BOREL EXTENSIONS OF BAIRE MEASURES IN ZFC

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Abstract. We prove:

(1) Every Baire measure on the Kojman-Shelah Dowker space [10] admits a Borel extension.
(2) If the continuum is not a real-valued measurable cardinal then every Baire measure on the M. E. Rudin Dowker space [16] admits a Borel extension.

Consequently, Balogh's space [3] remains as the only candidate to be a ZFC counterexample to the measure extension problem of the three presently known ZFC Dowker spaces.

1. Introduction

In the mid 1990-s, after two decades in which M. E. Rudin's space [16] had been the only known absolute Dowker space, two new ones were discovered [3, 10]. At a workshop on general topology, held in Budapest in 1999, the new spaces were presented, and D. Fremlin seized the opportunity to remind the speakers that only Dowker spaces could provide counter-examples to the measure extension problem. He expressed the hope that one of the three absolute spaces would eventually prove to be an absolute counterexample to the problem.

Below we prove that two of the three potential candidates are not absolute counterexamples to the measure extension problem, in fact, that a larger class of absolute Dowker spaces does not contain an absolute counterexample.

The *measure extension problem* is the following: given a normal topological space X and a probability measure μ on the minimal σ-algebra which makes all continuous real functions on X measurable, does μ admit an extension to the σ-algebra of all Borel subsets of X?

The σ-algebra which is generated over a topological space X by all zero-sets of continuous real-valued functions on X is called the *Baire algebra* of X and is denoted $\mathrm{Ba}(X)$. A probability measure on $\mathrm{Ba}(X)$ for a normal space X is called a *Baire measure.* All Baire measures are *regular*, that is, satisfy

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that the measure of a set is the supremum of measures of its measurable *closed* subsets [12, 412D].

In most normal spaces the measure extension problem is solved positively by Mařik's extension theorem [14]: if a normal topological space X is *countably paracompact* then every Baire measure on X admits a unique regular Borel extension. Mařik's theorem, then, restricts the measure extension problem to normal spaces which are not countably paracompact. By Dowker's theorem [6], these are exactly the normal spaces X for which $X \times [0,1]$ is not normal. Such spaces are called *Dowker spaces*. Whether Dowker spaces existed or not had been an open problem for quite some time (see [17] for the history of the subject, which began with Borsuk's work on homotopy theory).

The existence of Dowker spaces has been established on the basis of various additional (independent) axioms to the standard Zermelo-Fraenkel with Choice axiom system, ZFC (see [17, 25]), mostly axioms in the direction of Gödel's constructibility axiom $V = L$. In 1970 also an absolute Dowker space was constructed, that is, proved to exist in ZFC. Three absolute Dowker spaces are known presently [16, 3, 10]. The measure extension problem has so far not been decided in any one of them.

It is customary to call a normal space in which every Baire measure admits an inner regular Borel extension a *Mařik space* (see [26]). Mařik's theorem says, then, that every normal non-Mařik spaces is Dowker. Ohta and Tamano [15] call a normal space X *quasi-Mařik* if every Baire measure on X admits *some* Borel extension. In this terminology, a counter-example to the measure extension problem is a non quasi-Mařik Dowker space.

The existence of Dowker spaces with the following prescribed measure extendibility properties has been raised in the literature. Wheeler [26] asks if there are Dowker spaces that are Mařik. Ohta and Tamano [15] ask if quasi-Mařik non Mařik Dowker spaces exist. And Fremlin [12] asks for a non-Mařik Dowker space (namely, for a counter-example for the measure extension problem).

Each of these three questions has been provided with a *consistent* positive answer. Fremlin [12] constructs a non-quasi-Mařik Dowker space of cardinality $\aleph_1$ from the axiom $\clubsuit(\aleph_1)$, and thus establishes the consistency of a counterexample to the measure extension problem. Aldaz [2] uses the same axiom to establish the consistent existence of a quasi-Mařik non-Mařik Dowker space. He also proves, using a construction of M. G. Bell [4], that under Martin Axiom, or ven under the weaker axiom $P(\mathfrak{c})$, there exists a Mařik Dowker space (thus showing that it is impossible to prove that all Dowker spaces are not Mařik).

No absolute positive answers were known to any of these questions.

The following is a list of all presently known ZFC Dowker spaces:

(1) M. E. Rudin's space X^R [16], whose cardinality is $(\aleph_\omega)^{\aleph_0}$;
(2) Balogh's space [3], whose cardinality is $2^{\aleph_0}$;

(3) Kojman and Shelah's space [10], whose cardinality is $\aleph_{\omega+1}$.

P. Simon [18] proved that space (1) was not Mařik Shortly after its discovery. Quasi-Mařikness has not been decided in any of the three spaces (in no extension of ZFC).

1.1. **The results.** We introduce an infinite class of normal spaces — the class of *Rudin spaces*. Spaces (1) and (3) belong to this class. We prove in ZFC that every Rudin space is a non-Mařik Dowker space and:

(A) Space (3) and every other space of cardinality $\aleph_{\omega+1}$ in this class are quasi-Mařik.
(B) If the class of Rudin spaces contains a non-quasi-Mařik member then the continuum is real-valued-measurable.

By (A), space (3) is an absolute quasi-Mařik non-Mařik space; this provides a positive ZFC solution to Ohta and Tamano's question. Being quasi-Mařik, space (3) is not an absolute counterexample to the measure extension problem.

By (B), also space (1) is not an absolute counterexample to the measure extension problem. It is impossible to prove in ZFC that the continuum is real-valued measurable (unless ZFC is inconsistent). Therefore (B) implies that an absolute counter-example to the measure extension problem does not exist in the class of Rudin spaces. In particular, Rudin's space (1) is not such an example.

Furthermore, it follows from (B) and Solovay's theorem [24] that the consistency strength of the existence of a non quasi-Mařik Rudin space is that of a measurable cardinal. This means that if the statement "there exists a non-quasi-Mařik Rudin space" is consistent with ZFC, then this formal consistency will have to be established from the assumption that the theory ZFC + "there exists a measurable cardinal" is consistent. This is a much stronger assumption than the assumption that ZFC is consistent.

Balogh's space (2) remains now the only known candidate to be a ZFC counterexample to the measure extension problem.

1.2. **The method.** The main tool we use is some further development of Shelah's PCF theory, which we employ for an analysis of Baire measures on Rudin spaces. Nonextendible Baire measures are shown to necessarily come from a real valued measure on the cardinality of the space. PCF theory is used again to construct Rudin spaces whose cardinality is absolutely not real-valued measurable. In these constructions no infinite products may be used: an infinite product of (larger than singleton) sets has at least the cardinality of the continuum, which, by Solovay's work, can be real-valued measurable.

1.3. **Organization of the paper.** In Section 2 we introduce the class of *Rudin spaces* and develop their PCF-theoretic properties. Then we prove that every Rudin space is Dowker and prove that every Rudin space contains a (closed) Rudin subspace of cardinality $\aleph_{\omega+1}$. In Section 3 we prove that *cofinal Baire measures* on Rudin spaces do not admit regular Borel extensions, but always admit some Borel extensions, and prove the main Baire-measures decomposition-theorem: if the cardinality of a Rudin space X is not real-valued-measurable then every Baire measure on X is a countable sum of measures concentrated on singletons and of cofinal Baire measures supported on pairwise-disjoint Rudin subspaces. This suffices to prove that every Rudin spaces of non-real-valued-measurable cardinality is quasi-Mařik. We conclude with some remarks and a discussion of two issues in Section 4.

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2. Rudin spaces

We define Rudin spaces and develop their properties.

By ON we denote the class of ordinal numbers. The ordinal ω is the set of natural numbers. For an ordinal α, $\mathrm{cf}(\alpha)$ is the cofinality of α. By ON^ω we denote the class of all functions from ω to ON. For $f, g \in \mathrm{ON}^\omega$ we write $f \leqq g$ if $f(n) \leqq g(n)$ for all $n \in \omega$.

Let

$$\begin{aligned} P &= \prod_{n\in\omega} (\omega_{n+2} + 1) \\ &= \Big\{ f : f \in \mathrm{ON}^\omega \text{ and } \forall n [f(n) \leqq \omega_{n+2}] \Big\} \end{aligned}$$

Let

$$T = \Big\{ f : f \in P \text{ and } (\forall n)\ [\mathrm{cf}(f(n)) > \aleph_0] \Big\}$$

Finally,

$$X^R = \big\{ f \in T : (\exists l)\ (\forall n)\ [\mathrm{cf}\, f(n) < \aleph_l] \big\}$$

X^R is the underlying set of of Rudin's space (1). The topology on X^R is defined in 2.10 below.

2.1. **m-clubs and m-stationarity in X^g for $g \in T \setminus X^R$.** The topological properties of Rudin spaces follow from the PCF-theoretic properties of $(X^R, \leqq)$. We establish the required latter properties in this section.

The presentation is self contained, but familiarity with PCF theory is helpful. The reader may consult the short appendix at the end of the paper in which relevant PCF notation and results are summarized. Proofs of the standard PCF theorems we use can be found in [1, 9] and a general source for the theory is of course [20].

Definition 2.1. *Suppose that* $g \in T$.

(1) *For each* $m \in \omega$ *let*

$$C^g_m = \{n : \operatorname{cf} g(n) = \aleph_m\}.$$

Let

$$C^g_{\leqq m} = \bigcup_{m' \leqq m} C^g_{m'}$$

and

$$C^g_{>m} = \bigcup_{m' > m} C^g_{m'}.$$

(2) *Let*

$$X^g := \{f \in X^R : f \leqq g\}.$$

If $g \in X^R$, then by the definition of X^R, there is some m such that $C^g_{\leqq m} = \omega$ or, equivalently, $C^g_{>m} = \emptyset$ for all sufficiently large m. On the other hand, if $g \in T \setminus X^R$, then $C^g_m \neq \emptyset$ for infinitely many $m \in \omega$.

Fact 2.2. *Suppose* $g \in T \setminus X^R$. *The partially ordered set* $(X^g, \leqq)$ *is directed.*

Proof. Suppose $h_1, h_2 \in X^g$ and let $h = \max\{h_1, h_2\}$. For every n it holds that $\aleph_0 < \operatorname{cf} h(n) \leqq \max\{\operatorname{cf} h_1(n), \operatorname{cf} h_2(n)\}$, thus there is some ℓ so that $\operatorname{cf} h(n) < \aleph_\ell$ for all n, and $h \in X^R$. Also, $h \leqq g$, since $h_1 \leqq g$ and $h_2 \leqq g$. Thus $h \in X^g$ and $h_1, h_2 \leqq h$. □

Suppose $(P, \leqq)$ is any directed poset. Then for every $p \in P$ the set $\{q \in P : p \leqq q\}$ is *cofinal* in $(P, \leqq)$, that is, for every $t \in P$ there is some $q \in P$ so that $t, p \leqq q$. If $S \subseteq P$ is not cofinal in $(P, \leqq)$ then there is $p \in P$ so that $S \cap \{q \in P : p \leqq q\} = \emptyset$ and thus $P \setminus S$ is *cofinal* in S. The following follows immediately from this observation and Fact 2.2:

Fact 2.3. *Suppose* $g \in T \setminus X^R$. *Then for every subset* $D \subseteq X$, *either* D *or* $X \setminus D$ *is cofinal in* $(X^g, \leqq)$.

Definition 2.4. *Suppose* $g \in T \setminus X^R$. *For* $m \in \omega$ *let*

(i) $$D^g_m = \left\{h \in X^g : \exists n \in C^g_{>m}\, [h(n) = g(n)]\right\}$$

(ii) $$X^g_m = X^g \setminus D^g_m = \left\{h \in X^g : \forall n \in C^g_{>m}\, [h(n) < g(n)]\right\}$$

If $g \in T \setminus X^R$ and $h \in X^g_m$ satisfies $f \leqq h$ then $f \in X^g_m$. Since $X^g \subseteq X^R$, for every $h \in X^g$ it holds that $h(n) < g(n)$ for all $n \in C^g_{>m}$ for some m. We have, then, for every $g \in T \setminus X^R$:

(iii) $$\bigcap_{m \in \omega} D^g_m = \emptyset$$

(iv) $$\bigcup_{m \in \omega} X^g_m = X^g.$$

Definition 2.5. *Suppose that $g \in T \setminus X^R$ and $m \in \omega$, $m > 0$. An element $f \in X^g$ is called* m-normal in X^g *if*

1. $f(c) = g(c)$ *for all* $c \in C^g_{\leqq m}$
2. $\operatorname{cf} f(c) = \omega_m$ *for all* $c \in C^g_{>m}$

Fact 2.6. *Suppose that $g \in T \setminus X^R$ and $m > 0$. Then:*

1. *If $f \in X^g$ is m-normal in X^g then $f \in X^g_m$.*
2. *for every $h \in X^g_m$ there is an m-normal $f \in X^g_m$ such that $h \leqq f$.*
3. *For every m-normal $h \leqq g$ the cofinality of*
$$\big(\{f : f \in P \text{ and } f < h\}, \leqq\big)$$
is equal to $\aleph_m$.
4. *The set of m-normal elements in X^g is $\leqq$-directed.*

Proof. Suppose $h \in X^g_m$. If f is an m-normal element in X^g then $f(c) < g(c)$ for all $c \in C^g_{>m}$ and hence $f \in X^g_m$.

To prove (2), suppose that $h \in X^g_m$. For every $n \in C^g_{>m}$ let $f(n) \in g(n) \setminus (h(n) + 1)$ be an ordinal of cofinality ω_m. Let $f(n) = g(n)$ for all $n \in C^g_{\leqq m}$. The element f is m-normal in X^g and $h \leqq f$.

To prove (3), observe first that if $C \subseteq \omega$ is any nonempty set, $m > 0$ and $h : C \to \mathrm{ON}$ satisfies $\operatorname{cf} h(n) = \aleph_m$ for all $n \in C$, then $\operatorname{cf}\big(\{h \in \mathrm{ON}^C : f < g\}, \leqq\big) = \aleph_m$. This can be seen by fixing, for every $n \in C$, a $<$-increasing sequence $\langle \zeta^n_\alpha : \alpha < \omega_m\rangle$ with $\sup\{\zeta^n_\alpha : \alpha < \omega_m\} = h(n)$, and for every $\alpha < \omega$ letting $f_\alpha(n) = \zeta^n_\alpha$. If $f < h$ is any ordinal function on C, then $\operatorname{cf} h(n) > \aleph_0$ for $n \in C$, there is some $\alpha < \omega_m$ so that $f(n) < f^n_\alpha$ for all $n \in C$.

Next observe that the cofinality of a product of finitely many posets, each with an infinite cofinality, is the maximum of their cofinalities. Now (3) follows from the fact that $\big(\{f : f \in P \text{ and } f < h\}, \leqq\big)$ is isomorphic to the product of $\big(\{(f \restriction A_i) : f \in P \text{ and } f < h\}, \leqq\big)$ over all $i \leqq m$ such that there is some n with $\operatorname{cf} h(n) = \aleph_i$.

For part (4) it suffices to observe that $\max\{h_1, h_2\}$ is m-normal if h_1, h_2 are m-normal. □

Definition 2.7. *Suppose $g \in T \setminus X^R$ and $0 < m \in \omega$.*

1. *An m-club in X^g is a subset $D \subseteq X^g$ which satisfies:*
   (a) *Every $h \in D$ is m-normal;*
   (b) *D is* cofinal *in $(X^g_m, \leqq)$, that is, for every $h \in X^g_m$ there is some $f \in D$ such that $h \leqq f$;*
   (c) *If $h_\zeta \in D$ for $\zeta < \omega_m$ and $\langle (h_\zeta \restriction C^g_{>m}) : \zeta < \omega_m\rangle$ is $<$-increasing, then* $\sup\{h_\zeta : \zeta < \omega_m\} \in D$.
2. *A set $S \subseteq X^g$ is* m-stationary in X^g *if S has a non-empty intersection with every m-club in X^g.*

Example. Suppose $g \in T \setminus X^R$, $0 < m \in \omega$. Assume, that $h \in X^g_m$. Then the set
$$\big\{f \in X^g : f \text{ is } m\text{–normal and } h \leqq f\big\}$$

is an m–club in X^g.

Fact 2.8. *Suppose that $g \in T \setminus X^R$ and $m > 0$.*

(1) *Every intersection of $\aleph_m$ m-clubs is an m-club.*
(2) *If $S \subseteq X^g$ is m-stationary then S is cofinal in $(X^g_m, \leqq)$.*
(3) *Suppose S is a set of m-normal elements in X^g and that S is not cofinal in X^g_m. Then $X^g \setminus S$ contains an m-club of X^g.*

Proof. Suppose $m > 0$ and that D_α is a given m-club for every $\alpha < \omega_m$. The intersection $D := \bigcap_{\alpha<\omega} D_\alpha$ clearly satisfies conditions (a) and (c). To see that it satisfies (b) let h be an arbitrary m-normal element in X^g. By induction on $\alpha < \omega_m$ choose an m-normal element h_α as follows. For $\alpha = 0$ let $h_0 = \sup\{h^0_\beta : \beta < \omega_m\}$ where $h_0 \leqq h^0_0$, $h^0_\beta \in D_\beta$ and $\beta < \gamma \implies (h^0_\beta \restriction C_{>m}) < (h^0_\gamma \restriction C_{>m})$. This is clearly possible, as each D_β is an m-club. At limit $\alpha < \omega_m$ let $h_\alpha = \sup\{h_\beta : \alpha < \beta\}$ and for $\alpha + 1 < \omega_m$ let $h_{\alpha+1}$ be defined from h_α in the same way h_0 is defined from h.

Let $h^* = \sup\{h_\alpha : \alpha < \omega_m\}$. Since $\langle h^\alpha_\beta : \alpha < \omega_m\rangle$ is $<$-increasing on $C_{>m}$, has supremum h^* and each h^α_β belongs to D_β, if follows by (b) that $h^* \in D_\beta$ for all $\beta < \omega_m$. Clearly, $h \leqq h^*$, so we are done.

Part (2) is follows from the fact that for every $h \in X^g_m$, the set $\{f \in X^g : f$ is m-normal and $h \leqq f\}$ is an m-club.

(3) follows from the fact that the set of m-normal elements in X^g_m is cofinal in X^g_m and is directed. □

Lemma 2.9 (Fodor lemma for m-clubs)**.** *Suppose that $g \in T \setminus X^R$, $0 < m \in N$ and that D is an m-club in X^g. Suppose that $F : D \to P$ satisfies that $F(h) < h$ for all $h \in D$. Then there is some $f_0 \in P$ and an m-stationary $S \subseteq D$ so that $f_0 < g$ and $F(h) < f_0$ for all $h \in S$.*

Proof. Suppose that D is an m-club in X^g and $F : D \to P$ is given and satisfies $F(h) < h$ for all $h \in D$, but that, contrary to the lemma, for every $f < g$ in P some m-club D_f is fixed so that $F(h) \not< f$ for all $h \in D_f$. By intersecting each D_f with D we assume that $D_f \subseteq D$ for all $f < g$ in P.

By induction on $\zeta < \omega_m$ define h_ζ and A_ζ so that the following hold:

(1) $A_\zeta \subseteq \{f \in P : f < h_\xi\}$ is cofinal in $\big(\{f \in P : f < h_\xi\}, \leqq\big)$, $|A_\zeta| = \aleph_m$ and $\xi < \zeta \implies A_\xi \subseteq A_\zeta$.
(2)
$$h_\zeta \in \bigcap\{D_f : f \in \bigcup_{\xi<\zeta} A_\xi\}$$
and
$$\xi < \zeta \implies (h_\xi \restriction C^g_{>m}) < (h_\zeta \restriction C^g_{>m}).$$

Suppose $\zeta < \omega_m$ and that h_ξ and A_ξ are defined for all $\xi < \zeta$. Pick $h_\zeta \in \bigcap\{D_f : f \in \bigcap_{\xi<\zeta} D_f\}$ so that $(h_\xi \restriction C^g_{>m}) < (h_\zeta \restriction C^g_{>m})$ for all $\xi < \zeta$. Since $|\bigcup_{\xi<\zeta} A_\xi| = \aleph_m$ and the intersection of $\aleph_m$ m-clubs is an m-club, it is possible to pick h_ζ. (For $\zeta = 0$ let $h_\zeta \in X^g_m$ be arbitrary.) Fix now a cofinal set $B \subseteq \{f \in P : f < h_\zeta\}$ satisfying $|B| = \aleph_m$ and let $A_\zeta = B \cup \bigcup_{\xi<\zeta} A_\xi$.

Let $h = \sup\{h_\zeta : \zeta < \omega_m\}$. Since $h_\zeta \in D$ for all $\zeta < \omega_m$ and D is an m-club, $h \in D$. Denote now $t := F(h) < h$. Since $\langle (h_\zeta \restriction C^g_{>m}) : \zeta < \omega_m \rangle$ is strictly increasing with supremum $(h \restriction C^g_{>m})$, there is some $\zeta < \aleph_m$ so that $(t \restriction C^g_{>m}) < (h_\zeta \restriction C^g_{>m})$. Notice, that $(t \restriction C^g_{\leqq m}) < (h_\zeta \restriction C^g_{\leqq m}) = (h \restriction C^g_{\leqq m})$. This means that $t < h_\zeta$. Since A_ζ is cofinal in $\{f \in P : f < h_\zeta\}$ and $t \in P$ satisfies $t < h_\zeta$, there is some $f \in A_\zeta$ so that $t < f$. Finally, since $h_\xi \in D_f$ for all $\zeta < \xi < \omega_m$, it follows that $h \in D_f$. Now contradiction follows, since $F(h) = t < f$. □

2.2. **Topologically closed cofinal subsets of X^g for $g \in T \setminus X^R$.** For $f < g$ from P let

$$(f, g] = \{h : h \in P \text{ and } f < h \leqq g\}$$

The family of all sets $(f, g]$ for $f < g$ in P constitutes a basis for the *box product topology* on P. In this topology, the basic open set $(f, g]$ is actually clopen, for every $f < g$ in P.

All spaces that we shall consider are subspaces of T taken with the induced box product topology from P. The first space we consider is M. E. Rudin's Dowker space from [16]:

Definition 2.10. *The Rudin space $X^R \subseteq T$ is defined as*

$$X^R = \left\{f \in T : \exists l\ \forall n\, [\operatorname{cf} f(n) < \aleph_l]\right\}$$

with the induced topology from the box product topology on P.

If $h \in X^R$ and $X \subseteq X^R$, then h belongs to the closure of X if and only if for all $t \in P$ that satisfies $t < h$ there is some $h \in X \cap (t, h]$.

Lemma 2.11. *Suppose $g \in T \setminus X^R$, that $X \subseteq X^g$ for all $h < g$ there is $f \in X$ such that $h \leqq f$. Then for some $m_0 \in \omega$ it holds that for all $h < g$ there exists $f \in X \cap X^g_{m_0}$ so that $h < f$.*

Proof. Suppose $X \subseteq X^g$ satisfies the assumption but that for each m there is some $h_m < g$ so that $h_m \nleqq f$ for all $f \in X \cap X^g_m$. Let $h = \sup\{h_m : m \in \omega\}$. Since $\operatorname{cf} g(c) > \omega_0$ for all $c \in \mathbb{N}$, it holds that $h < g$. By the definition of h, $h \nleqq f$ for all $f \in D \cap X^g_m$ and $m \in \omega$. Since $\bigcup_m X^g_m = X^g$ this contradicts the assumption on X. □

Lemma 2.12. *Suppose $g \in T \setminus X^R$, $X \subseteq X^g$, $m_0 > 0$ satisfies that for all $h < g$ there is some $f \in X \cap X^g_{m_0}$ so that $h < f$. Then the set of m-normal elements in the closure of X is cofinal in X^g_m for all $m \geqq m_0$.*

Proof. Let $m \geqq m_0$ be given. Let $m' = \max\{n : n \leqq m \text{ and } C^g_n \neq \emptyset\}$ and fix a cofinal set $A \subseteq (\prod_{c \in C^g_{\leqq m}} g(c), \leqq)$ of cardinality $\aleph_{m'}$. Fix an enumeration $\langle t_\alpha : \alpha < \omega_m \rangle$ of A in which every $t \in A$ appears $\aleph_m$ times. Thus, the set $\{t_\alpha : \beta < \alpha < \omega_m\}$ is cofinal in $(\prod_{c \in C^g_{\leqq m}} g(c), \leqq)$ for every $\beta < \omega_m$.

Let $h \in X^g_m$ be given. By induction on $\alpha < \omega_m$ find $f_\alpha \in X \cap X^g_{m_0}$ so that $\alpha < \beta < \omega_m$ implies that $(f_\alpha \restriction C^g_{>m}) < (f_\beta \restriction C^g_{>m})$ and $t_\alpha < (f_\alpha \restriction C^g_{\leqq m})$.

Let $f_0 \in D$ be chosen so that $t_0 \cup (h \restriction C^g_{>m}) < f_0$. This is possible by the assumption on X because $t_0 \cup (h \restriction C^g_{>m}) < g$. At stage $\alpha > 0$ let $h_\alpha = t_\alpha \cup \sup\{(f_\beta \restriction C^g_{>m}) : \beta < \alpha\}$ and find $f_\alpha \in D \cap X^g_m$ so that $h_\alpha < f_\alpha$.

Let $f := (g \restriction C^g_{\leqq m}) \cup \sup\{f_\alpha \restriction C^g_{>m} : \alpha < \omega_m\}$. Clearly, $f \in X_g$, is m-normal and $h \leqq f$. To see that f belongs to the closure of X let $t < f$ be arbitrary. Find some $\beta < \omega_m$ so that $(t \restriction C^g_{>m}) < (f_\beta \restriction C^g_{>m})$. Then find some $\beta < \alpha < \omega_m$ so that $(t \restriction C^g_{\leqq m}) < t_\alpha$. Now $f_\alpha \in (t, f]$. □

Theorem 2.13. *Suppose $g \in T \setminus X^R$ and $X \subseteq X^g$ is closed in X^R. Then X is cofinal in $(X^g, \leqq)$ if and only if there is some $m_0 > 0$ so that X contains an m-club of X^g for all $m \geqq m_0$.*

Proof. If $X \subseteq X^g$ is any cofinal set in $(X^g, \leqq)$, then by Lemma 2.11 there exists some m_0 so that for all $f < g$ there is $h \in X \cap X^g_{m_0}$ such that $f < h$. If X is also closed in X^R, then by Lemma 2.12, for all $m \geqq m_0$, the set of all m-normal elements in X — let us denote it by A_m — is cofinal in X^g_m. Hence A_m satisfies conditions (a) and (b) in Definition 2.7 of an m-club. It satisfies also condition (c), since X is closed, and therefore $A_m \subseteq X$ is an m-club.

Conversely, suppose $X \subseteq X^g$ is closed and contains an m-club for every $m \geqq m_0$ for some $m_0 > 0$. Let $h \in X^g$ be given. Since $X^g = \bigcup_m X^g_m$ there exists some m so that $h \in X^g_m$. By increasing m, we may assume $m \geqq m_0$. Since X contains an m-club, there is some m-normal $f \in X$ so that $h \leqq f$. □

Remark: Taking $g(n) = \omega_{n+2}$ for all n, the space D^g_m for $m > 1$ is closed and cofinal in $X^g = X^R$ but contains no m-clubs for $m' < m$. This shows that the restriction to $m \geqq m_0$ for some m_0 is necessary in Theorem 2.13.

Lemma 2.14. *Suppose $g \in T \setminus X^R$ and that $X \subseteq X^g$ is closed in X^R and cofinal in $(X^g, \leqq)$. Suppose $m > 0$ and X contains an m-club in X^g. Then for every closed $D \subseteq X$, either D contains an m-club of X^g or $X \setminus D$ contains an m-club of X^g.*

Proof. Suppose $m > 0$, that $E \subseteq X$ is an m-club of X^g and that $D \subseteq X$ is closed. As D is closed, $D \cap E$ satisfies condition (c) in the definition of m-club. Thus, if $D \cap E$ is cofinal in $(E, \leqq)$, then $D \cap E$ is an m-club.

Otherwise, there is some $f \in E$ so that $\{h : h \in E \text{ and } f \leqq h\} \cap D = \emptyset$, which implies that $X \setminus D$ contains an m-club. □

2.3. **Rudin spaces.**

Definition 2.15. *A space X is a* Rudin space *if there exists $g \in T \setminus X^R$ so that $X \subseteq X^g$ is closed in X^R and cofinal in $(X^g, \leqq)$.*

Observe that X^g is clopen in X^R for every $g \in T \setminus X^R$. This means that for $X \subseteq X^g$, X is closed in X^R iff X is closed in X^g. From now on we refer to this situation just by "$X \subseteq X^g$ is closed".

Fact 2.16. *X^R is a Rudin space and for every $g \in T \setminus X^R$, X^g is a Rudin space.*

Proof. The second part is obvious as X^g is clopen in X^R and cofinal in $(X^g, \leqq)$. For the first part let g be defined by $g(n) = \omega_{n+2}$. Now $X^g = X^R$. □

Fact 2.17 ([16], Lemma 4)**.** *X^R is a P-space, that is, every countable intersection of open subsets of X^R is open.*

Proof. Suppose $U_m \subseteq X$ is open for each $m \in \mathbb{N}$ and suppose $f \in \bigcap U_m$. For each m there is some $h_m < f$ so that $(h_m, f] \subseteq U_f$. Since cf $f(n) > \aleph_0$ for all $n \in \mathbb{N}$, it holds that $h = \sup\{h_m : m \in \mathbb{N}\}$ satisfies $h < f$ and clearly $(h, f] \subseteq \bigcap U_m$. □

Corollary 2.18. *Every Rudin space is a P-space.*

Definition 2.19.
(1) *A topological space X is* collectionwise normal *if for every discrete family $\{H_j : j \in J\}$ of closed subsets of X there exists a family $\{U_j : j \in J\}$ of open pairwise disjoint subsets of X such that for every $j \in J$ there is $H_j \subseteq U_j$. Every collectionwise normal space is normal.*
(2) *A normal topological space X is* countably paracompact *if for every family $\{D_n : n \in \omega\}$ of closed subsets of X, if $\bigcap_{n\in\omega} D_n = \emptyset$, then there exists a family $\{U_n : n \in \omega\}$ of open subsets of X such that $D_n \subseteq U_n$ for every $n \in \omega$ and $\bigcap_{n\in\omega} U_n = \emptyset$.*
(3) *A topological space X is* Dowker*, if it is normal and not countably paracompact.*

Theorem 2.20 (M. E. Rudin,[16])**.** *X^R is collectionwise normal.*

Every Rudin space is a closed subspace of X^R, therefore:

Corollary 2.21. *Every Rudin space is collectionwise normal.*

Lemma 2.22. *Suppose that $g \in T \setminus X^R$ and that $X \subseteq X^g$ is closed and cofinal in $(X^g, \leqq)$. The collection of closed subsets of X which are cofinal in $(X, \leqq)$ satisfies the finite intersection property.*

Proof. Suppose $D_1, D_2 \subseteq X$ are closed and cofinal in $(X, \leqq)$. Then they are closed and cofinal in $(X^g, \leqq)$. By Theorem 2.13 applied to D_1 and D_2, there is some $m > 0$ so that both D_1 and D_2 contain m-clubs of X^g. Now by Fact 2.8 $D_1 \cap D_2$ contains an m-club of X^g too and is therefore nonempty. □

Definition 2.23. *Suppose $g \in T \setminus X^R$ and $X \subseteq X^g$ is closed and cofinal in $(X^g, \leqq)$. For every $m > 0$ we define, analogously to Definition 2.4:*

(v) $$D^X_m = X \cap D^g_m = \left\{h \in X : \exists n \in C^g_{>m}\, [h(n) = g(n)]\right\}$$

(vi) $$X_m = X \cap X^g_m = X \setminus D^X_m = \left\{h \in X : \forall n \in C^g_{>m}\, [h(n) < g(n)]\right\}$$

If $X \subseteq X^g$ is closed and cofinal in $(X^g, \leqq)$ and $h \in X_m$, then for every $t \in X$ such that $t \leqq h$ it holds that $t \in X_m$. This makes X_n an open subset of X and D_m a closed subset of X for all $m > 0$. The set D^X_m is clearly cofinal in X for all $m > 0$. Finally, $\bigcap_m D^X_m = \emptyset$ $X = \bigcup_m X_m$.

Fact 2.24. *Suppose $g \in T \setminus X^R$ and $X \subseteq X^g$ is closed and cofinal in $(X^g, \leqq)$. The collection of closed and cofinal subsets in $(X, \leqq)$ does not satisfy the countable intersection property.*

Proof. For every $m > 0$ the set D^X_m is closed and cofinal in $(X, \leqq)$ and $\bigcap_{m>0} D^X_m = \emptyset$. □

Lemma 2.25. *Suppose that $g \in T \setminus X^R$ that $D \subseteq X^g$ is closed and cofinal in $(X^g, \leq)$. For every open $U \subset X^g$ such that $D \subseteq U$ there is $f \in P$ so that $f < g$ and $(f, g] \cap X^g \subseteq U$.*

Proof. Suppose that $g \in T \setminus X^R$ and $U \subseteq X^g$ is open and that $D \subseteq U$ is closed and cofinal in $(X^g, \leqq)$. For every $h \in D$, let $F(h) \in P$ be fixed so that $F(h) < h$ and $(F(h), h] \cap X^g \subseteq U$. This is possible because $h \in U$ and U is open.

By Theorem 2.13 there exists some $m_0 > 0$ and, for every $m \geqq m_0$, an m-club $D_m \subseteq D$ of X^g. By the Fodor Lemma for m-clubs, there is an m-stationary $S_m \subseteq D_m$ and $f_m < g$ so that for all $h \in S_m$ it holds that $F(h) < f_m$. Let $f = \sup\{f_m : m \geqq m_0\}$. Since $\mathrm{cf}(g(n)) > \aleph_0$ for all n, it holds that $f(n) < g(n)$ for all n, namely that $f < g$.

Suppose now that $h \in (f, g] \cap X^g$ and we shall show that $h \in U$. There is some $m \geqq m_0$ so that $h \in X^g_m$, and since S_m is cofinal in X^g_m, there is some $t \in S_m$ such that $h \leqq t$. Now

$$h \in (f, t] \cap X^g \subseteq (f_m, t] \cap X^g \subseteq U.$$

Thus $X^g \cap (t, g] \subseteq U$. □

Theorem 2.26. *Every Rudin space is Dowker.*

Proof. Every Rudin space is normal by 2.21.

Given a closed and cofinal $X \subseteq X^g$ for some $g \in T \setminus X^R$, it holds that $\bigcap_m D^X_m = \emptyset$ and that D^X_m is closed and cofinal in $(X, \leq)$ for every m. Therefore, each D^X_m is also closed and cofinal in $(X^g, \leq)$. Suppose $U_m \subseteq X$ is given for each m so so that U_m is open and $D^X_m \subseteq U_m$, and let $U^*_m \subseteq X^g$ be open in X^g so that $U_m = U^*_m \cap X$.

By Lemma 2.25 there is $f_m \in P$ so that $f_m < g$ and $(f_n, g] \cap X^g \subseteq U^*_m$. So $(f_m, g] \cap X \subseteq U_m$. Let $f = \sup_{m \in \omega} f_m$. It holds that $f < g$, and that $X \cap (f, g] \subseteq \bigcap_{m\in\omega} U_m$. Thus, by cofinality of X in $(X^g, \leq)$, $\bigcap_{m\in\omega} U_m \neq \emptyset$. This shows that X is not countably paracompact. □

Theorem 2.27. *Suppose $g \in T \setminus X^R$ and $X \subseteq X^g$ is closed and cofinal in $(X^g, \leqq)$. Then the collection of clopen and cofinal subsets of $(X, \leqq)$ is a σ-ultrafilter of clopen sets.*

Proof. Suppose that for each $i \in \omega$ the set $D_i \subseteq X$ is clopen and cofinal in $(X, \leqq)$. By Lemma 2.25, for each i there is some $f_i < g$ so that $X \cap (f_i, g] \subseteq D_i$. Now $f = \sup\{f_i : i \in \omega\} < g$ and $X \cap (f, g] \subseteq \bigcap_i D_i$. This establishes that a countable intersection of clopen and cofinal subsets of X is cofinal in $(X, \leqq)$. It is also clopen, since X is a P-space.

From Fact 2.3 it follows that for every clopen set $D \subseteq X$ either D or $X \setminus D$ is cofinal in $(X, \leqq)$. Thus clopen and cofinal sets form a σ-ultrafilter of clopen sets in X. □

2.4. **Rudin spaces of bounded cardinality.** In this section we prove that every Rudin space contains a Rudin subspace of cardinality $\aleph_{\omega+1}$.

We assume familiarity with the following concepts: exact upper bound (eub) of a given sequence of elements of ON^ω with respect to a given ideal on ω, flatness of a given sequence of elements of ON^ω with respect to a given ideal on ω, true cofinality and boundedness of a given subset of ON^ω with respect to a given ideal on ω. All these concepts are defined in the Appendix. See Definitions 5.1, 5.2, 5.5, together with formulations of Fact 5.3 and Lemma 5.4.

Definition 2.28. *Suppose $g \in T \setminus X^R$ and that $X \subseteq X^g$ is closed and cofinal in X^g.*

1. *Let I_g be the ideal generated over ω by $\{C^g_{\leqq m} : m \in \mathbb{N}\}$.*
2. *For each m with $C^g_m \neq \emptyset$ fix a strictly increasing and continuous sequence $\langle t^m_\alpha : \alpha \leqq \omega_m \rangle$ of functions on C^g_m with $t^m_{\omega_m} = (g \restriction C^g_m)$ and such that for every $\alpha \leqq \omega_m$ and for every $n_1, n_2 \in C^g_m$ it holds that $\operatorname{cf} t^m_\alpha(n_1) = \operatorname{cf} t^m_\alpha(n_2)$. Let*

$$D_g = \{h \in X^g : (\forall m)(\exists \alpha \leqq \omega_m)[(h \restriction C^g_m) = t^m_\alpha]\}.$$

Claim 2.29. *D_g is closed and cofinal in X^g.*

Proof. Assume that $f \in X^g \setminus D_g$. Find $m \in \omega$ and $n \in C^g_m$ such that

$$\forall\ \alpha \leqq \aleph_m\ f(n) \neq t^m_\alpha(n). \tag{vii}$$

By continuity of $\langle t^m_\alpha(n) : \alpha \leq \omega_m \rangle$ there exists a largest $\alpha_0 < \omega_m$ satisfying $t^m_{\alpha_0}(n) < f(n)$. The set $\{h : h \in X^g \text{ and } t^m_{\alpha_0}(n) < h(n) \leq f(n)\}$ is open, disjoint to D_g and contains f. This shows that D_g is closed.

To see that D_g is cofinal in $(X^g, \leq)$ suppose $f \in X^g$ is arbitrary. Suppose $m \in \omega$ is such that $C^g_m \neq \emptyset$ and let $t^m = (f \restriction C^g_m)$. Since $f \in X^g$, for all but finitely many $n \in \omega$ it holds that $t^m < (g \restriction C^g_m)$. Fix $m_0 \in \omega$ so that $m_0 > 0$ and for all $m > m_0$, if t^m is defined, then $t^m < (g \restriction C^g_m)$. For each $m > m_0$ for which $C^g_m \neq \emptyset$ find $\alpha_m < \omega_m$ such that $t^m < t^m_{\alpha_m}$. Since $m \geq 2$ it holds that $\operatorname{cf} g(n) = \omega_m \geq \omega_2$, so we may increase $\alpha_m < \omega_m$ and assume that $\operatorname{cf} \alpha_m = \omega_1$. For each $m \leq m_0$ let $\alpha_m = \omega_m$, so $t^m_{\alpha_m} = (g \restriction C^g_m)$. Now let $h = \bigcup_{m \in \omega_m C^g_m \neq \emptyset} t^m_{\alpha_m}$. It holds that $f \leq h \leq g$ and since $\operatorname{cf} h(n) \geq \omega_1$ for all n, it holds that $h \in X^g$ and by definition of h, also $h \in D_g$. □

Fact 2.30. *Suppose $g \in T \setminus X^R$. Then every subset of X^g of cardinality $\aleph_\omega$ is bounded in $(X^g, \leqq_{I_g})$ and the least cardinality of an unbounded subset of $(X^g, \leqq_{I_g})$, denoted by $\mathfrak{b}(X, \leqq_{I_g})$, is a regular cardinal.*

Proof. Let $B = \{f_\alpha : \alpha < \aleph_\omega\} \subseteq X^g$ be given. For every $m > 0$ and $n \in C^g_m$ let

$$f'(n) = \sup\bigl\{f_\alpha(n) : \alpha < \omega_{m-1} \text{ and } f_\alpha(n) < g(n)\bigr\}.$$

Since for $n \in C^g_m$ it holds that $\operatorname{cf} g(n) = \omega_m$, we have by the definition of f' that $f' < g$.

Let $\alpha < \aleph_\omega$ be given. Then for some m_α, which, without loss of generality, satisfies $\alpha < \omega_{m_\alpha}$, for all $m > m_\alpha$ and $n \in C^g_m$ it holds that $f_\alpha(n) < g(n)$; therefore $f_\alpha(n) \leq f'(n)$.

We showed that for every $\alpha < \aleph_\omega$ there is some m_α so that $(f_\alpha \restriction C^g_m) < (f' \restriction C^g_m) < (g \restriction C^g_m)$ for all $m \geq m_\alpha$. Find some $f \in X^g$ so that $f' \leq f$, and now we have $f_\alpha <_{I_g} f$ for all $\alpha < \aleph_\omega$ as required.

The proof that $\mathfrak{b}(X^g, \leqq_{I_g})$ is regular is straightforward. □

Claim 2.31. *Suppose $g \in T \setminus X^R$ and that $X \subseteq X^g$ is closed and cofinal in $(X^g, \leqq)$ and suppose that for some $k \in \omega$ the sequence $\{(h_\beta \restriction C^g_{>k}) : \beta < \omega_m\}$ is strictly increasing in $<$. Then there exists $h \in X$ and $A \in [\omega_m]^{\aleph_m}$ such that $\langle h_\alpha : \alpha \in A\rangle$ converges to h.*

Proof. For each $2 \leqq i \leqq k$ and for each $\alpha < \aleph_m$ there exists $f_i(\alpha) < \aleph_i$ such that $(h_\alpha \restriction C^g_i) = t^i_{f_i(\alpha)}$. That is, f_i is a mapping from $\aleph_m$ into $\aleph_i$.

Assume that for some i, $2 \leqq i < k$ we already constructed $A_i \in [\omega_m]^{\aleph_m}$ such that $\{t^j_{f_j(\alpha)} : \alpha \in A_i\}$ is a sequence converging to some t^j $(2 \leqq j \leqq i)$. We notice that either the set $f_{i+1}[A_i]$ is bounded in $\aleph_{i+1}$ and then there exists $A_{i+1} \in [A_i]^{\aleph_m}$ such that $f_{i+1} \restriction A_{i+1}$ is constant, or $f_{i+1}[A_i]$ is unbounded in $\aleph_m$ and then there exists $A_{i+1} \in [A_i]^{\aleph_m}$ such that for every $\alpha_1 < \alpha_2 < \aleph_m$ we have $f_m(\alpha_1) < f_m(\alpha_2)$. The sequence $\{t^{i+1}_{f_{i+1}(\alpha)} : \alpha \in A_{i+1}\}$ is either constant, or increasing in $<$. In both cases we define t^{i+1} as the supremum of $\{t^{i+1}_{f_{i+1}(\alpha)} : \alpha \in A_{i+1}\}$.

Let $A = A_k$. From our assumption we know that the sequence $\{(f^\alpha \restriction C^g_{>k}) : \alpha \in A_k\}$ is strictly increasing and it has a supremum t. We define $h = \bigcup_{i=2}^k t^i \cup t$. If $f < h$ then for each $n \in \omega$ there exists $2 \leqq i < \infty$ such that $n \in C^g_i$. We notice that the sequence $(f_\alpha \restriction C^g_i) = t^i_{f_i(\alpha)}$ $(\alpha \in A)$ is either constant or strictly increasing in $<$. In both cases there exists $\alpha_n \in A$ such that $f(n) < f_\alpha(n)$ for each $\alpha \geqq \alpha_n$. Let $\alpha = \sup_{n\in\omega} \alpha_n$. It follows that for each $n \in \omega$ we have $f(n) < f_\alpha(n)$. □

Theorem 2.32. *Suppose $g \in T \setminus X^R$ and X is closed and cofinal in $(X^g, \leqq)$ and that $\operatorname{tcf}(X, \leqq_{I_g}) = \lambda$. Then there is a cofinal Rudin subspace $Y \subseteq X$ of cardinality λ.*

Proof. We generalize here the proof from [10]. First, since $D_g = D^X_g$ is a closed and cofinal subset of X, by intersecting X with D_g, we may assume that $X \subseteq D_g$. Since $\lambda = \mathrm{tcf}(X, \leqq_{I_g})$, λ is regular by Fact 2.30, and greater than $\aleph_\omega$. We can fix a sequence $(h_\alpha : \alpha < \lambda)$, of elements of X which is $<_{I_g}$-increasing and $<_{I_g}$-cofinal in X such that for every $\alpha < \lambda$ with $\mathrm{cf}(\alpha) = \aleph_n$, if $\overline{h} \restriction \alpha$ is flat, then h_α is an eub of $\overline{h} \restriction \alpha$. Let

(viii) $$Y = \{h \in X : (\exists \alpha < \lambda)[h =_{I_g} h_\alpha]\}$$

For every $k, h \in Y$ we have either $k \leqq_{I_g} h$ or $h \leqq_{I_g} k$.

Fact 2.33. $|Y| = \lambda$.

Proof. (of the Fact) If $h \in Y$ there is a (unique) $\alpha < \lambda$ such that $h =_{I_g} h_\alpha$. This means that there is some m so that $h_\alpha(n) \neq h(n) \Longrightarrow n \in C^g_{\leq m}$. Since $X \subseteq D$, for every $m' \leq m$ the restriction $(h \restriction C^g_{m'})$ is one of $\omega_{m'}$ fixed functions. In total, the number of possible $h \in X$ which satisfy $h =_{I_g} h_\alpha$, for a given $\alpha < \lambda$, is $\leq \aleph_\omega$. This shows that $|Y| \leq \lambda \times \aleph_\omega = \lambda$. The converse inequality holds too, since $h_\alpha \in Y$ for all $\alpha < \lambda$. □

In the subsequent Fact 2.34, Fact 2.35 and Claim 2.36 we will show that Y is a closed subspace of X, and since Y is cofinal in X_g, it will finish the proof of the Theorem.

Fact 2.34. *Let $\langle \alpha(\beta) : \beta < \omega_m \rangle$ be a strictly increasing sequence of ordinals with supremum δ. If $\langle g_\beta : \beta < \omega_m \rangle$ is a sequence of functions in Y such that $\langle g_\beta \restriction C^g_{>k} \rangle$ is increasing in $<$ and $g_\beta =_{I_\beta} f_{\alpha(\beta)}$ for every $\beta < \omega_m$, then*

1. *$g = \sup\{g_\beta : \beta < \aleph_m\} \in \Pi_{c \in C^g_{>k}} \aleph_c$ is an eub of $\overline{f} \restriction \delta$,*
2. *$\mathrm{cf}(g(n)) = \aleph_m$ for $n \in C^g_{>k}$ and*
3. *$g =_{I_g} f_\delta$.*

Proof. (of the Fact) This is a modification of Claim 4 from [10]. For the reader's convenience we include a proof.

Condition (2) of the Fact is satisfied automatically. Also from assumptions of the Fact and from assumption about the sequence $\overline{f}$ follows immediately, that $\overline{f} \restriction \delta$ is flat and f_δ is an eub of $\overline{f} \restriction \delta$. Summarizing, to finish the proof of the Fact it is enough to show that Condition (3) is satisfied.

First, we will show that $f_\delta \geqq_I g$. Assume otherwise, that $(g \restriction A) >_I (f_\delta \restriction A)$ on a set $A \notin I$. It means that for some $\alpha < \delta$ we have

$$(g_\alpha \restriction C^g_{>k} \cap A) > (f_\delta \restriction C^g_{>k} \cap A).$$

However $\omega \setminus C^g_{>k} \in I$, hence $C^g_{>k} \cap A \notin I$ and as a consequence

$$(f_\alpha \restriction B) > (f_\delta \restriction B)$$

for some set $B \notin I$, a contradiction.

Assume now that $(f_\delta \restriction A) >_I (g \restriction A)$ on some $A \notin I$. Define h equal to 0 on $\omega \setminus A$ and g on A. From flatness of $f \restriction \delta$ follows that there exists $\alpha < \delta$ such that $f_\alpha >_I h$, hence $(f_\alpha \restriction B) > (g \restriction B)$ for some $B \notin I$, a contradiction with the fact that $g >_I f_\alpha$. □

Fact 2.35. *Let $\overline{h} = \langle h_\beta : \beta < \omega_m \rangle$ is a sequence of elements of Y such that for some $k \in \omega$ the sequence $\{(h_\beta \restriction C^g_{>k}) : \beta < \omega_m\}$ is strictly increasing in $<$. Then there exists $A \in [\omega_m]^{\aleph_m}$ such that* $\sup\{h_\beta : \beta \in A\} \in Y$.

Proof. (of the Fact) We apply Claim 2.31 and find $A \in [\omega_m]^{\aleph_m}$ such that $h = \sup\{h_\beta : \beta \in A\}$ is an element of X.

Fix $\alpha(\beta)$ such that $h_\beta = f_{\alpha(\beta)}$ and $\delta = \sup_{\beta<\aleph_m} \alpha(\beta)$. We apply Fact 2.34 and notice that $h =_I f_\delta$, hence $h \in Y$. □

Now we are ready to prove that if $t \in \mathrm{cl}_X(Y)$, then $t \in Y$. Let $M_k = \{n \in \omega : \mathrm{cf}(t(n)) = \aleph_k\}$. Since $t \in D_g$, from Definition 2.28 it follows that $C^g_k \subseteq \bigcup_{i \leqq k} M_i$, because if $n \in C^g_k$ then $t(n)$ is a limit of $\{t^k_\alpha : \alpha \in I\}$ for some $I \subseteq \aleph_k$. In particular $\mathrm{cf}(t(n)) \leqq \omega_k$.

For $k \geq 1$ define $T_k = \{f \in Y : f \leq t,\ \forall_{n\in\omega}\ \mathrm{cf}(f(n)) \leq \aleph_k$ and $(f \restriction \bigcup_{i\leqq k} M_i) = (t \restriction \bigcup_{i \leqq k} M_i)\}$.

Claim 2.36. *For every $k \in \omega$*

1. *For every $i \in \omega$ either $C^g_i \subseteq M_k$ or $C^g_i \cap M_k = \emptyset$ and*
2. *if $k \geq 1$ then the set T_k is a $\leqq$–cofinal subset of $\{f \in X : f \leqq t\}$ and*
3. *the set $L_k = \{i \in \omega : C^g_i \subseteq M_k\}$ is finite or co–finite.*

Before we begin the proof we observe that the Claim completes the proof of Theorem 2.32. Indeed, since $t \in X$, there exists $k \in \omega$ such that L_k is infinite, or equivalently, co–finite. Fix $f \in T_k$ and notice that $(f \restriction \bigcup_{i \in L_k} C^g_i) = (t \restriction \bigcup_{i\in L_k} C^g_i)$, in particular $f =_{I_g} t$ and from definition of Y, $t \in Y$.

Proof. (of the Claim) Statement (1) follows immediately from Definition 2.28. Indeed, fix $k \in \omega$. If $n, l \in C^g_i$ and $n \in M_k$, then there exists a sequence $\langle \alpha(\beta) : \beta < \aleph_k \rangle$ such that $\langle t^i_{\alpha(\beta)}(n) : \beta < \aleph_k \rangle$ is an increasing sequence converging to $t(n)$. However, in such a case $\langle t^i_{\alpha(\beta)}(l) : \beta < \aleph_k \rangle$ is an increasing sequence converging to $t(l)$ and $l \in M_k$.

Now we will prove Statement (2) only for natural numbers $k \geq 1$ smaller or equal to the first number number k_0 such that L_{k_0} is infinite. We proceed by induction with respect to $k \geq 1$. For $k = 0$ we define an auxiliary $T_0 = \{f \in P : f \leqq t\}$.

Assume that $k = 0$ or that Statement (2) was already proved for some k. We will prove Statement (2) for $k + 1$. Take $g < t$ and find a sequence $\overline{h} = \langle h_\alpha : \alpha < \omega_{k+1} \rangle$ such that

- (I) $h_\alpha \in T_k$ and
- (II) $h_\alpha > g$ and
- (III) $\overline{h} \restriction (\omega \setminus \bigcup_{i \leq k} M_i)$ is $<$–increasing and
- (IV) for every $n \in M_{k+1}$ the sequence $\langle h_\alpha(n) : \alpha < \omega_{k+1} \rangle$ converges to $t(n)$.

Notice, that the sequence $\overline{h}$ fulfills conditions of Fact 2.35 and from the Fact follows that T_{k+1} is a $\leqq$–cofinal subset of $\{f \in X : f \leqq t\}$. It shows that Statement (2) holds for all natural numbers k smaller or equal to the first number number k_0 such that L_{k_0} is infinite.

Since the families $\{C^g_k : k \in \omega\}$, $\{M_k : k \in \omega\}$ and $\{L_k : k \in \omega\}$ are pairwise disjoint, Statement (3) requires a proof only when k is the first natural number such that L_k is infinite. Assume that Statement (3) does not hold and that $k \in \omega$ is the smallest natural number such that L_k and $\omega \setminus L_k$ are infinite. Take any $f \in T_k$. We define

$$h(n) = \begin{cases} 0 & \text{if } n \in \bigcup_{i \in \bigcup_{j \leq k} L_j} C^g_i, \\ f(n) & \text{otherwise.} \end{cases}$$

Since $\bigcup_{i \in \bigcup_{j \leq k} L_j} C^g_i = \bigcup_{i \leq k} M_i$, if $n \in \omega \setminus \bigcup_{i \in \bigcup_{j \leq k} L_j} C^g_i$, we have $\mathrm{cf}(t(n)) > \aleph_k$. Since $f \in T_k$, in particular $\mathrm{cf}(f(n)) \leq \aleph_k$ and since $f \leq t$, we infer that $f(n) < t(n)$. It implies that $h < t$, hence there exists $w \in Y$ such that $w \in (h, t]$. Since $\omega \setminus \bigcup_{j \leq k} L_j$ is infinite and for every $n \in \bigcup_{i \in \bigcup_{j \leq k} L_j} C^g_i$ we have $w(n) > h(n) = f(n)$, it follows that $w \not<_{I_g} f$. It means that $w >_{I_g} f$, but for every $n \in \bigcup_{m \in L_k} C^g_m$ we have $w(n) \leqq t(n) = f(n)$, a contradiction (see formula (viii) defining space Y and a comment following the formula).

From Statement (3) and Statement (2) for natural numbers k smaller or equal to the first number number k_0 such that L_{k_0} is infinite, from definition of the space Y trivially follows Statement (2) for all natural numbers. □

As was mentioned earlier, the proof of Claim 2.36 completes the proof of Theorem 2.32. □

Lemma 2.37. *Suppose $g \in T \setminus X^R$ and that $X \subseteq X^g$ is closed and cofinal in $(X^g, \leqq)$. If $Y \subseteq D_g$ and g^* is an eub of Y, then there exists a sequence $\langle \alpha(m) : m \in \omega \rangle$ such that $\alpha(m) \leq \omega_m$ for all m and there exists some and $m_0 \in \omega$ such that for every $m \in \omega$, $m \geqq m_0$ it holds*

$$(g^* \restriction C^g_m) = t^m_{\alpha(m)}.$$

Proof. For every $m \in \omega$ we define

$$\alpha(m) = \sup\{\beta : t^\beta_m \leqq (g^* \restriction C^g_m)\}.$$

From Definition 2.28 follows

$$t^{\alpha(m)}_m \leqq (g^* \restriction C^g_m).$$

For $n \in C^g_m$ define

$$h(n) = t^{\alpha(m)}_m(n) \leqq g^*(n).$$

The proof of the Fact will be finished when we establish the equality

$$g^* =_{I_g} h.$$

From Fact 5.3 in the Appendix, in order to prove that $g^* =_{I_g} h$ it is enough to check, that h is an upper bound of Y modulo I_g. Assume, that there exists $f \in Y$ such that $f \nleqq_{I_g} h$. Let

$$L = \{m \in \omega : (f \restriction C^g_m) \nleqq (h \restriction C^g_m)\} = \{m \in \omega : (f \restriction C^g_m) > h(\restriction C^g_m)\}.$$

Since $f \not\leqq_{I_g} h$, the set L is infinite. The function g^* is an upper bound of Y, hence there exists $M_0 \in \omega$ such that

$$(f \restriction C_g^{>M_0}) \leqq (g^* \restriction C_g^{>M_0}). \tag{ix}$$

Let $m \in L$, $m > M_0$. Fix $\beta \leqq \aleph_m$ such that $f \restriction C_m^g = t_m^\beta$ and notice, that from inequality (ix) follows that $\beta \leqq \alpha(m)$ and in particular

$$(h \restriction C_m^g) = t_m^{\alpha(m))} \geqq t_m^\beta = (f \restriction C_m^g),$$

but $m \in L$, and from the definition of L follows that

$$(f \restriction C_m^g) > (h \restriction C_m^g),$$

a contradiction. □

The following is a Rudin spaces analog of Shelah's theorem that an $\aleph_{\omega+1}$-scale always exists [20, Theorem 2.5, page 50].

Theorem 2.38. *For every $g \in T \setminus X^R$ and a Rudin space X closed and cofinal in X^g there is some $g^* \leqq g$ in $T \setminus X^R$ so that*

(1) $I_{g^*} = I_g$
(2) $\operatorname{tcf}(X \cap X^{g^*}, \leqq_{I_{g^*}}) = \aleph_{\omega+1}$.

Proof. We assume that $X \subseteq D^g$.

First we shall define a $<_{I_g}$–increasing sequence $\overline{h} = \langle h_\alpha : \alpha < \aleph_{\omega+1}\}$ of members of X with an eub. In order to obtain an eub we will apply Theorem 5.6 from the Appendix. In particular the Theorem says that a $<_{I_g}$-sequence $\overline{h}$ has an eub if for all $k > 0$ the set

$$\{\alpha < \aleph_{\omega+1} : \operatorname{cf}\alpha = \aleph_k \wedge \overline{h} \restriction \alpha \text{ is flat mod } I_g\}$$

is stationary in $\aleph_{\omega+1}$.

Let $S = \bigcup_{n>0} S_n \subseteq \aleph_{\omega+1}$ be fixed so that $S \in I[\aleph_{\omega+1}]$ and $S_n \subseteq \{\alpha < \aleph_{\omega+1} : \operatorname{cf}\alpha = \aleph_n\}$ is stationary. This is possible due to Shelah's Theorem 5.8 (see the Appendix). Let $P_\alpha \subseteq \mathcal{P}(\alpha)$ be fixed for all $\alpha < \aleph_{\omega+1}$ so that $|P_\alpha| \leqq \aleph_\omega$, $\alpha < \beta < \aleph_{\omega+1}$ implies $P_\alpha \subseteq P_\beta$, $\operatorname{otp} c < \aleph_\omega$ for all $c \in P_\alpha$ and such that for all $\delta \in S_n$ there is $c \subseteq \delta$ of ordertype ω_n so that for all $\beta < \delta$ it holds that $c \cap \beta \in \bigcup_{\alpha<\delta} P_\alpha$.

By induction on $\alpha < \aleph_{\omega+1}$ construct a $<_{I_g}$-increasing sequence $\langle h_\alpha : \alpha < \aleph_{\omega+1}\rangle$ of members of X.

Let $h_0 \in X$ be arbitrary.

Suppose $\alpha < \aleph_{\omega+1}$ and h_β is defined for all $\beta < \alpha$. For all $c \in P_\alpha$ let $t_c = \sup\{h_{\beta+1} : \beta \in c\}$. Since $\operatorname{otp} c < \aleph_\omega$, it holds that $t_c <_{I_g} g$. Since $|P_\alpha| \leqq \aleph_\omega$ and $|\alpha| \leqq \aleph_\omega$, by Fact 2.30 it is possible to choose $h_\alpha \in X$ so that $t_c <_{I_g} h_\alpha$ for all $c \in P_\alpha$ and $h_\beta <_{I_g} h_\alpha$ for all $\beta < \alpha$.

Claim 2.39. *For $n > 0$ and every $\delta \in S_n$, $\overline{h} \restriction \delta$ is flat.*

Proof. Let $\delta \in S_n$ and fix $c \subseteq \delta$ cofinal in δ with $\operatorname{otp} c = \omega_n$ such that for all $\beta \in c$ there is some $\gamma < \delta$ so that $c \cap \beta \in P_\gamma$. According to Lemma 5.4

from the Appendix it suffices to show that $\overline{h} \restriction \delta$ is equivalent mod I_g to $\langle t_{c\cap\beta} : \beta \in c\rangle$ where, according to the definition, $t_{c\cap\beta} = \sup\{h_\alpha : \alpha \in c \cap \beta\}$.

First, if $\alpha < \delta$ is arbitrary, find $\beta \in c$ and $\eta \in c \cap \beta$ satisfying

$$\alpha < \eta < \beta.$$

Clearly,

$$h_\alpha \leqq_{I_g} h_\eta \leqq t_{c\cap\beta},$$

thus we see that for all $\alpha < \delta$ there is some $\beta \in c$ so that $h_\alpha \leqq_{I_g} t_{c\cap\beta}$.

Conversely, suppose $\beta \in c$ is given. There is some $\alpha < \delta$ so that $c\cap\beta \in P_\alpha$, therefore by the inductive construction of h_α it holds that $t_{c\cap\beta} <_{I_g} h_\alpha$. □

By Theorem 5.6 from the Appendix, there exists an exact upper bound g' of the sequence $\overline{h} = \langle h_\alpha : \alpha < \aleph_{\omega+1}\rangle$ so that

(x) $$\forall k \text{ the set } \{n : \mathrm{cf}(g'(n)) < \aleph_k\} \in I_g,$$

that is the set $\{n : \mathrm{cf}(g'(n)) < \aleph_k\}$ is contained in some $C^g_{\leqq m}$. Clearly, we may assume that $g' \leqq g$ and that $\mathrm{cf}(g'(n)) > \aleph_0$ for all n.

For each $\alpha < \aleph_{\omega+1}$ there is some m so that $h_\alpha(n) < g'(n)$ for all $n \in C^g_{<m}$, so there is some m so that

$$\{h \in X : \forall n \in C^g_{>m}\ h(n) < g'(n)\}$$

is cofinal mod $<_{I_g}$ below g'. Let now $g^* = (g \restriction C^g_{\leqq m}) \cup (g' \restriction C^g_{>m})$. Since g^* differs from g' only on some $C^g_{\leqq m}$, g^* is also an eub of $\overline{h}$ modulo I_g.

Claim 2.40. $I_{g^*} = I_g$.

Proof. We have to prove that

(1) for every $k \in \omega$ we have $C^g_k \in I_{g^*}$, or equivalently for every $k \in \omega$ there exists $m \in \omega$ such that $C^g_k \subseteq C^{g^*}_{\leqq m}$ and
(2) for every $k \in \omega$ we have $C^{g^*}_k \in I_g$, or equivalently for every $k \in \omega$ there exists $m \in \omega$ such that $C^{g^*}_k \subseteq C^g_{\leqq m}$.

First we will prove Statement (1). From Lemma 2.37 we may assume, that for every $k \in \omega$ there exists $\alpha(k) \leqq \aleph_k$ such that

$$(g^* \restriction C^g_k) = t^{\alpha(k)}_k.$$

We fix $k \in \omega$. Since from Definition 2.28 cofinality of $t^{\alpha(k)}_k$ is fixed, equal to some $\aleph_m$, we have $C^g_k \subseteq C^{g^*}_m$.

Statement (2) follows from Condition (x).

□

As $\overline{h}$ demonstrates, $\mathrm{tcf}(\{h \in X : h \leqq g^*\}, <_{I_g}) = \aleph_{\omega+1}$.

□

Theorem 2.41. *For every Rudin space X there is a Rudin space $Y \subseteq X$ with $|Y| = \aleph_{\omega+1}$.*

Proof. Suppose $g \in T \setminus X^R$ and $X \subseteq X^g$ is a Rudin space. By 2.38 there is some $g^* \leq g$ in $T \setminus X^R$ so that $I_{g^*} = I_g$ and $\mathrm{tcf}(X \cap X^{g^*}, \leq_{I_{g^*}}) = \aleph_{\omega+1}$. By 2.32, $X \cap X^{g^*}$ contains a closed and cofinal subspace Y of cardinality $\aleph_{\omega+1}$. □

3. Baire measures and their Borel extensions in Rudin spaces

In this section we prove that all Rudin spaces are non-Mařik and that every Rudin space whose cardinality is not real-valued-measurable is quasi-Mařik.

Definition 3.1. *For a given space X and a σ–field of subsets of X, a function $\mu : \Sigma \to \mathbb{R}$ is a* measure *if*

(1) $\emptyset, X \in \Sigma$, $\mu(\emptyset) = 0$, $0 \leqq \mu(X) < \infty$,
(2) *for every $A, B \in \Sigma$, if $A \subseteq B$ then $\mu(A) \leqq \mu(B)$,*
(3) *for every pairwise-disjoint family $\{A_n : n \in \omega\} \subseteq \Sigma$ it holds that*

$$\mu(\bigcup_{n\in\omega} A_n) = \sum_{n\in\omega} \mu(A_n).$$

Definition 3.2. *For a given space X and a measure μ on a σ–field Σ of subsets of X, we say that μ is concentrated on a singleton if there exists $x \in X$ such that for every $A \in \Sigma$ it holds that $\mu(A) = \mu(X)$ if and only if $x \in A$.*

Definition 3.3. *Let X be a topological space.*

(1) *A set $A \subseteq X$ is* functionally closed *if there exists a continuous function $f : X \to [0,1]$ such that $A = f^{-1}[\{0\}]$.*
(2) *The* Baire σ–field $\mathrm{Ba}(X)$ *on X is the σ–field generated by all functionally closed sets.*
(3) *A probability measure defined on* $\mathrm{Ba}(X)$ *is called a* Baire measure.

Recall that in a normal space X a closed set $D \subseteq X$ is *functionally closed* if and only if D is G_δ in X, and that X is called *perfectly normal* if every closed subset of X is functionally closed. Dowker spaces are never perfectly normal, since a perfectly normal space is countably paracompact.

Fact 3.4. *Let X be a Rudin space. The σ–field* $\mathrm{Ba}(X)$ *consists exactly of all clopen subsets of X.*

Proof. By Fact 2.17 every countable intersection of clopen sets is clopen. Thus, the family of all clopen subsets of X is a σ-field of sets.

Each clopen set is functionally closed trivially. Conversely, a functionally closed set is closed and G_δ hence clopen. □

Definition 3.5. *Let X be a topological space.*

(1) *The* Borel σ–field $\mathrm{Bo}(X)$ *on X is the σ–field generated all closed subsets of X.*
(2) *A measure on* $\mathrm{Bo}(X)$ *is called a Borel measure.*

Definition 3.6. *Let X be a topological space and let μ be a measure on* $\mathrm{Bo}(X)$. *A Borel measure μ we call a* regular *Borel measure if for every $A \in \mathrm{Bo}(X)$ it holds that*

$$\mu(A) = \sup\{\mu(F) : F \subseteq A,\ F \text{ closed}\}.$$

Definition 3.7. *Let X be a normal topological space. We call X a* Mařik *space if for every Baire measure $\mu : \mathrm{Ba}(X) \to [0,1]$ there exists a regular Borel measure which extends μ.*

Definition 3.8. *Suppose that X is a Rudin space. A Baire measure μ on X is called a* cofinal Baire measure *if there exists $0 < r \in \mathbb{R}$ such that for every set $A \in \mathrm{Ba}(X)$*

$$\mu(A) = r \text{ iff } A \text{ is cofinal in } X.$$

By Theorem 2.27 the collection of all clopen and cofinal subsets of a Rudin space X forms a σ-ultrafilter of clopen sets. Since the family of all clopen subsets of X coincides with the σ-field $\mathrm{Ba}(X)$, the clopen and cofinal subsets of X form a σ-ultrafilter of Baire sets. A cofinal Baire measure is, then, a measure that assigns a constant value $r > 0$ to all sets in this σ-ultrafilter and constant value 0 to all Baire sets which are not in this σ-ultrafilter.

Theorem 3.9. *Suppose $g \in T \setminus X^R$ and $X \subseteq X^g$ closed and cofinal in X^g. If μ is a cofinal Baire measure on X then μ does not admit a regular Borel extension.*

This theorem generalizes Simon's Theorem [18] that X^R is not Mařik. See also Wheeler [26].

Proof. Let μ be a cofinal Baire measure. We assume for simplicity that $\mu(X) = 1$. Assume to the contrary that there exists a regular Borel extension of μ and denote this extension also by μ.

According to Definition 2.23 for every $n \in \omega$ we have

$$D^X_m = X \cap D^g_m = \{h \in X : (\exists n \in C^g_{>m})\,[h(n) = g(n)]\}.$$

Recall that for every $m \in \omega$ the set D^X_m is closed and $\bigcap_{m\in\omega} D^X_m = \emptyset$. In particular, $\lim_{m\to\infty} \mu(D^X_m) = 0$. We fix $m_0 \in \omega$ such that $\mu(D^X_{m_0}) < \frac{1}{2}$.

By Definition 2.23, $X_{m_0} = X \setminus D^X_{m_0}$ and according to our choice of m_0 it holds that $\mu(X_{m_0}) > \frac{1}{2}$. The set X_{m_0} is open and from the regularity of μ we conclude that there exists a closed subset $F \subseteq X_{m_0}$ such that $\mu(F) \geqq \frac{1}{2}$. By normality of X there are open sets U and W such that $D^X_{m_0} \subseteq U$, $F \subseteq W$ and $U \cap W = \emptyset$.

Since $D^X_{m_0}$ is closed and cofinal and $D^X_{m_0} \subseteq U$ and U open, according to Lemma 2.25 there exists $f \in P$ so that $f < g$ and $X \cap (f, g] \subseteq U$. Thus U contains a clopen and cofinal set $X \cap (f, g]$ whose measure is 1. Consequently, $\mu(W) = 0$ and as $F \subseteq W$ also $\mu(F) = 0$, contrary to $\mu(F) \geqq \frac{1}{2}$. □

Theorem 3.10. *Suppose $g \in T \setminus X^R$ and $X \subseteq X^g$ is closed and cofinal in $(X^g, \leqq)$. Let μ be a cofinal Baire measure on X. Then there is some $m_0 > 0$ so that for all $m \geqq m_0$, μ extends to a Borel measure μ^m via the definition "$\mu^m(B) = \mu(X)$ if and only if B contains an m-club".*

Proof. Since $X \subseteq X^g$ is closed and cofinal in $(X^g, \leqq)$, by Fact 2.13 there exists $m_0 > 0$ so that X contains an m-club in X^g for all $m \geqq m_0$. We show that for all $m \geqq m_0$ the condition "$\mu^m(B) = \mu(X)$ iff B contains an m-club" defines a Borel measure μ^m which extends μ.

By Lemma 2.25 and the Example after Definition 2.7, we know that every clopen and cofinal subset B of X has the property, that for every $m \geqq m_0$ the set B contains an m–club. This shows, that

$$\mu^m(B) = \mu(X) = \mu(B)$$

for every set B belonging to $\mathrm{Ba}(X)$. Let

$$\mathcal{C} = \big\{B \subseteq X : (\forall m \geqq m_0)\ [B \text{ or } X \setminus B \text{ contains an } m\text{–club of } X^g]\big\}.$$

To finish the proof it is enough to show that $\mathcal{C}$ contains all Borel sets. We prove this by showing that $\mathcal{C}$ is a σ–field to which all closed subsets of X belong.

Suppose that $D \subseteq X$ is closed. By Lemma 2.14, either D or $X \setminus D$ contains an m-club of X^g for every $m \geqq m_0$, hence $D \in \mathcal{C}$.

Obviously, if $B \in \mathcal{C}$, then $X \setminus B \in \mathcal{C}$. To see that $\mathcal{C}$ is closed under countable intersections, suppose we are given $B_n \in \mathcal{C}$ for each $n \in \omega$ and that $m \geqq m_0$ is fixed. Either for every $n \in \omega$ the set B_n contains an m-club, and then by Fact 2.8, the intersection $\bigcap_n B_n$ also contains an m–club, or for some $n \in \omega$ the set B_n does not contain an m–club, and then $X \setminus B_n$ contains an m–club, and since

$$X \setminus B_n \subseteq X \setminus \bigcap_{n\in\omega} B_n,$$

also $X \setminus \bigcap_{n\in\omega} B_n$ contains an m–club. □

Let us also comment that the extension described in Theorem 3.10 is not unique. If $m_2 > m_1 \geqq m_0$ then $\mu^{m_1} \neq \mu^{m_2}$, since any m_1-club is disjoint from any m_2-club. So for a given Baire measure μ, every sequence $\overline{r} = \{r_m\}_{m \geqq m_0}$ of nonnegative reals with $\sum_{m\geqq m_0} r_m = 1$ corresponds injectively to a Borel extension $\mu_{\overline{r}}$ of μ via $\mu_{\overline{r}}(B) = \sum_{m\geqq m_0} r_m \cdot \mu^m(B)$.

Definition 3.11. *A cardinal κ is real–valued measurable if there exists a measure $\mu : P(\kappa) \to [0, 1]$ which is 0 on singletons and such that $\mu(\kappa) = 1$.*

If $\kappa < \lambda$ are cardinals and κ is real-valued measurable, then clearly also λ is real-valued measurable.

Theorem 3.12. *(S. Ulam,* [13, Theorem 1Dc]*) The smallest real–valued-measurable cardinal is weakly inaccessible, that is, is a regular limit cardinal. In particular, if $\aleph_\alpha$ is the least real-valued-measurable cardinal it holds that $\aleph_\alpha = \alpha$, namely, $\aleph_\alpha$ is a fixed point of the $\aleph$ function.*

From Ulam's theorem it follows that $\aleph_{\omega+1}$ is not real-valued-measurable. (The smallest fixed point of the $\aleph$ function is much larger than $\aleph_{\aleph_\omega}$, while $\aleph_{\omega+1} < \aleph_{\aleph_\omega}$.) Thus, we know by Theorem 2.41 that every Rudin space contains a Rudin subspaces whose cardinality is not real-valued measurable.

In the next theorem the reason for working with the full class of Rudin spaces becomes clear. In the generality of this class we can prove a structure theorem for all Baire measure on sufficiently small Rudin spaces.

Theorem 3.13. *Suppose $g \in T \setminus X^R$ and $X \subseteq X^g$ is closed and cofinal in $(X^g, \leqq)$. Suppose that $|X|$ is not real-valued measurable. Suppose μ is a Baire measure on X. Then there are countable sets I and J, elements $f_i \in X$ for all $i \in I$, clopen Rudin subspaces $X_j \subseteq X$ for all $j \in J$ and measures μ_i for $i \in I$ and μ_j for $j \in J$ such that:*

1. *for every $i \in I$, μ_i is a measure on X concentrated on the singleton $\{f_i\}$;*
2. *if $j_1, j_2 \in J$ and $j_1 \neq j_2$ then $X_{j_1} \cap X_{j_2} = \emptyset$;*
3. *for every $j \in J$, μ_j is a cofinal Baire measure on X_j.*

Finally:

$$\mu = \sum_{i \in I} \mu_i + \sum_{j \in J} \mu_j$$

Proof. Fix a Baire measure μ on X and assume for simplicity that $\mu(X) = 1$.

For $n \in \omega$ and $\alpha \leqq g(n)$ let $U_{n,\alpha} := \{f \in X : f(n) \leqq \alpha\}$. This is a clopen set in X, and therefore belongs to $\mathrm{Ba}(X)$.

For each $n \in \omega$ we define by induction on $\xi < \xi_n$, for some ordinal $\xi_n < \omega_1$ which will be specified below, a strictly increasing and continuous countable sequence of ordinals $\alpha^n_\xi \leq g(n)$ Assuming, that we already know a^n_ξ, we define a real number $r^n_\xi \in [0,1]$ by

$$r^n_\xi := \mu(U_{n,\alpha^n_\xi}). \tag{i}$$

Let $\alpha^n_0 = 0$. Since $f(n) > 0$ for all $f \in X$ we have $U_{n,0} = \emptyset$, so $r^n_0 := \mu(U_{n,\alpha_0}) = 0$.

When $\xi < \omega_1$ is limit, let $\alpha^n_\xi = \sup\{\alpha^n_\zeta : \zeta < \xi\}$. Since $\mathrm{cf}\, \alpha^n_\xi = \aleph_0$, it follows that $f(n) \neq \alpha^n_\xi$ for all $f \in X$ and therefore $\bigcup_{\zeta<\xi} U_{n,\alpha^n_\zeta} = U_{n,\alpha^n_\xi}$. Hence $r^n_\xi = \sup\{r^n_\zeta : \zeta < \xi\}$.

If r^n_ζ is defined and $r^n_\zeta < 1$, then necessarily $\alpha^n_\xi < g(n)$, since $\mu(U_{n,g(n)}) = \mu(X) = 1$. Let

$$\alpha^n_{\xi+1} = \min\{\alpha \leqq g(n) : \mu(U_{n,\alpha}) > r^n_{\alpha_\xi}\}. \tag{ii}$$

If $r^n_\xi = 1$ we cease the induction and put $\xi_n = \xi + 1$. The induction has to terminate at some $\xi_n < \omega_1$, or else $\{r^n_\zeta : \zeta < \omega_1\} \subseteq [0,1]$ would be order isomorphic to ω_1, which is impossible.

Claim 3.14. *For each $n \in \omega$ and $\xi < \xi_n$, $\mathrm{cf}\, \alpha^n_\xi > \aleph_0$ if and only if ξ is a successor ordinal.*

Proof. If $\xi < \xi_n$ is limit, then by continuity α^n_ξ has cofinality $\aleph_0$.

Suppose that $\xi = \zeta + 1 < \xi_n$. First we observe that $\alpha^n_{\zeta+1}$ cannot be a successor, since if $\alpha^n_{\zeta+1} = \beta + 1$ then $U_{n,\beta+1} = U_{n,\beta}$, contrary to the minimality of $\alpha^n_{\zeta+1}$. We know then that $\alpha^n_{\zeta+1}$ is limit, and need only prove that its cofinality is uncountable. Suppose to the contrary that $\langle \beta_i : i \in \omega \rangle$ is strictly increasing with limit $\alpha^n_{\zeta+1}$ and that $\beta_0 > \alpha^n_\zeta$. By the definition of $\alpha^n_{\zeta+1}$ (see formula ii) it holds that

$$\mu(U_{n,\beta_i}) = \mu(U_{n,\alpha^n_\zeta})$$

for all $i \in \omega$. Since μ is σ–additive,

$$\mu(\bigcup_i U_{n,\beta_i}) = \mu(U_{n,\alpha^n_\zeta})$$

and since cf $\alpha^n_{\zeta+1} = \aleph_0$ it holds that

$$\bigcup_i U_{n,\beta_i} = U_{n,\alpha^n_{\zeta+1}}.$$

This contradicts

$$\mu(U_{n,\alpha^n_{\zeta+1}}) > \mu(U_{n,\alpha^n_\zeta}).$$

□

For each $n \in \omega$ let $S_n = \{\alpha^n_\xi : \xi < \xi_n\}$.

Suppose that $x \in \prod_{n\in\omega}(S_n \setminus \sup\{S_n\})$; then for every $n \in \omega$, $\min\{S_n \setminus x(n)+1\}$ is well defined. Denote by x^s the function in $\prod_{n\in\omega} S_n$ that satisfies

$$x^s(n) = \min\{S_n \setminus (x(n) + 1)\}$$

for all $n \in \omega$.

For every $x \in \prod_{n\in\omega}(S_n \setminus \{\sup S_n\})$ let

$$U_x = (x, x^s] = \big\{f \in X : (\forall n \in \omega)\ [x(n) < f(n) \leqq x^s(n)]\big\}.$$

Let $x \in \prod_{n\in\omega} S_n$ and suppose that for some $n \in \omega$ it holds that $x(n) = \max S_n$. In that case let

$$U_x = (x, g].$$

If $x(n) = g(n)$ for some $n \in \omega$ then $U_x = \emptyset$. Thus U_x is a basic clopen set of X for all $x \in \prod_{n\in\omega} S_n$.

In the case that $x(n) = \max S_n$ and $\max S_n < g(n)$ it holds that $U_x \subseteq \{f \in X : f(n) > \max S_n\}$ and since $\mu(U_{n,\max S_n}) = 1$ it follows that $\mu(U_x) = 0$. Hence,

Claim 3.15. *If $x \in \prod_{n\in\omega} S_n$ and for some $n \in \omega$ it holds that $x(n) = \max S_n$ then $\mu(U_x) = 0$.*

It is obvious that $X = \bigcup_{x\in\prod_{n\in\omega} S_n} U_x$. If $x \neq y$ and $x, y \in \prod_{n\in\omega} S_n$ then clearly $U_x \cap U_y = \emptyset$.

Since $|X|$ is not real–valued measurable, also the cardinality of the set

$$A = \{x \in \prod_{n\in\omega} S_n : U_x \neq \emptyset\}$$

is not real–valued measurable. Given an arbitrary $D \subseteq A$, both $\bigcup_{x\in D} U_x$ and $\bigcup_{x\in A\setminus D} U_x$ are open, hence each of them is also clopen and μ-measurable.

By letting $\mu'(D) = \mu(\bigcup_{x\in D} U_x)$ we define a measure μ' on $\mathcal{P}(A)$. Since $|A|$ is not real valued measurable, according to [12, Lemma 438Bb], we conclude that μ' is a countable union of measures concentrated on singletons, in particular that there exists a countable subset $H \subseteq A$ such that

$$\mu(U_x) > 0 \text{ iff } x \in H$$

and

$$\mu\big(\bigcup\{U_x : x \in A \setminus H\}\big) = 0.$$

From now on we work with fixed x, x^s, assuming that:

- $\mu((x, x^s]) = \mu(U_x) = \epsilon > 0$
- for every $n \in \omega$ and $\alpha \in \big(x(n), x^s(n)\big)$ it holds that $\mu\big(\{t \in X : x(n) < t(n) \leqq \alpha\}\big) = 0$.

The second item follows from the minimality of $x^s(n)$. From this and from countable additivity of μ follows:

Claim 3.16. *For every function h which satisfies $x < h < x^s$ it holds that $\mu\big((h, x^s]\big) = \epsilon$; in particular, $X \cap (h, x^s] \neq \emptyset$.*

The set H is countable and now we decompose it into two subsets I and J, as promised in the statement of the Theorem, according to the following two cases:

Case 1: $x^s \in X^R$. In this case, since X is closed in X^R, from the Claim 3.16 follows that for each $h < x^s$ it holds that $X \cap (h, x^s] \neq \emptyset$, in particular $x^s \in X$. Thus, for every clopen and cofinal $U \subseteq (x, x^s]$ it holds that $x^s \in U$ and μ restricted to $(x, x^s]$ is concentrated on a singleton, because on a clopen $U \subseteq (x, x^s]$ we have $\mu(U) = \epsilon$ if and only if $x^s \in U$.

Case 2: $x^s \notin X^R$. Since $\aleph_0 < \operatorname{cf} x^s(n) < \aleph_\omega$ for all $n \in \omega$, $x^s \in T \setminus X^R$. From Claim 3.16 we conclude that $\mu \restriction \mathrm{Ba}(X \cap U_x)$ is a cofinal Baire measure on a Rudin space $X \cap U_x = X \cap (x, x^s]$.

We define

$$I = \{x : x^s \in H \cap X^R\} \tag{iii}$$

$$J = \{x : x^s \in H \setminus X^R\}. \tag{iv}$$

For every $i = x \in I$ let $\mu_i := \mu \restriction \mathrm{Ba}(X \cap (x, x^s]$. This is indeed a measure concentrated on the singleton $\{f_i\} := \{x^s\}$.

For each $j = x \in J$ let $\mu_j = \mu \restriction \mathrm{Ba}(X \cap (x, x^s])$. This is indeed a cofinal Baire measure on the Rudin space $X_j := X \cap (x, x^s]$. For $j_1 \neq j_2$ in J we have established that $X_{j_1} \cap X_{j_2} = \emptyset$.

It remains to show that $\mu = \sum_{i\in I}\mu_i + \sum_{j\in J}\mu_j$. Let $B \subseteq X$ be an arbitrary Baire set. As $\mu(\bigcup_{x\in A\setminus H} U_x) = 0$, it holds that $\mu(B) = \mu(B \cap \bigcup_{x\in H} U_x)$. Since $\{U_x : x \in H\}$ is a pairwise-disjoint family of sets from $\mathrm{Ba}(X)$, it holds that $\mu(B) = \sum_{x\in H}\mu(B\cap U_x)$ which is exactly what we need. □

Theorem 3.17. *If X is a Rudin space and $|X|$ is not a real-valued measurable cardinal, then X is quasi-Mařik.*

Proof. Suppose X is a Rudin spaces and $|X|$ is not real-valued-measurable. Let I, J, μ_i, μ_j and X^i, X^j for $i \in I, j \in J$ and f_i for $i \in I$ be as stated in Theorem 3.13. It is enough to extend each of measures μ_i, μ_j to Borel measures on X^i and X^j respectively.

If $i \in I$ then μ_i is concentrated on the singleton $\{f_i\}$. We define the extension of measure μ_i on the whole $\mathcal{P}(X)$ by the formula $\mu_i(A) = \mu_i(X)$ if and only if $f_i \in A$ for each $A \in \mathcal{P}(X)$.

For $j \in J$ the extension of μ_j to a Borel measure on X_j exists by Theorem 3.10. □

Corollary 3.18 (ZFC)**.** *There are quasi-Mařik non-Mařik Dowker space. In fact, every Rudin space contains a quasi-Mařik non Mařik Rudin subspace of cardinality $\aleph_{\omega+1}$.*

Proof. Suppose $X \subseteq X^R$ is a Rudin space. By Theorem 2.41 fix a Rudin subspace $Y \subseteq X$ with $|Y| = \aleph_{\omega+1}$. From Theorem 3.12 the cardinal $|Y|$ is not real–valued measurable. Now apply Theorem 3.17. □

Theorem 3.19. *If the continuum is not real-valued measurable then every Rudin space is quasi Mařik.*

Proof. The cardinality of every Rudin space is at most $|X^R| = (\aleph_\omega)^{\aleph_0}$. Shelah's ω_4-inequality is as follows:

$$(\aleph_\omega)^{\aleph_0} < \max\{(2^{\aleph_0})^+, \aleph_{\omega_4}\}.$$

Since the smallest fixed point of the $\aleph$ function is larger than $\aleph_{\omega_4}$, no cardinal below $\aleph_{\omega_4}$ is real-valued-measurable. Thus, $(\aleph_\omega)^{\aleph_0}$ is real-valued measurable if and only if $2^{\aleph_0}$ is.

Since we assume that $2^{\aleph_0}$ is not real-valued measurable, $(\aleph_\omega)^{\aleph_0}$ and hence also the cardinality of every Rudin space is not real-valued measurable. It now follows that every Rudin space is quasi-Mařik by Theorem 3.17. □

4. Discussion and concluding remarks

We conclude with a short discussion of two general issues in set theoretic topology and in set theory.

4.1. **The small Dowker space problem.** Since the discovery of Rudin's space (1), M. E. Rudin herself has argued that the cardinality $(\aleph_\omega)^{\aleph_0}$ of her space was too big, and has repeatedly promoted the problem of finding an absolute Dowker space with small cardinal characteristics (cardinality, weight, local character, etc.). This problem is referred to in the literature as "the Small Dowker space problem". The question is, of course, what "small" means exactly.

Balogh's space (2) of cardinality $2^{\aleph_0}$ was accepted by many topologists as a solution to the small Dowker space problem. Kojman and Shelah have argued that their space (3) was in fact a more adequate solution to this problem because its cardinality, weight and local character are absolute cardinals ($\aleph_{\omega+1}$, $\aleph_\omega$ and $\aleph_\omega$ respectively).

ZFC allows a proper class of $\aleph$-s as consistent values to $2^{\aleph_0}$, so it is not a priori clear how to compare $2^{\aleph_0}$ with $\aleph_{\omega+1}$. If one goes by the *smallest* possible value of $2^{\aleph_0}$, which is $\aleph_1$, then $2^{\aleph_0}$ is indeed smaller than $\aleph_{\omega+1}$; if, on the other hand, one chooses to measure $2^{\aleph_0}$ in the "sup norm", then it is much bigger than $\aleph_{\omega+1}$, and is actually equal to $(\aleph_\omega)^{\aleph_0}$ (if $2^{\aleph_0} > \aleph_\omega$ then $2^{\aleph_0} = (\aleph_\omega)^{\aleph_0}$).

The measure theoretic properties of Dowker spaces, dealt with in this paper, reveal a crucial largeness property that $2^{\aleph_0}$ and $(\aleph_\omega)^{\aleph_0}$ possess, but $\aleph_{\omega+1}$ does not. This is the possibility of being real-valued measurable: if measurable cardinals can exist, then $2^{\aleph_0}$ and $(\aleph_\omega)^{\aleph_0}$ could be real-valued measurable; but $\aleph_{\omega+1}$ is never real-valued measurable. We feel that this supports the thesis that among the three Dowker spaces under discussion, it is space (3) which is "small".

4.2. **PCF and Verifiable consequences of Gödel's constructibility axiom.** The second issue we address is the relation between ZFC and Gödel's constructibility axiom $V = L$. .

Foreman, in his recent discussion [11], lays out criteria for evaluating axiom systems for set theory, among which he lists Gödel's criterion of *prediction*, phrased by Gödel in his discussion of the continuum hypothesis [7] as the property of having *verifiable consequences*. Foreman considers the combinatorial principles that Jensen discovered in the constructible universe as having at least "methodological predictions" in ZFC, in the form of their weaker ZFC forms discovered by Shelah (see [11]). Shelah's combinatorics plays a central role in the development of PCF theory, and are also used in the further development of the theory for the purposes this paper.

We suggest that the existence of a quasi-Mařik non Mařik space is a verifiable consequence of $V = L$ in measure theory. The existence of such a space on $\aleph_1$ follows from $\clubsuit(\omega_1)$ by Aldaz [2] and is now proved also in ZFC — on the larger cardinal $\aleph_{\omega+1}$. The role of PCF theory in this demonstrates the theory's typical ability to act as an agent of the following verifiability relation between $V = L$ and ZFC: $V = L$ theorems resurface via pcf theory as ZFC theorems — after some delay. In the present case the $V = L$

combinatorics for Aldaz' constructions on ω_1 is replaced by PCF theory on $\aleph_{\omega+1}$ and the delay is from $\aleph_1$ to $\aleph_{\omega+1}$. There are other examples as well, the most spectacular of which is Shelah's revised GCH theorem, that states that a form of the GCH holds after $\beth_\omega$ [22, 23].

It will be interesting to know whether PCF technology is necessary for answering the measure extension problem.

Let us conclude with three problems. The first problem is old and well known. The second is a weaker form of the first, localized to the measure-theoretic context.

Problem 4.1. *Is it consistent with ZFC that there are no Dowker spaces of cardinality $\aleph_1$?*

Problem 4.2. *Is it consistent that there is no counter example to the measure extension problem of cardinality $\aleph_1$?*

Problem 4.3. *Is it provable in ZFC that every Rudin space is quasi-Mařik?*

5. Appendix: PCF notation preliminaries

We set some standard PCF theory terminology and preliminary PCF facts. The reader may consult [1, 5, 9, 20] for additional details.

Let ON^ω denote the class of all functions from ω to the Ordinal Numbers. Let $0 \in \mathrm{ON}^\omega$ stand for the constant function 0.

Definition 5.1. *Suppose $I \subseteq \mathcal{P}(\omega)$ is an ideal on ω*

(1) *For $f, g \in \mathrm{ON}^\omega$ we write*

$$f \leqq_I g \text{ if } \{n : f(n) > g(n)\} \in I,$$

$$f <_I g \text{ if } \{n : f(n) \geqq g(n)\} \in I$$

and

$$f =_I g \text{ if } \{n : f(n) \neq g(n)\} \in I.$$

In the case $I = \{\emptyset\}$ we omit I from the notation and write just $<, \leqq$ and $=$.

(2) *A function $h \in \mathrm{ON}^\omega$ is an* upper bound *of a set $A \subseteq \mathrm{ON}^\omega$ with respect to $<_I$ (modulo $<_I$, mod $<_I$, modulo I) if for every $f \in I$ we have $f \leqq_I h$. For $A \subseteq X \subseteq \mathrm{ON}^\omega$ we say that X is* unbounded in X *with respect to $\leqq_I$ there is no upper bound of A in X with respect to $\leqq_I$.*

(3) *For $X \subset \mathrm{ON}^\omega$ we define $\mathfrak{b}(X, \leqq_I)$ as the smallest cardinality of $\leqq_I$-unbounded subset of X, if X has no maximum, and as ∞ otherwise, where ∞ is taken to be larger then every cardinal.*

(4) *For $A \subseteq X \subset \mathrm{ON}^\omega$, A is* cofinal *in $(X, \leqq_I)$ if for every $f \in X$ there exists $h \in A$ such that $f \leqq_I h$. The* cofinality *of $(X, \leqq_I)$, denoted* $\mathrm{cf}(X, \leqq_I)$*, is the smallest cardinality of $A \subseteq X$ which is cofinal in $(X, \leqq_I)$.*

(5) *For* $X \subseteq \mathrm{ON}^\omega$ *we say that* $(X, \leqq_I)$ has true cofinality *if* $\mathfrak{b}(X, \leqq_I) = \mathrm{cf}(X, \leqq_I)$. *If* $(X, \leqq_i)$ *has true cofinality we define the* true cofinality *of* $(X, \leqq_I)$, *denoted* $\mathrm{tcf}(X, < I)$, *by*

$$\mathrm{tcf}(X, \leqq_I) = \mathfrak{b}(X, \leqq_I).$$

We remark that unless $\mathfrak{b}(X, \leqq_I) = \infty$ it holds that $\mathfrak{b}(X, \leqq_I) \leqq \mathrm{cf}(X, \leqq_I I)$ and unless $\mathfrak{b}(X, \leqq_I)$ is finite, it is an infinite *regular* cardinal.

Let $I \subseteq \mathcal{P}(\omega)$ be an ideal over ω which contains all finite subsets of ω.

Definition 5.2. *Let* $I \subseteq \mathcal{P}(\omega)$ *be an ideal over* ω. *A function* $h \in \mathrm{ON}^\omega$ *is an exact upper bound (eub) of* $A \subseteq \mathrm{ON}^\omega$ *with respect to* I *if*

(1) *h is an upper bound of A with respect to* $\leqq_I$ *and*
(2) *for every* $w \in \mathrm{ON}^\omega$, *if* $w <_I h$ *there exists* $f \in A$ *such that* $w <_I f <_I h$.

Fact 5.3. *Let* $I \subseteq \mathcal{P}(\omega)$ *be an ideal over* ω. *If* $A \subseteq \mathrm{ON}^\omega$ *contains some* h *such that* $0 <_I h$ *and both* $g, h \in \mathrm{ON}^\omega$ *are eubs of* A *with respect to* $<_I$ *then* $g =_I h$.

Lemma 5.4. *(see* [8, Claim 5]*) Let* $I \subseteq \mathcal{P}(\omega)$ *be an ideal over* ω. *Let* κ *be a regular uncountable cardinal. Let* $\overline{f} = \langle f_\alpha \in \mathrm{ON}^\omega : \alpha < \delta\rangle$ *be an* $<_I$*-increasing sequence of functions. The following conditions are equivalent:*

(1) *There is an eub* $f \in \mathrm{ON}^\omega$ *of* $\overline{f}$ *such that* $\{n \in \omega : \mathrm{cf}(f(n)) \neq \kappa\} \in I$.
(2) *There exists a sequence* $\overline{h} = \langle h_\alpha \in \mathrm{ON}^\omega\rangle$ *such that the sequence* $\langle h_\beta : \beta < \kappa\rangle$ *is* $<$*–increasing and*
    (a) *for every* $\alpha < \delta$ *there exists* $\beta < \kappa$ *such that*
    $$f_{\alpha(\beta)} <_I h_\beta$$
    and
    (b) *for every* $\beta < \kappa$ *there exists* $\alpha < \delta$ *such that*
    $$h_\beta <_I f_\alpha.$$

Definition 5.5. *A given* $<_I$*– increasing sequence of functions* $\langle f_\alpha \in \mathrm{ON}^\omega : \alpha < \lambda\rangle$ *is* flat of cofinality κ *if one of the equivalent conditions of Lemma 5.4 is satisfied.*

Theorem 5.6. *(*[8],[9, Theorem 20]*) Let* $I \subseteq \mathcal{P}(\omega)$ *be an ideal over* ω. *Let* $\lambda > \aleph_1$ *be a regular cardinal and let* $\overline{f} = \langle f_\alpha \in \mathrm{ON}^\omega : \alpha < \lambda\rangle$ *be a* $<_I$*– increasing sequence of functions. For every regular* κ *such that* $\omega < \kappa < \lambda$ *the following conditions are equivalent:*

(1) *The sequence* $\overline{f}$ *has an eub* f *and*
    $$\{n \in \omega : \mathrm{cf}(f(n)) \leqq \kappa\} \in I.$$
(2) *The set* $\{\delta < \lambda : \mathrm{cf}(\delta) = \kappa,\ \overline{f} \restriction \delta$ *is flat of cofinality* $\kappa\}$ *is stationary in* λ.

Definition 5.7. *(Shelah's $I[\lambda]$ ideal, see* [20, Definition 2.3, page 14]*) For a regular uncountable cardinal λ we define an ideal $I[\lambda]$ as the family of all $S \subseteq \lambda$ such that there exists a sequence of sets $\langle P_\alpha : \alpha < \lambda \rangle$ and a club $E \subseteq \lambda$ with the following properties:*

1. *$P_\alpha \subseteq P(\alpha)$, $|P_\alpha| < \lambda$,*
2. *for every $\delta \in E \cap S$ there exists $c \subseteq \delta$, $\sup(c) = \delta$, $\mathrm{otp}(c) = \mathrm{cf}(\delta) < \delta$ and for every $\beta \in c$ we have $c \cap \beta \in \bigcup_{\beta<\delta} P_\delta$.*

Theorem 5.8 (Shelah,[21])**.** *For any two regular cardinals κ and λ such that $\kappa^+ < \lambda$ there exists a stationary set $S \subseteq \{\alpha < \lambda : \mathrm{cf}(\alpha) = \kappa\}$ such that $S \in I[\lambda]$.*

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