x_1^4 + x_1^2y_1 + y_1^4 \) | \((X_6)\) |
| \( 30 \) | \( y_1(x_1 - y_2)(x_1 - 2y_2) + x_1^4 \) | \((X_6)\) |

Periodicities in Arnold's List of Singularities

| \(-\) | Other singularities of type \( X \) cannot occur. |
| \( 31 \) | \( x_1^4 + x_1^2y_1 + y_1^4 \) | \((Y_{n+1})\) |
| \( 26 + k \) | \( x_1^4 + x_1^2y_1 + y_1^4 \) | \((Y_{n+1})\) |

| \(-\) | Other singularities of type \( Y \) and other tangential directions don't occur. |
| \( 31 \) | \( x_1^4 + x_1^2y_1 \) | \((Z_1)\) |
| \( 32 \) | \( x_1^4 + x_1^2y_1 + x_1y_1 \) | \((Z_2)\) |
| \( 33 \) | \( x_1^4 + x_1^2y_1 + y_1^4 \) | \((Z_3)\) |

| \(-\) | Also all other singularities of type \( Z \) occur. |
| \( 32 \) | \( x_1^4 + y_1^4 \) | \((W_{n+1})\) |

| \(-\) | Other singularities of type \( W \) don't occur. |

Next \( x(y_1y_2) = 5 \) and because of (4.3) we find all singularities of \( \Pi - \Pi[5] \). So the 'period' is:

\[
\begin{align*}
\text{\( x_1^4 + x_1^2y_1 + y_1^4 \)} & \quad \text{\( x_1^4 + x_1^2y_1 + y_1^4 \)} \\
\text{\( x_1^4 + x_1^2y_1 \)} & \quad \text{\( x_1^4 + x_1^2y_1 \)} \\
\text{\( x_1^4 \)} & \quad \text{\( x_1^4 \)}
\end{align*}
\]

Next, the diagram shows the relationships between the singularities and their periodicities.
8. Morsifications and intersection forms

The general reference for this paragraph is A. Campol [1].

(8.1) A morsification $h$ of $f$ is a nearby germ $h$, with only non-degenerate critical points.

A. Campol constructed for every $\mu$-class of $f: \mathbb{C}^n \to \mathbb{C}$ a morsification with real critical points, using the blowing up construction.

The intersection $h^{-1}(O) \cap \mathbb{R}^2$ consists of a curve $C$ with only double points and:

$(\text{number of double points}) + (\text{number of regions}) = \mu.$

The vanishing cycles and their intersections are determined by the curve $C$.

**Example.** (Eu)

We simplify this picture to

In the following pictures (and those of table III) one has to replace the multiple intersecting lines by lines in general position, as follows:

(8.2) The periodicity of the classification induces a periodicity of morsifications and intersection forms.

**Examples**

$$
\begin{array}{cccc}
D_4 & I_{1,0} & I_{3,0} & I_{3,0} \\
D_4 & I_{1,0} & I_{3,0} & I_{3,0} \\
\end{array}
$$

(one has to contract along the dotted line to get the next picture.)

$$
\begin{array}{cccc}
D_{4,4} & I_{1,0} & I_{3,0} & I_{3,0} \\
D_{4,4} & I_{1,0} & I_{3,0} & I_{3,0} \\
I_{1} & I_{14} & I_{30} & \text{etd.} \\
I_{1} & I_{14} & I_{30} & \text{etd.} \\
E_6 & E_{11} & E_{20} & \text{etd.} \\
\end{array}
$$
TABLE III. List of monodromies.

<table>
<thead>
<tr>
<th>A</th>
<th>A_1</th>
<th>A_2</th>
<th>A_3</th>
<th>A_4</th>
<th>A_{n-1}</th>
<th>A_{n+1}</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>D_2</td>
<td>D_3</td>
<td>D_4</td>
<td>D_5</td>
<td>D_6</td>
<td>D_{n+1}</td>
</tr>
<tr>
<td>E</td>
<td>E_6</td>
<td>E_7</td>
<td>E_8</td>
<td>E_9</td>
<td>E_{10}</td>
<td>J_{1,0}</td>
</tr>
<tr>
<td>F</td>
<td>F_{12}</td>
<td>F_{13}</td>
<td>F_{14}</td>
<td>F_{15}</td>
<td>F_{16}</td>
<td>J_{1,1}</td>
</tr>
<tr>
<td>G</td>
<td>G_{12}</td>
<td>G_{13}</td>
<td>G_{14}</td>
<td>G_{15}</td>
<td>G_{16}</td>
<td>J_{1,2}</td>
</tr>
<tr>
<td>H</td>
<td>H_{12}</td>
<td>H_{13}</td>
<td>H_{14}</td>
<td>H_{15}</td>
<td>H_{16}</td>
<td>D_{1,4}</td>
</tr>
</tbody>
</table>

Diagram showing various monodromies associated with different classifications and indices.
MIXED HODGE STRUCTURE ON THE VANISHING COHOMOLOGY

J. H. M. Steenbrink*

Abstract

We construct a mixed Hodge structure on the cohomology of the Milnor fiber associated to an isolated singularity of a complex hypersurface. We determine the relations it has with monodromy, intersection form and local cohomology.


Keywords: mixed Hodge structure, vanishing cycles, monodromy, De Rham cohomology, Hodge filtration, weight filtration, intersection form.

Introduction

Let $P \in \mathbb{C}^{n}$ with $P(0) = 0$. Assume that $0 \in \mathbb{C}^{*}$ is a critical point of $P$. Denote $B$ the open ball in $\mathbb{C}^{*}$ with center 0 and radius $\varepsilon > 0$. There exists $\eta > 0$ such that $0 < \eta < \eta$ implies that $B = \mathbb{C}^{*} \setminus B$ is a complex manifold. In this paper we follow a suggestion of Deligne and construct a mixed Hodge structure on the cohomology of $B$, (the vanishing cohomology) in the case that $P$ has an isolated critical point at 0.

Let $S$ be the disk with center 0 and radius $\eta$. Denote $X = \mathbb{C}^{*} \setminus B$. Let $\mu: X \to X'$ be a resolution of $P$, i.e. a proper map which is an isomorphism outside $\mu^{-1}(0)$ such that $(\mu P)^{-1}(0)$ is a union of smooth divisors on $X$ with normal crossings. Let $\sigma$ be the least common multiple of the multiplicities occurring in the fiber $(\mu P)^{-1}(0)$. Let $S$ be the disk with radius $\eta^{1/\sigma}$ and define $\sigma: S \to S$ by $\sigma(t) = t^{\sigma}$. Let $X$ be the normalization of the fiber product $X \times_{S} S$ and let $\nu: \tilde{X} \to X$ be the natural map. Let $D = (\nu P)^{-1}(0)$ and denote $D_{0}, \ldots, D_{n}$ its irreducible components. The cohomology groups of the $D_{i}$

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