ON IRREGULARITY AND GEOMETRIC GENUS OF ISOLATED SINGULARITIES

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1. Introduction. Let $(V, 0)$ be a normal surface singularity. Wahlreich first defined an invariant geometric genus $p_g$ for the singularity $(V, 0)$. It turns out that this is an important invariant for the theory of normal surface singularities. In this paper we shall introduce another invariant called irregularity $q$ of the singularity $(V, 0)$. This invariant is interesting for the following reason. It is a long-term conjecture that Gorenstein surface singularities are not rigid, i.e. $\dim T^1_1 \geq 1$ (cf. §3 for the definition of $T^1_1$). In the case of Gorenstein surface singularities this irregularity actually gives a lower bound for $\dim T^1_1$ (cf. Remark 3.2). Therefore it is of great interest to understand this irregularity more closely.

In §2 we give formulae for the irregularity in case $(V, 0)$ is a hypersurface singularity (cf. Theorem 2.1) or a singularity with $C^*$-action (cf. Theorem 2.2). An example will be computed explicitly by using these formulae.

In §3 we give lower estimate for $\dim T^1_1$ in terms of geometric genus (cf. Corollary 3.5). We also have lower estimate for irregularity in terms of geometric genus (cf. Theorem 3.7). It is well known that Gorenstein surface singularities with geometric genus equal to zero are rational double points $A_1, D_4, E_6, E_7, E_8$. These singularities admit a $C^*$-action. The natural question is to classify Gorenstein singularities with $C^*$-action and irregularity equal to zero. Theorem 3.8 gives a very simple answer to this question. In most of the above results, the fact that $(V, 0)$ admits a $C^*$-action plays an important role.

The purpose of this paper is two-fold. On the one hand, we describe some of our previous work [11, 12, 13]. On the other hand, we study irregularity for those singularities which do not admit $C^*$-action (cf. Theorem 3.9). This is a beginning step towards classifying hypersurface singularities with irregularity equal to zero.

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2. Irregularity and geometric genus. Let \( (V, 0) \) be a normal Stein analytic space of dimension \( n (n \geq 2) \) with 0 as its only isolated singularity. Let \( \sigma : M \to V \) be a resolution of the singularity of \( V \) with exceptional set \( A = \sigma^{-1}(0) \). We define invariants \( \sigma^{i,i} \), \( 0 \leq i \leq n \), of the singularity 0 to be \( \dim \Gamma (M - A, O_M^{(i)}) / \Gamma (M, O_M^{(i)}) \).

Let \( D_1 \) be the 0th direct image sheaf \( \sigma_* D_1 \). By Grauert's direct image theorem, \( D_1 \) is a coherent sheaf. Let \( \theta : V - \{ 0 \} \to V \) be the inclusion map. Then the 0th direct image sheaf \( D_2 := \theta_* D_2^{(-0)} \) is coherent (cf. [7]). Hence the quotient sheaf \( D_2 / D_1 \) is coherent and supported on \( \{ 0 \} \). \( x^{(i)} \) is exactly \( \dim D_2 / D_1 \). Therefore the invariants \( x^{(i)} \), \( 1 \leq i \leq n \), are indeed invariants of isolated singularities. Wahlreich defined geometric genus \( p_g \) of the singularity \( (V, 0) \) to be \( \dim H^{n-2} (M, \mathcal{O}_M) \). It is proved in [2, 10] that \( p_g = x^{(n-1)} \). Hence it is quite natural to define irregularity \( q \) of the singularity \( (V, 0) \) to be \( x^{(n-1)} \). In this section we shall assume that \( (V, 0) \) is a surface singularity. The following theorem which relates the irregularity \( q \) and \( \dim H^1 (M, \Omega^1) \) was proved in [11].

**Theorem 2.1.** Let \( f(x, y, z) \) be holomorphic in \( N \), a Stein neighborhood of \( (0, 0, 0) \) with \( f(0, 0, 0) = 0 \). Let \( V = \{ (x, y, z) \in N : f(x, y, z) = 0 \} \) be as its only singular point. Let

\[
\mu = \dim C[[x, y, z]]/\left\{ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\}
\]

and

\[
\tau = \dim C[[x, y, z]]/\left\{ f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\}.
\]

Let \( \sigma : M \to V \) be a resolution of \( V \) and \( A = \sigma^{-1}(0) \), then

\[
q + \dim H^1 (M, \Omega^1) = \mu + 1 + \chi (\sigma(A)) + 2 \dim H^1 (M, \mathcal{O}_M)
\]

where \( q \) is irregularity of the singularity and \( \chi (\sigma(A)) \) is the topological Euler characteristic of \( A \).

The following theorem gives us an explicit way to compute the irregularity and geometric genus of any isolated singularity \( (V, 0) \) with \( C^* \)-action.

**Theorem 2.2.** Suppose \( V \subseteq C^n \) is an analytic variety of dimension two with the origin as its only isolated singularity. Suppose \( \sigma \) is a \( C^* \)-action leaving \( V \) invariant, defined by

\[
\sigma (t, (z_1, \ldots, z_n)) = (t^n z_1, \ldots, t^n z_n),
\]

\( n \) is a positive integer. Let \( \psi : \mathbb{C}^n \to \mathbb{C}^n \) be defined by \( \psi (z_1, \ldots, z_n) = (t^n z_1, \ldots, t^n z_n) \) and let \( V' = \psi^{-1} (V) \) be the cone over \( V \). Then \( V'' \) has a natural \( C^* \)-action defined by \( \sigma (t, (z_1, \ldots, z_n)) = (t^{n-1} z_1, \ldots, t^{n-1} z_n) \) and the induced map \( \psi' : V'' \to V' \) commutes with the \( C^* \)-action. Let \( A' = V' - \{ 0 \} \). Let \( N' \) be the universal subbundle (i.e. dual of the hyperplane bundle) of \( P^{n-1} \) restricted to \( A' \).

Identify \( \mathbb{Z}_n \) with the group of \( qz \)-roots of 1, \( G = \mathbb{Z}_n \oplus \cdots \oplus \mathbb{Z}_n \) acts on \( V'' \) by coordinatewise multiplication. \( G \) also acts on \( A' \) and \( N' \). Let \( \sigma : A' = A' \)

normalization and \( N' = p_t (N') \), pull back of \( N' \) by \( \sigma \). Then the irregularity \( q \)

and the geometry genus \( p_g \) of the singularity \( (V, 0) \) can be computed by the following formulae.

\[
q = \sum_{i=1}^{n} \dim \Gamma (A^n, K_{A^n} N^{m+1})^G \quad \text{if} \quad g'' = 1,
\]

\[
q = 0 \quad \text{if} \quad g'' > 1.
\]

\[
p_g = \sum_{i=1}^{n} \dim \Gamma (A^n, K_{A^n} N^{m+1})^G \quad \text{if} \quad g'' = 1,
\]

\[
p_g = 0 \quad \text{if} \quad g'' > 1.
\]

where \( g'' \) is the genus of \( A' \), \( K_{A^n} \) is the canonical line bundle of \( A'' \) and \( \Gamma (A^n, K_{A^n} N^{m+1})^G \) denotes the \( G \)-invariant sections.

The proof of the above theorem we refer the reader to [11].

**Corollary 2.3.** Suppose \( V \subseteq C^n \) is an analytic variety of dimension two which admits a \( C^* \)-action. Let \( G, A' \) and \( K_{A^n} \) be defined as in Theorem 2.2. Then

\[
P_g = q + \dim \Gamma (A^n, K_{A^n} G).
\]

In particular

\[
P_g > q.
\]

**Corollary 2.4.** \( q \) is equal to zero for any 2-dimensional rational singularity with \( C^* \)-action.

In fact, for any 2-dimensional rational singularity \( q \) is equal to zero. This was proved by Pinkham [4] and Wahl [9]. In view of formula (2.0), if we know \( \dim H^1 (M, \mathcal{O}_M) \) then the irregularity \( q \) of the singularity can be computed explicitly. Recall that in [3] Narasimhan proved that given a finitely generated abelian group \( G \) and integers \( k \geq 1, n \geq k + 3 \), there is a Runge domain \( D \) in \( C^n \) with \( H^1 (D, \mathcal{O}_D) = 0 \). However, for a strongly pseudoconvex 2-dimensional manifold we have the following [11].

**Theorem 2.5.** Let \( M \) be a two-dimensional strongly pseudoconvex manifold in which the exceptional set may admit arbitrary singularities. Then \( \dim H^1 (M, \mathcal{O}_M) \geq b_2 \) where \( b_2 \) is the second betti number of the tubular neighborhood of the exceptional set \( A \) of \( M \).

**Theorem 2.6.** Let \( M \) be a two-dimensional strongly pseudoconvex manifold with a nonsingular Riemann surface \( A \) of genus \( g \) as its maximal compact analytic set. Then \( \dim H^1 (M, \mathcal{O}_M) \geq \dim \Gamma (M, \mathcal{O}_M \otimes \Theta_{\text{reg}} (n_0, A)) \) where \( n_0 = \max ((2 - 2g) / (A \cdot A) + 1, 2), (2 - 2g) / (A \cdot A) < 1 \), then \( \dim H^1 (M, \mathcal{O}_M) = 1 \). In particular, if \( A \) is a rational curve or an elliptic curve, then \( \dim H^1 (M, \mathcal{O}_M) = 1 \). Hence irregularity for any simple elliptic singularity is equal to zero.
2. Irregularity and geometric genus. Let \((V, \mathcal{O})\) be a normal Stein analytic space of dimension \(n (n \geq 2)\) with 0 as its only isolated singularity. Let \(\pi: M \to V\) be a resolution of the singularity of \(V\) with exceptional set \(A = \pi^{-1}(0)\). We define the invariants \(s^{(i)}, 0 \leq i \leq n\), of the singularity 0 to be dim \(\Gamma(M - A, \mathcal{O}_M/\mathcal{O}_A)\).

Let \(\mathcal{O}_A\) be the 0th direct image sheaf of \(\mathcal{O}_M\). By Grauert’s direct image theorem, \(\mathcal{O}_A\) is a coherent sheaf. By \(\theta: V - \{0\} \to V\) be the inclusion map. Then the 0th direct image sheaf \(\mathcal{O}_A = \mathcal{O}_A|_{V - \{0\}}\) is coherent (cf. [7]). Hence the quotient sheaf \(\mathcal{O}_M\mathcal{O}_A\) is coherent and supported on \(\{0\}\). \(s^{(i)}\) is exactly dim \(\mathcal{O}_A/\mathcal{O}_A\).

Therefore the invariants \(s^{(i)}, 1 \leq i \leq n\), are indeed invariants of isolated singularities. Wahlreich defined geometric genus \(p_g\) of the singularity (\(V, 0\)) to be dim \(H^{n-1}(M, \mathcal{O}_M)\). It is proved in [2, 10] that \(p_g = s^{(n)}\). Hence it is quite natural to define irregularity \(q\) of the singularity (\(V, 0\)) to be \(s^{(n-1)}\). In this section we shall assume that (\(V, 0\)) is a surface singularity. The following theorem which relates the irregularity \(q\) and dim \(H^1(M, \mathcal{O}_M)\) was proved in [11].

**Theorem 2.1.** Let \(f(x, y, z)\) be holomorphic in \(N\), a Stein neighborhood of \((0, 0, 0)\) with \(f(0, 0, 0) = 0\). Let \(V = \{(x, y, z) \in N; f(x, y, z) = 0\}\) have \((0, 0, 0)\) as its only singular point. Let

\[
\mu = \dim C[[x, y, z]]/\left(\frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \frac{\partial f}{\partial z}\right)
\]

and

\[
\tau = \dim C[[x, y, z]]/\left(\frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \frac{\partial f}{\partial z}\right).
\]

Let \(\pi: M \to V\) be a resolution of \(V\) and \(A = \pi^{-1}(0, 0, 0, 0)\).

\[
q = \dim H^1(M, \mathcal{O}_M) = \tau - (\mu + 1) + \chi(c_t, A) + 2 \dim H^1(M, \mathcal{O}_M)
\]

where \(q\) is irregularity of the singularity and \(\chi(c_t, A)\) is the topological Euler characteristic of \(A\).

The following theorem gives us an explicit way to compute the irregularity and geometric genus of any isolated singularity (\(V, 0\)) with \(C\)-action.

**Theorem 2.2.** Suppose \(V \subset C^n\) is an analytic variety of dimension two with the origin as its only isolated singularity. Suppose \(\sigma\) is a \(C\)-action leaving \(V\) invariant, defined by

\[
\sigma(t, z_1, \ldots, z_n) = (t^q_1, \ldots, t^q_nz_n),
\]

\(q_i\)'s are positive integers. Let \(\psi: C^n \to C^n\) is defined by \(\psi(z_1, \ldots, z_n) = (z_1^q, \ldots, z_n^q)\) and let \(V' = \psi(V)\) be the cone over \(V\). Then \(V'\) has a natural \(C\)-action defined by \(\sigma(t, z_1, \ldots, z_n) = (t^q_1z_1, \ldots, t^q_nz_n)\). Let \(\Lambda' = (V' - \{0\})/C^n \cong P^{n-1}\). Let \(\Lambda\) be the universal subbundle (i.e. dual of the hyperplane bundle of \(P^{n-1}\) restricted to \(A\)). Identify \(Z_q\) with the group of \(q\)-th roots of 1. \(G = Z_q \oplus \cdots \oplus Z_{q^n}\) acts on \(V'\) by coordinatewise multiplication. \(G\) also acts on \(A\) and \(\Lambda\). Let \(\tau: A' = A\) be the normalization and \(\Lambda' = \pi'(\Lambda')\), the pull back of \(\Lambda'\) by \(\tau\). Then the irregularity \(q\) and the geometric genus \(p_g\) of the singularity (\(V, 0\)) can be computed by the following formulas.

\[
\begin{align*}
q &= \sum_{a=1}^{\infty} \dim \Gamma(A^n, K_{A^n} - a\Lambda')^G, & \text{if } g^* = 1, \\
&= 0, & \text{if } g^* < 1, \\
&= \sum_{a=1}^{\infty} \dim \Gamma(A^n, K_{A^n} - a\Lambda')^G, & \text{if } g^* > 1, \\
&= 0, & \text{if } g^* = 0.
\end{align*}
\]

**Corollary 2.3.** Suppose \(V \subset C^n\) is an analytic variety of dimension two which admits a \(C\)-action. Let \(G, A^n\) and \(K_{A^n}\) be defined as in Theorem 2.2. Then

\[
P_g = q + \dim \Gamma(A^n, K_{A^n} - \Lambda').
\]

In particular

\[
P_g > q.
\]

**Corollary 2.4.** \(q\) is equal to zero for any 2-dimensional rational singularity with \(C\)-action.

In fact, for any 2-dimensional rational singularity \(q\) is equal to zero. This was proved by Pinkham [4] and Wahl [9]. In view of formulas (2.0), if we know \(\dim H^1(M, \mathcal{O}_M)\) then the irregularity \(q\) of the singularity can be computed explicitly. Recall that in [3] Narasimhan proved that given a finitely generated abelian group \(G\) and integers \(k \geq 1, n \geq k + 3\), there is a Runge domain \(D\) in \(C^n\) with \(H_\infty(D, \mathcal{O}) = G\). However, for a strongly pseudoconvex 2-dimensional manifold we have the following [11].

**Theorem 2.5.** Let \(M\) be a two-dimensional strongly pseudoconvex manifold in which the exceptional set may admit arbitrary singularities. Then \(\dim H^1(M, \mathcal{O}_M) \geq b_2\) where \(b_2\) is the second betti number of the tubular neighborhood of the exceptional set \(A\) of \(M\).

**Theorem 2.6.** Let \(M\) be a two-dimensional strongly pseudoconvex manifold with a nonsingular Riemann surface \(A\) of genus \(g\) as its maximal compact analytic set. Then \(\dim H^1(M, \mathcal{O}_M) = \dim \Gamma(M, \mathcal{O}_M) - \delta_{\infty}(\mathcal{O}_A)\) where \(n_0 = \max \{[(2 - 2g)/A - A]\} + 1, 2\). If \((2 - 2g)/A - A < 1/2\), then \(\dim H^1(M, \mathcal{O}_M) = 1\). In particular, if \(A\) is a rational curve or an elliptic curve, then \(\dim H^1(M, \mathcal{O}_M) = 1\). Hence irregularity for any simple elliptic singularity is equal to zero.
The proof of Theorem 2.6 can be found in [11].
If \( M \) is a resolution of 2-dimensional rational singularity, then J. Wahl [9] has proved the following theorem.

**Theorem 2.7.** Let \( M \) be a two-dimensional strongly pseudoconvex manifold \( M \) with \( A \) as its maximally compact analytic set. Let \( b_2 \) be the second betti number of \( A \). If \( H'(M, \mathbb{Q}) = 0 \), then \( b_2 = \dim H'(M, \mathbb{Q}) \).

**Example 2.8.** Let \( V = \{ z^2 = y(x^4 + y^6) \} \subseteq \mathbb{C}^3 \) as in Theorem 2.2. \( V \) admits a \( C^* \)-action \( \sigma: \mathbb{C}^* \times V \to V \),
\[ \sigma((r, x, y, z)) = (r^2 x, r^2 y, r^2 z) \]
Then \( V' = \{ (x', y', z'): x'^{12} y'^{13} + y'^{14} z'^{14} = 0 \} \). Identify \( Z_1, Z_2 \) and \( Z_3 \) with the groups of 3rd roots, 2nd roots and 7th roots of 1 respectively. \( G = Z_1 \oplus Z_2 \oplus Z_3 \) acts on \( V' \) by coordinate multiplication. Let \( A' \) be the curve defined by \( x'^{12} y'^{13} + y'^{14} z'^{14} = 0 \) in \( \mathbb{C} P^2 \). In \( (\theta, x, y, z) \) coordinate patch, equation of \( A' \) is
\[ x^2 + y^4 - z^4 = 0 \]
Clearly, any holomorphic one forms on \( A' \) are of the forms \( P(y_1, z_1) dy_1/14z_1^{13} \) where \( P \) is a polynomial of degree \( \leq 11 \). Let \( g_1(0, 1, 1), g_2(1, 1, 1) \) and \( g_3 = (1, 1, 1) \) be the three generators in \( G \), where \( \xi = 2\pi i/3, \eta = -1 \) and \( \zeta = 2\pi i/7 \). Then the actions by \( g_1, g_2 \) and \( g_3 \) look like
\[ (x, y, z) \mapsto (\xi x, y, z) \mapsto (x, \eta y, z) \mapsto (x, y, z) \mapsto (x, y, z) \]
In \( (\theta, x, y, z) \) coordinate patch
\[ \omega = \sum_{j+k \leq 11} P(y_1, z_1) dy_1/14z_1^{13} = \sum_{j+k \leq 11} a_{jk} y_1^j z_1^k dy_1 \]
\[ \xi \mapsto g_1 \ast \sigma = \frac{\sum_{j+k \leq 11} a_{jk} \xi^{-j-k} y_1^j z_1^k dy_1}{14z_1^{13}} \]
\[ = \sum_{j+k \leq 11} a_{jk} \xi^{12-j-k} y_1^j z_1^k dy_1 \]
\[ = \frac{\sum_{j+k \leq 11} a_{jk} y_1^j z_1^k \xi^{12-j-k} dy_1}{14z_1^{13}} \]
Hence \( \omega \) is invariant under \( g_1 \) if and only if \( a_{jk} = 0 \) for \( 12 - j - k = 3, 6, 9, 12 \), i.e. \( \omega \) is invariant under \( g_1 \) if and only if \( \omega \) is one of the following forms.
\[ \omega_1 = \frac{a_{00} dy_1}{14z_1^{13}}, \quad \omega_2 = \frac{\sum_{j+k \leq 11} a_{jk} y_1^j z_1^k dy_1}{14z_1^{13}} \]
\[ \omega_3 = \frac{a_{00} dy_1}{14z_1^{13}}, \quad \omega_4 = \frac{\sum_{j+k \leq 11} a_{jk} y_1^j z_1^k dy_1}{14z_1^{13}} \]

Hence (i) \( \omega_1 \) is invariant under \( g_2 \).
(ii) \( \omega_2 \) is invariant under \( g_2 \) if and only if \( \omega_2 \) is one of the forms
\[ \omega_2 = \frac{a_{00} y_1^2 z_1^2 dy_1}{14z_1^{13}}, \quad \omega_3 = \frac{a_{00} y_1^3 dy_1}{14z_1^{13}} \]
(iii) \( \omega_3 \) is invariant under \( g_2 \) if and only if \( \omega_3 \) is one of the forms
\[ \omega_3 = \frac{a_{00} y_1^4 z_1^4 dy_1}{14z_1^{13}}, \quad \omega_4 = \frac{a_{00} y_1^5 dy_1}{14z_1^{13}} \]
(iv) \( \omega_4 \) is invariant under \( g_2 \) if and only if \( \omega_4 \) is one of the forms
\[ \omega_4 = \frac{a_{00} y_1^6 z_1^6 dy_1}{14z_1^{13}}, \quad \omega_5 = \frac{a_{00} y_1^7 dy_1}{14z_1^{13}} \]
It is easy to see that among \( \omega_1, \omega_2, \omega_3, \omega_4, \omega_6, \omega_7, \omega_8, \omega_9, \omega_{10}, \omega_{11}, \omega_{12}, \omega_{13}, \omega_{14} \)
\[ \omega_{14} = \frac{a_{00} y_1^8 dy_1}{14z_1^{13}} \]
\[ \omega_{14} = \frac{a_{00} y_1^9 dy_1}{14z_1^{13}} \]
is invariant under the action of \( g_3 \). Hence
\[ \Gamma(A', K_{\mathfrak{m}})^G = \left\{ \frac{a_{00} y_1^3 dy_1}{14z_1^{13}} : a_{00} \in \mathbb{C} \right\} \]
By Theorems 2.1 and 2.2 we have
\[ \dim H'(M, \mathbb{Q}) = \chi(A) - 1 + \dim H'(M, \mathbb{Q}) + \dim \Gamma(A', K_{\mathfrak{m}})^G \]
\[ = (3 + 1 - 2) - 1 + 2 + 1 = 4 \]
The proof of Theorem 2.6 can be found in [11].
If $M$ is a resolution of 2-dimensional rational singularity, then J. Wahl [9] has proved the following theorem.

**Theorem 2.7.** Let $M$ be a two-dimensional strongly pseudoconvex manifold with $A$ as its maximally compact analytic set. Let $b_2$ be the second betti number of $A$. If $H^1(M, \mathcal{O}) = 0$, then $b_2 = \dim H^1(M, \mathcal{O})$.

**Example 2.8.** Let $V = \{x^2 = y(x^4 + y^6)\} \subseteq \mathbb{C}^3$ as in Theorem 2.2. $V$ admits a $\mathbb{C}^*$-action $\sigma : \mathbb{C}^* \times V \to V$,
\[
(\lambda, (x, y, z)) \mapsto (\lambda^2 x, \lambda^2 y, \lambda^2 z).
\]
Then $V' = ((x', y', z') : x'^2 y'^2 + y'^4 - z'^4 = 0)$. Identify $\mathbb{Z}_3$, $\mathbb{Z}_2$ and $\mathbb{Z}_2$ with the groups of 3rd roots, 2nd roots and 7th roots of 1 respectively. $G = \mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ acts on $V'$ by coordinate multiplication. Let $\mathcal{A'}$ be the curve defined by $x'^2 y'^2 + y'^4 - z'^4 = 0$ in $\mathbb{C}P^2$. In $(\mathfrak{h}, y, z)$ coordinate patch, equation of $\mathcal{A'}$ is
\[
y^2 + y^4 - z^4 = 0.
\]
Clearly, any holomorphic one forms on $\mathcal{A'}$ are of the forms $P(y, z) \frac{dy}{14z^3}$, where $P$ is a polynomial of degree $\leq 11$. Let $g_1(1, 1, 1)$, $g_2 = (1, 1, 1)$ and $g_3 = (1, 1, 1)$ be the three generators in $G$, where $\xi = 2\pi i/3$, $\eta = -1$ and $\xi = 2\pi i/7$. Then the actions by $g_1$, $g_2$ and $g_3$ look like
\[
(\mathfrak{h}, y, z) \mapsto (\mathfrak{h}, y, z) \mapsto (\mathfrak{h}, y, z) \mapsto (\mathfrak{h}, y, z).
\]
In $(\mathfrak{h}, y, z)$ coordinate patch
\[
\omega = \sum_{j+k \leq 11} b_j y^j z^k \frac{dy}{14z^3} = \sum_{j+k \leq 11} b_j y^j z^k \frac{dy}{14z^3},
\]
\[
\omega_{g_1} = \sum_{j+k \leq 11} b_j y^j z^k \frac{dy}{14z^3},
\]
\[
\omega_{g_2} = \sum_{j+k \leq 11} b_j y^j z^k \frac{dy}{14z^3},
\]
\[
\omega_{g_3} = \sum_{j+k \leq 11} b_j y^j z^k \frac{dy}{14z^3}.
\]
Hence $\omega$ is invariant under $g_1$, $g_2$, and $g_3$. Therefore, $\omega$ is invariant under $G$.

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\[
\omega_3 = \frac{\sum_{j+k} a_{jk} y^{j+k} + dy}{14z^3},
\]
\[
\omega_3 = \frac{\sum_{j+k} a_{jk} y^{j+k} + dy}{14z^3}.
\]

Hence (i) $\omega_3$ is invariant under $g_2$.

(ii) $\omega_2$ is invariant under $g_2$ if and only if $\omega_2$ is one of the forms
\[
\omega_2 = \frac{a_{j} y^{j} + dy}{14z^3},
\]
\[
\omega_2 = \frac{a_{j} y^{j} + dy}{14z^3}.
\]

(iii) $\omega_3$ is invariant under $g_2$ if and only if $\omega_3$ is one of the forms
\[
\omega_3 = \frac{a_{j} y^{j} + dy}{14z^3},
\]
\[
\omega_3 = \frac{a_{j} y^{j} + dy}{14z^3}.
\]

(iv) $\omega_4$ is invariant under $g_2$ if and only if $\omega_4$ is one of the forms
\[
\omega_4 = \frac{a_{j} y^{j} + dy}{14z^3},
\]
\[
\omega_4 = \frac{a_{j} y^{j} + dy}{14z^3}.
\]

It is easy to see that among $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8, \omega_9, \omega_{10}, \omega_{11}, \omega_{12}, \omega_{13},$ $\omega_{14}$ and $\omega_{15}$, only
\[
\omega_2 = \frac{a_{j} y^{j} + dy}{14z^3},
\]
is invariant under the action of $g_3$. Hence
\[
G(A, X^*) G = \left\{ \alpha, y^j + dy : \alpha \in \mathbb{C} \right\}.
\]

By Theorems 2.1 and 2.2 we have
\[
\dim H^1(M, \mathcal{O}) = \chi_f(A) + 1 + \dim H^1(M, \mathcal{O}) + \dim G(A, X^*) G = (3+1-2) - 1 + 2 + 1 = 4.
\]
and
\[ \dim \Gamma(M \setminus A, \Omega^1) / \Gamma(M, \Omega^1) = \dim H^1(M, \mathcal{O}) - \dim \Gamma(A^n, K_{n'})^G \]
\[ = 2 - 1 = 1. \]

3. Lower estimate of \( \dim T^1 \). Let \( V \subset C^n \) be a local complex space. A deformation of \( V \) is a flat map \( \pi: X \rightarrow T \) of local complex spaces, together with an isomorphism \( \pi^{-1}(0) \cong V \). Let \( I = \mu_1, \ldots, \mu_n \) be the ideal in \( \mathcal{E}_{C^n} \) defining the local complex space \( V \subset C^n \). Thus \( \mathcal{E}_V / I \). If we take as parameter the (one point) space \( T = \text{spec} C(\mathcal{E}_V / I) \), then a deformation \( X \rightarrow T \) of \( V \) is a first-order infinitesimal deformation of \( V \). \( X \) will be given by equations
\[ f_\mu(x) + e_\mu(x) = 0 \]
and condition for flatness is simply that \( g_\mu = 0 \), determining elements of the normal sheaf \( N_V = \text{Hom}_{\mathcal{E}_V}(L, \mathcal{E}_V) \). The deformation \( X \rightarrow T \) is trivial if and only if there is an automorphism \( x_i \rightarrow x_i + \delta_i(x) \) of \( C^n \times T \) over \( T \) such that \( (f_\mu(x) + e_\mu(x)) \) and \( (f_\mu + g_\mu(x)) \) determine the same ideal in \( \mathcal{E}_{C^n \times T} \); in other words
\[ \sum_\mu \frac{\partial f_\mu}{\partial x_\mu}(x) = g_\mu(x) \quad (\text{mod } I). \]

Now there is a homomorphism \( \rho: \mathcal{E}_{C^n} \rightarrow N_V \) (\( \Theta \) denoting tangent sheaf) defined by mapping the vector field \( \Theta = \sum_\mu \delta_i(x)\partial / \partial x_\mu \) to the homomorphism \( f_\mu \rightarrow \Theta f_\mu \) in \( N_V \), and the element \( g \in \mathcal{E}_V \) induces a trivial deformation of \( V \) if and only if it lies in the image of \( \rho \).

Following Schlessinger [6], we define an \( \mathcal{E}_V \)-module \( T^1 \) by the exact sequence
\[ 0 \rightarrow \Theta V \rightarrow \mathcal{E}_{C^n} \rightarrow N_V \rightarrow T^1 \rightarrow 0. \]

Then \( T^1 \) is the set of isomorphism classes of first order infinitesimal deformations of \( V \), analogous to \( H^1(Y, \Theta_V) \) for a manifold \( Y \). In [8] Tyurina shows that the \( T^1 \) may be replaced by \( \text{Ext}_G(\mathcal{E}_V, \mathcal{E}_V) \) (\( \mathcal{E}_V \) denoting \( \text{Kähler differentials} \) when \( V \) has positive depth along local singular locus, e.g. when \( V \) is reduced of positive dimension. In [1] Grauert constructs a versal deformation \( V \rightarrow S \) of \( V \) from which every other deformation \( W \rightarrow T \) may be induced, up to isomorphism, by a map \( \varphi: T \rightarrow S \), with \( \varphi^* (X) = W \). Moreover, the map \( \tau : \tau : t \rightarrow t \) between Zariski tangent spaces is uniquely determined by the isomorphism class of \( W \). As Grauert shows, the Zariski tangent space of \( S \) is isomorphic to \( T^1 \).

\( V \) is rigid when every deformation is trivial, or \( S \) is reduced to a point. Thus, \( T^1 = 0 \) is the necessary and sufficient condition for rigidity.

In [6] Schlessinger proves that quotient singularities of dimension \( \geq 3 \) are rigid. It is a long-standing conjecture that there is no rigid normal surface singularity. The normality condition is important because the singularity obtained by taking two planes in \( C^3 \) which meet at a point is rigid.

**Definition 3.1.** Let \( V(0) \) be a normal isolated singularity of dimension \( n \geq 2 \).

Let \( \overline{\Omega}^i \) and \( \overline{\Omega}^i \) be as defined in §2. Clearly, there are natural maps \( \varphi_i: \overline{\Omega}^i \rightarrow \overline{\Omega}^i \) and \( \varphi_i^*: \overline{\Omega}^i \rightarrow \overline{\Omega}^i \). The cokernels of these maps are finite-dimensional vector spaces over \( 0 \). In [11 and 13] the invariants \( r^i \) and \( s^i \) are defined to be the dimensions of coker \( \varphi_i \) and coker \( \varphi^i \) respectively.

**Remark 3.2.** In case \( (V, 0) \) is a Gorenstein surface singularity, then
\[ \dim T^1 = \dim \text{Ext}_G(\Omega_V, \mathcal{E}_V) \]
\[ = \dim H^1_0(V, \Omega^1_V) \quad (\text{Gorenstein duality}) \]
\[ = \delta(n) \geq r^1 \]
\[ \geq s^1, \]
where \( g \) is the irregularity of the singularity \( (V, 0) \).

In view of the above remark, it is important to know the lower estimates for \( g(n) \) and \( q \). In [13] we have proved the following theorems.

**Theorem 3.3.** Suppose that \( (V, 0) \) is a normal isolated singularity of dimension \( n \) which admits a \( C^* \)-action. Then \( g^{n-1} \geq r^n \) and \( s^{n-1} \geq s^n \).

**Theorem 3.4.** Let \( \varphi: \mathcal{M} \rightarrow V \) be any resolution of normal isolated singularity \( (V, 0) \) of dimension \( n \). Then:

(a) For \( n = 2, g^2 \geq 1 \) and \( s^2 \geq 1 + \dim H^1(M, \mathcal{E}_M) \).

(b) For \( n \geq 3, g^n \geq n - 1 \) and \( s^n \geq n - 1 + \dim H^{n-1}(M, \mathcal{E}_M) \) if \( \dim H^{n-1}(M, \mathcal{E}_M) > 0 \).

**Corollary 3.5.** Let \( (V, 0) \) be a normal surface singularity with \( C^* \)-action. Then
\[ g^{n-1} \geq 1 + \dim H^1(M, \mathcal{E}_M). \]

In particular, if \( (V, 0) \) is Gorenstein, then
\[ \dim T^1 = 1 + \dim H^1(M, \mathcal{E}_M). \]

**Remark 3.6.** In case \( (V, 0) \) is a Gorenstein isolated singularity with \( C^* \)-action, Wahl has informed us that he has proved that \( \dim T^1 \geq \dim H^1(M, \mathcal{E}_M) \). Ours in formula (3.2) gives the fact that there exists a holomorphic 2-form \( \omega \) defined on \( V - \{0\} \) which is \( L^2 \)-integrable in a neighborhood of \( 0 \); but \( \omega \) is not holomorphic on \( V \).

The second approach to the rigidity problem for Gorenstein surface singularities is to get lower estimate for irregularity. In [12] we have proved the following.

**Theorem 3.7.** Let \( (V, 0) \) be a normal isolated singularity of dimension \( n \) with \( C^* \)-action, \( M \rightarrow V \) an equiariant resolution whose exceptional divisor \( A \) has normal crossings. Then:

(a) \( s^{n-1} > \dim H^{n-1}(M, \mathcal{E}_M) - \dim H^{n-1}(A, \mathcal{E}_A) \).

(b) If \( V \) is Gorenstein and \( \dim H^{n-1}(M, \mathcal{E}_M) \geq 2 \), then \( s^{n-1} > 0 \).

As an immediate application of the above theorem, we have the following classification theorem.
and

\[ \dim \Gamma(M, \mathcal{A}, \mathcal{O}) \setminus \Gamma(M, \mathcal{O}) \setminus \Gamma(A, K) = 2 - 1 = 1. \]

3. Lower estimate of \( \dim T^1_v \). Let \( V \subset \mathbb{C}^n \) be a local complex space. A deformation of \( V \) is a flat map \( \pi: X \rightarrow T \) of local complex spaces, together with an isomorphism \( \pi^{-1}(0) \cong V \). Let \( I = (f_1, \ldots, f_k) \) be the ideal in \( \mathcal{O}_x \) defining the local complex space \( V \subset \mathbb{C}^n \). Thus \( \mathcal{O}_x = \mathcal{O}_{\mathbb{C}^n}/I \). If we take as parameter the (one point) space \( T = \text{spec}(\mathbb{C}[e]/(e^2)) \), then a deformation \( X \rightarrow T \) of \( V \) is a first-order infinitesimal deformation of \( V, X \) will be given by equations

\[ f_i(x) + e\partial f_i(x) = 0 \]

and condition for flatness is simply that \( g \) determines an element of the normal sheaf \( N_x = \text{Hom}_{\mathcal{O}_{\mathbb{C}^n}}(I, \mathcal{O}_x) \). The deformation \( X \rightarrow T \) is trivial if and only if there is an automorphism \( x_i \rightarrow x_i + e\delta_i(x) \) of \( \mathbb{C}^n \times T \) over \( T \) such that \((f_i(x) + e\delta_i(x))\) and \((f_i + eg)\) determine the same ideal in \( \mathcal{O}_{\mathbb{C}^n \times T} \) in other words

\[ \sum \frac{\partial f_i}{\partial \lambda_i}(x) = g_i(x) \quad (\text{mod } I). \]

Now there is a homomorphism \( \rho: \mathcal{O}_{\mathbb{C}^n} \rightarrow N_x \) \((\mathcal{O} \text{ denoting tangent sheaf})\) defined by mapping the vector field \( \Theta = \sum \delta_i(x)\partial/\partial x_i \) to the homomorphism \( f_i \mapsto \Theta f_i) \) in \( N_x \), and the element \( g \) in \( N_x \) induces a trivial deformation of \( V \) if and only if it lies in the image of \( \rho \).

Following Schlessinger [6], we define an \( \mathcal{O}_x \)-module \( T^1_v \) by the exact sequence

\[ 0 \rightarrow \mathcal{O}_x \rightarrow \mathcal{O}_{\mathbb{C}^n \times T^1} \rightarrow T^1_v \rightarrow 0. \]

Then \( T^1_v \) is the set of isomorphism classes of first order infinitesimal deformations of \( V \), analogous to \( H^1(Y, \mathcal{O}) \) for a manifold \( Y \). In [8] Tyrina shows that the \( T^1_v \) may be replaced by \( \text{Ext}^1_{\mathcal{O}_x}(\mathcal{O}_x, \mathcal{O}_x) \) \((\mathcal{O} \text{ denoting Kähler differentials})\) when \( V \) has positive depth along singular locus, e.g. when \( V \) is reduced of positive dimension. In [1] Grauert constructs a versal deformation \( S \rightarrow T \) from which every other deformation \( W \rightarrow T \) may be induced, up to isomorphism, by a map \( \Phi: T \rightarrow S \), with \( \Phi^*(x) = W \). Moreover, the map \( t_x: t \rightarrow t_x \) between Zariski tangent spaces is uniquely determined by the isomorphism class of \( W \). As Grauert shows, the Zariski tangent space of \( S \) is isomorphic to \( T^1_v \).

\( V \) is rigid when every deformation is trivial, or \( S \) is reduced to a point. Thus, \( T^1_v = 0 \) is the necessary and sufficient condition for rigidity.

In [6] Schlessinger proves that quotient singularities of dimension \( \geq 3 \) are rigid. It is a long-standing conjecture that there is no rigid normal surface singularity. The normality condition is important because the singularity obtained by taking two planes in \( \mathbb{C}^3 \) which meet at a point is rigid.

Definition 3.1. Let \( (V, 0) \) be a normal isolated singularity of dimension \( n \geq 2 \). Let \( \mathcal{O}_x \) and \( \mathcal{O}_x^\prime \) be as defined in \( \S 2 \). Clearly, there are natural maps \( \mathcal{O}_x \rightarrow \mathcal{O}_x^\prime \) and \( \mathcal{O}_x^\prime \rightarrow \mathcal{O}_x \). The cokernels of these maps are finite-dimensional vector spaces over \( 0 \). In [11] and [13] the invariants \( g^{(p)} \) and \( \delta^{(p)} \) are defined to be the dimensions of coker \( \phi \) and coker \( \psi \), respectively.

Remark 3.2. In case \( (V, 0) \) is a Gorenstein surface singularity, then

\[ \dim \chi \delta^{(p)} = \dim \text{Ext}^1_{\mathcal{O}_x}(\mathcal{O}_x, \mathcal{O}_x) = \dim H^1(\mathbb{C}[e]/(e^2), \mathcal{O}_x) \quad (\text{Gorenstein duality}) \]

\[ \delta^{(n-1)} \geq g^{(n-1)} \geq g, \]

where \( g \) is the irregularity of the singularity \( (V, 0) \).

In view of the above remark, it is important to know the lower estimates for \( g^{(1)} \) and \( g \). In [13] we have proved the following theorems.

Theorem 3.3. Suppose that \( (V, 0) \) is a normal isolated singularity of dimension \( n \) which admits a \( \mathbb{C}^* \)-action. Then \( \delta^{(n-1)} \geq g^{(n-1)} \) and \( \delta^{(n)} \geq g^{(n)} \).

Theorem 3.4. Let \( \pi: M \rightarrow V \) be any resolution of normal isolated singularity \( (V, 0) \) of dimension \( n \). Then:

(a) For \( n = 2 \), \( g^{(2)} \geq 1 \) and \( \delta^{(2)} = 1 + \dim H^1(M, \mathcal{O}) \).
(b) For \( n \geq 3 \), \( g^{(n)} \geq n - 1 \) and \( \delta^{(n)} \geq n - 1 + \dim H^{n-1}(M, \mathcal{O}) \) if \( \dim H^{n-1}(M, \mathcal{O}) > 0 \).

Corollary 3.5. Let \( (V, 0) \) be a normal surface singularity with \( \mathbb{C}^* \)-action. Then

\[ \delta^{(1)} \geq 1 + \dim H^1(M, \mathcal{O}). \]

In particular, if \( (V, 0) \) is Gorenstein, then

\[ \dim \chi \delta^{(2)} = 1 + \dim H^1(M, \mathcal{O}). \]

Remark 3.6. In case \( (V, 0) \) is a Gorenstein isolated singularity with \( \mathbb{C}^* \)-action, Wahl has informed us that he has proved that \( \dim \chi \delta^{(2)} = \dim H^1(M, \mathcal{O}) \). Ours in formula (3.2) gives the fact that there exists a holomorphic 2-form \( \omega \) defined on \( V \) which is \( L^2 \)-integrable in a neighborhood of 0; but \( \omega \) is not holomorphic on \( V \).

The second approach to the rigidity problem for Gorenstein surface singularities is to get lower estimate for irregularity. In [12] we have proved the following.

Theorem 3.7. Let \( (V, 0) \) be a normal isolated singularity of dimension \( n \) with \( \mathbb{C}^* \)-action, \( M \rightarrow V \) an equivariant resolution whose exceptional divisor \( A \) has normal crossings. Then:

(a) \( \delta^{(n-1)} > \dim H^{n-1}(M, \mathcal{O}_x) - \dim H^{n-1}(A, \mathcal{O}_x) \).
(b) If \( V \) is Gorenstein and \( \dim H^{n-1}(M, \mathcal{O}_x) > 2 \), then \( \delta^{(n-1)} > 0 \).

As an immediate application of the above theorem, we have the following classification theorem.
Theorem 3.8. Suppose that \((V,0)\) is a Gorenstein surface singularity with \(C^\ast\)-action. Then the irregularity is equal to zero if and only if \((V,0)\) is either a rational double point or a simple elliptic singularity.

For singularities without \(C^\ast\)-action, it is quite difficult to get lower estimate of irregularity. However, the technique of the proof of the following theorem should be useful.

Theorem 3.9. Suppose that \(V = \{(x, y, z) \in C^3; \ z^m = g(x, y)\}\) has isolated singularity at origin. If \((V,0)\) does not admit any \(C^\ast\)-action and \(m \geq \text{multiplicity of } g(x, y)\), then the irregularity \(q\) of the singularity is strictly greater than zero.

Proof. Since \((V,0)\) does not admit any \(C^\ast\)-action, it is clear that \(\{(x, y) \in C^2; g(x, y) = 0\}\) defines a reduced place curve singularity at origin without any \(C^\ast\)-action. \(\{(x, y) \in C^2; g = 0, \partial g/\partial x = 0, \partial g/\partial y = 0\}\) defines a codimension 2 subvariety in \(C^2\). Consequently, it is determinantal [5]. There exists an exact sequence of the form

\[
0 \rightarrow C(x, y)^2 \xrightarrow{(a, b)} C(x, y)^3 \xrightarrow{(g_x, g_y, g_z)} C(x, y) \rightarrow 0
\]

with the following properties:

\[
\begin{align*}
& (a_1, b_1) = g_x, \\
& (a_2, b_2) = g_y, \\
& (a_3, b_3) = g_z.
\end{align*}
\]

Exactness of (3.3) implies

\[
\begin{align*}
& a_1g_x + a_2g_y + a_3g_z = 0, \\
& b_1g_x + b_2g_y + b_3g_z = 0.
\end{align*}
\]

Set \(f(x, y, z) = z^m - g(x, y)\). Since \(g = z^m - f = \frac{1}{m}\partial f/\partial z - f\); we have

\[
\begin{align*}
& a_1\frac{\partial f}{\partial x} + a_2\frac{\partial f}{\partial y} + a_3\frac{\partial f}{\partial z} = af, \\
& b_1\frac{\partial f}{\partial x} + b_2\frac{\partial f}{\partial y} + b_3\frac{\partial f}{\partial z} = bf.
\end{align*}
\]

By a biholomorphic change of coordinate, we may assume that the least order terms of \(g(x, y)\) involves the term \(x^t\) where \(r = \text{multiplicity of } g(x, y)\).

\[
\omega = \frac{dy \wedge dz}{f_x} = \frac{dx \wedge dy}{f_y} = \frac{dz \wedge dx}{f_y} = \frac{dz \wedge dx}{g_y}
\]

is a holomorphic 2-form on \(V - \{0\}\) which cannot be extended across 0 holomorphically.

\[
a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{m} \text{ and } b_1\frac{\partial}{\partial x} + b_2\frac{\partial}{\partial y} + b_3\frac{\partial}{m} \text{ are holomorphic vector fields on } (V,0). \text{ By contracting } \omega \text{ with the above two holomorphic vector fields, we get two holomorphic 1-forms on } V - \{0\}\]

\[
\gamma = \frac{az \, dy + mz \, dz}{gs} \quad \text{and} \quad \delta = \frac{bz \, dy - mz \, dz}{gs}.
\]

Let \(\sigma(t) = (x_0(t), y_0(t), z_0(t))\) be a normalization of a curve in \(V\). We can choose \(\sigma(t)\) such that \(O_\sigma(t) = O_{(0,0)}\) where \(O_{(0,0)}\) denotes the vanishing order of \(g(x, y)\) at origin. In fact, if \(m \geq r = \text{multiplicity of } g(x, y)\), we can choose \(O_{\sigma(t)} = m \geq O_{(0,0)} = r\).

\[
\sigma^*\gamma = -\frac{(a_2 \sigma^* a) \, dx_0 \, dt}{g_x \sigma^* a} \quad \text{and} \quad \sigma^*\delta = -\frac{(b_2 \sigma^* a) \, dx_0 \, dt}{g_x \sigma^* a}.
\]

By (3.4) we have

\[
O(a_2 \sigma^* a) - (b_2 \sigma^* a)/(a \sigma^* a) = g_x \sigma^* a.
\]

Case 1. \(O((a_2 \sigma^* a)/(b_2 \sigma^* a)) + O((b_2 \sigma^* a)/(a \sigma^* a))\). Then

\[
O(g_x \sigma^* a) - O \left( (a_2 \sigma^* a) \left( \frac{dx_0}{dt} \right) \right) = O(a_2 \sigma^* a) + O(b_2 \sigma^* a) - O(a_1 \sigma^* a) - O(a_3 \sigma^* a) + 1 = O(b_2 \sigma^* a) - O(a_3 \sigma^* a) + 1.
\]

Observe that since the singularity does not admit any \(C^\ast\)-action, \(b\) vanishes at the origin. It follows that

\[
O(b_2 \sigma^* a) - O(a_3 \sigma^* a) + 1 > 0.
\]

This implies that \(\gamma\) is not in \(\Gamma(M, \Omega^1)\).

Case 2. \(O((a_2 \sigma^* a)/(b_2 \sigma^* a)) + O((b_2 \sigma^* a)/(a \sigma^* a))\).

Similar arguments as above will show that \(\delta\) is not in \(\Gamma(M, \Omega^1)\). Q.E.D.

References
5. M. Schaps, Deformations of Cohen-Macaulay schemes of codimension two, Tel Aviv University.
THEOREM 3.8. Suppose that \((V, 0)\) is a Gorenstein surface singularity with \(C^*\)-action. Then the irregularity is equal to zero if and only if \((V, 0)\) is either a rational double point or a simple elliptic singularity.

For singularities without \(C^*\)-action, it is quite difficult to get lower estimate of irregularity. However, the technique of the proof of the following theorem should be useful.

THEOREM 3.9. Suppose that \(V = \{(x, y, z) \in C^3; z^m = g(x, y)\}\) has isolated singularity at origin. If \((V, 0)\) does not admit any \(C^*\)-action and \(m \gg \text{multiplicity of } g(x, y)\), then the irregularity \(q\) of the singularity is strictly greater than zero.

PROOF. Since \((V, 0)\) does not admit any \(C^*\)-action, it is clear that \((x, y) \in C^2; g(x, y) = 0\) defines a reduced place curve singularity at origin without any \(C^*\)-action. \((x, y) \in C^2; g = 6g/6x = 6g/6y = 0\) defines a codimension 2 subvariety in \(C^2\). Consequently, it is determinantal [5]. There exists an exact sequence of the form

\[
0 \rightarrow C(x, y)^2 \rightarrow C(x, y)^{(e, r, z, t)} \rightarrow C(x, y) \rightarrow 0
\]

(3.3)

with the following properties:

\[
a_1 b_1 - b_1 a = s_x, \\
a_2 b_2 - a_2 b = s_y, \\
a_1 b_2 - a_2 b_1 = g, \\
a_2 b_1 - a_1 b_2 = g.
\]

Exactness of (3.3) implies

\[
0 = a_1 s_x + a_2 s_y + a g = 0, \\
b_1 s_x + b_2 s_y + b g = 0.
\]

Set \((x, y, z) = z^m - g(x, y)\). Since \(g = z^m - f = \frac{1}{2z} \frac{df}{dz} - f\), we have

\[
a_1 \frac{df}{dx} + a_2 \frac{df}{dy} + \frac{az}{m} \frac{df}{dz} = a f,
\]

\[
b_1 \frac{df}{dx} + b_2 \frac{df}{dy} + \frac{bz}{m} \frac{df}{dz} = b f,
\]

(3.10)

By a biholomorphic change of coordinate, we may assume that the least order terms of \(g(x, y)\) involves the term \(z^r\) where \(r = \text{multiplicity of } g(x, y)\).

\[
\omega = \frac{dy \wedge dz}{f_x} = \frac{dy \wedge dz}{g_x} = \frac{dz \wedge dx}{f_y} = \frac{dz \wedge dx}{g_y}
\]

is a holomorphic 2-form on \(V - \{0\}\) which cannot be extended across 0 holomorphically.

IRREGULARITY AND GEOMETRIC GENUS

\[
a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + \frac{az}{m} \frac{\partial}{\partial z} \quad \text{and} \quad b_1 \frac{\partial}{\partial x} + b_2 \frac{\partial}{\partial y} + \frac{bz}{m} \frac{\partial}{\partial z}
\]

are holomorphic vector fields on \((V, 0)\). By contracting \(\omega\) with the above two holomorphic vector fields, we get two holomorphic 1-forms on \(V - \{0\}\)

\[
\gamma = \frac{az \; dy/m - a_z \; dz}{g_x} \quad \text{and} \quad \delta = \frac{bz \; dy/m - b_z \; dz}{g_x}.
\]

Let \(o(t) = (o_y(t), 0, o_x(t))\) be a normalization of a curve in \(V\). We can choose \(o(t)\) such that \(O(o_x(t)) \ll O(o_y(t))\) where \(O(o_i(t))\) denotes the vanishing order of \(o_i(t)\) at origin. In fact, if \(m \gg r = \text{multiplicity of } g(x, y)\), we can choose \(O(o_x(t)) = m \gg O(o_y(t)) = r\).

\[
o^* \gamma = \frac{-(a_z \circ o) \; d o_y / d t}{g_x \circ o} \quad \text{and} \quad o^* \delta = \frac{-(b_z \circ o) \; d o_y / d t}{g_x \circ o}.
\]

(3.4)

By (3.4) we have

\[
(a_z \circ o)(b \circ o) - (b_z \circ o)(a \circ o) = g \circ o.
\]

Case 1. \(O(a_z \circ o)(b \circ o) < O((b_z \circ o)(a \circ o))\).

Then

\[
O(g \circ o) - O\left((a_z \circ o) \left(\frac{d o_y}{d t}\right)\right) = O(a_z \circ o) + O(b \circ o)
\]

\[
- O(a_z \circ o) - O(o_y) + 1
\]

\[
= O(b \circ o) - O(o_y) + 1.
\]

Observe that since the singularity does not admit any \(C^*\)-action, \(b\) vanishes at the origin. It follows that

\[
O(b \circ o) - O(o_y) + 1 > 0.
\]

This implies that \(\gamma\) is not in \(\Gamma(M, \Omega^1)\).

Case 2. \(O(a_z \circ o)(b \circ o) > O((b_z \circ o)(a \circ o))\).

Similar arguments as above will show that \(\delta\) is not in \(\Gamma(M, \Omega^1)\). Q.E.D.

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THE STRUCTURE OF STRATA $\mu = \text{CONST}$
IN A CRITICAL SET
OF A COMPLETE INTERSECTION SINGULARITY

YOSEF YOMDIN

This report is a summary of results of [9, 10, 11].

Let $f: M^n \to N^k$, $n > k$, be a flat mapping of regular complex manifolds, such that $f(\Sigma(f))$ is finite (where $\Sigma(f)$ is the critical set of $f$). Then for any $z \in M$ the germ of $f$ at $z$ defines a complete intersection singularity (SCI). Setting $\mu_f(z)$ to be equal to the Milnor number of this singularity, we define an integral function $\mu_f$ on $M$. The function $\mu_f$ is upper semicontinuous and $\mu_f(z) > 0$ if and only if $z \in \Sigma(f)$ (see e.g. [2, 3]).

Let $W^r_\mu(f) = \{ z \in M | \mu_f(z) > r \}$, $V^r_\mu(f) = \{ z \in M | \mu_f(z) = r \} = W^r - W^{r+1}$. Then each $W^r_\mu(f)$ is an analytic subset of $M$, $W^r_\mu(f) \neq M$, $W^r_\mu(f) = \Sigma(f)$ and if $M$ is compact, $W^r_\mu(f) = \emptyset$ for $r$ sufficiently big.

The symbol $f$ will be omitted below in the notation for $\Sigma(f)$, $\mu_f$, $V^r_\mu(f)$ and $W^r_\mu(f)$.

In the local case of SCI, $f: (C^n, 0) \to (C^k, 0)$, $\mu_\mu$, $V^r$, $W^r$, are germs at $0 \in C^n$.

Note that $V^r_\mu = W^r_\mu$ is an analytic germ.

Let $f: M^n \to N^k$, $\mu, V^r, W^r$ be as above, with $M$ and $N$ compact.

THEOREM 1 [9]. Let $F$ be a generic (nonsingular) fiber of $f$ and $\chi$ denote topological Euler characteristics. Then

$$\sum_{r > 0} r \cdot \chi(V^r) = \sum_{r > 0} \chi(W^r) = (-1)^{k-1} \left[ \chi(F) \cdot \chi(N) - \chi(M) \right].$$

In the local case of the germ $f: (C^n, 0) \to (C^k, 0)$ of SCI we have the following corollaries.