

ON THE ARITHMETIC OF DENSITY

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ABSTRACT. The κ -density of a cardinal $\mu \geq \kappa$ is the least cardinality of a dense collection of κ -subsets of μ and is denoted by $\mathcal{D}(\mu, \kappa)$. The *Singular Density Hypothesis* (SDH) for a singular cardinal μ of cofinality $\text{cf}\mu = \kappa$ is the equation $\overline{\mathcal{D}}(\mu, \kappa) = \mu^+$, where $\overline{\mathcal{D}}(\mu, \kappa)$ is the density of all unbounded subsets of μ of ordertype κ . The *Generalized Density Hypothesis* (GDH) for μ and λ such that $\lambda \leq \mu$ is:

$$\mathcal{D}(\mu, \lambda) = \begin{cases} \mu & \text{if } \text{cf}\mu \neq \text{cf}\lambda \\ \mu^+ & \text{if } \text{cf}\mu = \text{cf}\lambda. \end{cases}$$

Density is shown to satisfy Silver's theorem. The most important case is:

Theorem (Theorem 2.6). *If $\kappa = \text{cf}\kappa < \theta = \text{cf}\mu < \mu$ and the set of cardinals $\lambda < \mu$ of cofinality κ that satisfy the SDH is stationary in μ then the SDH holds at μ .*

A more general version is given in Theorem 2.8.

A corollary of Theorem 2.6 is:

Theorem (Theorem 3.5). *If the Singular Density Hypothesis holds for all sufficiently large singular cardinals of some fixed cofinality κ , then for all cardinals λ with $\text{cf}\lambda \geq \kappa$, for all sufficiently large μ , the GDH holds.*

1. INTRODUCTION

Eventual regularity is a recurring theme in cardinal arithmetic since the discovery of pcf theory. Arithmetic rules that do not necessarily hold for all cardinals, can sometimes be seen to hold in appropriate end-segments of the cardinals.

The most famous precursor of modern cardinal arithmetic is *Silver's theorem* [18], which says that if one of the arithmetic equations (1) the *Singular Cardinal Hypothesis* (SCH); or (2) the *Generalized Continuum Hypothesis* (GCH), holds sufficiently often below a singular of uncountable cofinality, then it holds at the singular itself.

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Silver's theorem came as a surprise in 1973, shortly after Solovay and Easton employed Forcing, that was discovered by Cohen in 1963, to prove that no non-trivial bound on the power of a regular cardinal could be deduced from information about the powers of smaller cardinals. At the time, all set theorists believed that no such implications existed and that further development of Forcing would clear the missing singular case soon (see [9] for the history of the subject and for a survey of other precursors of pcf theory, e.g. in topology).

The present article concerns the eventual regularity of the cardinal arithmetical function *density*. The density function $\mathcal{D}(\mu, \kappa)$ is defined for cardinals $\kappa \leq \mu$ as the least cardinality of a collection $\mathcal{D} \subseteq [\mu]^\kappa$ which is *dense* in $\langle [\mu]^\kappa, \subseteq \rangle$.

A detailed definition and basic properties of density appear in Section 2 below. Let us point out now, though, one crucial difference between $\mathcal{D}(\mu, \kappa)$ and the exponentiation μ^κ : the function $\mathcal{D}(\mu, \kappa)$ is *not* monotone increasing in the second variable. For example, if μ is a strong limit cardinal of cofinality ω then $\mathcal{D}(\mu, \aleph_0) = \mu^+ > \mathcal{D}(\mu, \aleph_1) = \mu$.

Recently, asymptotic results in infinite graph theory and in the combinatorics of families of sets [10, 11] — some of which were proved earlier with the GCH or with forms of the SCH [5, 3, 7, 12, 13] — were proved in ZFC by making use of an eventual regularity property of density: that density satisfies a version of Shelah's RGCH theorem. See also [17] on the question whether the use of RGCH in [10] is necessary.

1.1. The results. Three theorems about the eventual behaviour of density are proved below. Theorem 2.6 is a density version of the most popular case of Silver's theorem and Theorem 2.8 is a density version of the general Silver theorem. They deal with the way the behaviour of density at singular cardinals of cofinality κ below a singular μ of cofinality $\theta > \kappa$ bounds the θ -density at μ .

An elementary proof of Silver's theorem [2] was published by Baumgartner and Prikry soon after Silver's original proof was found. An almost identical elementary proof of Silver's theorem was discovered independently of [2] by Jensen in 1973, but was only circulated and not published (see the introduction to [2] and [9]).

Theorem 3.5 states that if the SDH holds eventually at some fixed cofinality κ then the GDH holds for all sufficiently large cardinals μ and $\lambda \leq \mu$ such that $\text{cf} \lambda \geq \kappa$. The proof is by induction, and employs Theorem 2.6 in the critical cases.

1.2. Notation and prerequisites. The notation used here is standard in set theory. In particular, the word *cardinal*, if not explicitly stated otherwise, is to be understood as "infinite cardinal". The variables $\kappa, \theta, \mu, \lambda$ stand for infinite cardinals and $\alpha, \beta, \gamma, \delta, i, j$ denote ordinals. By $\text{cf} \mu$ the *cofinality* of μ is denoted. For $\kappa < \mu$ the symbol $[\mu]^\kappa$ denotes the set of all subsets of μ

whose cardinality is κ . If $\kappa = \text{cf}\kappa < \text{cf}\mu$ then we denote by S_λ^μ the set of all ordinals below μ whose cofinality is κ . For a set of ordinals X , the set $\text{acc}(X)$ denotes the set of all accumulation points of X .

We assume familiarity with the basics of stationary sets and the non-stationary ideal and acquaintance with Fodor's pressing down theorem. This material is available in every standard set-theory textbook.

1.3. Potential use in topology. We conclude the introduction with the following illustration of the potential applicability of density to topology.

Suppose $G = \langle V, E \rangle$ is an arbitrarily large graph (one can assume that it is a proper class with no harm) and that G does not contain large bipartite graphs, say, for some cardinal λ there is no copy of the complete bipartite graph $K_{\lambda, \lambda}$ in G .

For every cardinal μ , let us define a topology on V by letting $U \subseteq V$ be open if for all $v \in U$ it holds that $|G[v] \setminus U| < \mu$ ($G[v]$ is the set of neighbours of v in G). Equivalently, $D \subseteq V$ is closed if every vertex $v \in V$ which is connected by edges to μ vertices from D belongs to D .

What can be said about the cardinalities of closed sets in this topology? Using the arithmetic properties of the density function, it was proved in [10] that if $\mu \geq \beth_\omega(\lambda)$ then the closure of every set of size $\theta \geq \mu$ has size θ .

2. DEFINITION AND BASIC PROPERTIES OF DENSITY

Definition 2.1. (1) If $\langle P, \leq \rangle$ is a partially ordered set and $A, B \subseteq P$ then A is dense in B if $(\forall y \in B)(\exists x \in A)(x \leq y)$. We say that $A \subseteq P$ is dense if A is dense in P .

(2) If $\langle P, \leq \rangle$ is a partially ordered set and $A, B \subseteq P$ then A is an antichain with respect to B if for all distinct $x, y \in A$ there is no $z \in B$ such that $z \leq x \wedge z \leq y$. We say that $A \subseteq B$ is an antichain if A is an antichain with respect to P .

Definition 2.2. Suppose $\theta \leq \mu$ are cardinals.

- (1) The θ -density of μ , denoted by $\mathcal{D}(\mu, \theta)$, is the least cardinality of a set $\mathcal{D} \subseteq [\lambda]^\theta$ which is dense in $\langle [\mu]^\theta, \subseteq \rangle$.
- (2) Let $\overline{[\mu]^\theta} = \{X \in [\mu]^\theta : \forall \alpha (\alpha < \theta \Rightarrow |X \cap \alpha| < \theta)\}$.
- (3) Let $\underline{[\mu]^\theta} = \bigcup \{[\alpha]^\theta : \alpha < \mu\}$ (the set of all members of $[\mu]^\theta$ which are bounded in μ).
- (4) Let $\overline{\mathcal{D}}(\lambda, \theta)$ be the least cardinality of a set $\mathcal{D} \subseteq \overline{[\mu]^\theta}$ which is dense in $\overline{[\mu]^\theta}$, and let us call it the upper θ -density of μ , and let $\underline{\mathcal{D}}(\mu, \theta)$ be the least cardinality of $\mathcal{D} \subseteq \underline{[\mu]^\theta}$ which is dense in $\underline{[\mu]^\theta}$, and let us call it the lower θ -density of μ .

Claim 2.3. Suppose $\theta \leq \mu$. Then $\mathcal{D}(\mu, \theta) = \overline{\mathcal{D}}(\mu, \theta) + \underline{\mathcal{D}}(\mu, \theta)$.

Proof. Given any $X \in [\mu]^\theta$, either there is some $\lambda < \mu$ such that $Y := X \cap \lambda$ is of cardinality θ or else $X \in \overline{[\mu]^\theta}$. Thus, $\overline{[\mu]^\theta} \cup \underline{[\mu]^\theta}$ is dense in $[\mu]^\theta$ and

therefore $\mathcal{D}(\mu, \theta) \leq \overline{\mathcal{D}}(\lambda, \theta) + \underline{\mathcal{D}}(\mu, \theta)$ by taking the union of dense subsets of $\overline{[\mu]^\theta}$ and of $[\mu]^\theta$ of minimal cardinalities.

Conversely, given a dense $\mathcal{D} \subseteq [\mu]^\theta$ of minimal cardinality let $\mathcal{D}_0 = \mathcal{D} \cap \overline{[\mu]^\theta}$ and let $\mathcal{D}_1 = \mathcal{D} \cap [\mu]^\theta$. Clearly, \mathcal{D}_0 is dense in $\overline{[\mu]^\theta}$. To see that \mathcal{D}_1 is dense in $[\mu]^\theta$ let $Y \in [\mu]^\theta$ be arbitrary. Since \mathcal{D} is dense, there is some $X \in \mathcal{D}$ such that $X \subseteq Y$. For all $\lambda < \mu$ it holds that $X \cap \lambda \subseteq Y \cap \lambda$, so $X \in \overline{[\mu]^\theta}$, and now $X \in \mathcal{D}_1$. \square

Remark: If $X \in \overline{[\mu]^\theta}$ then $\text{otp}X = \theta$ and X is cofinal in μ , so consequently $\text{cf}\theta = \text{cf}\mu$. Thus, if $\text{cf}\theta \neq \text{cf}\mu$ it holds that $\overline{[\mu]^\theta} = \emptyset$ and that $\mathcal{D}(\mu, \theta) = \underline{\mathcal{D}}(\mu, \theta)$.

Claim 2.4 (Folklore). *Suppose $\theta = \text{cf}\mu < \mu$. Then:*

- (1) *Every maximal antichain in $\langle \overline{[\mu]^\theta}, \subseteq \rangle$ has cardinality $\geq \mu^+$.*
- (2) *$\overline{\mathcal{D}}(\mu, \theta) = |\mathcal{A}| + \mathcal{D}(\theta, \theta)$ whenever $\mathcal{A} \subseteq \overline{[\mu]^\theta}$ is a maximal antichain in $[\mu]^\theta$.*

Proof. The first item is proved by standard diagonalization.

To prove the second let $\mathcal{A} \subseteq \overline{[\mu]^\theta}$ be a maximal antichain in $\overline{[\mu]^\theta}$. Since the intersection of two distinct members from $\overline{[\mu]^\theta}$ belongs to $[\mu]^\theta$ if and only if it belongs to $[\mu]^\theta$, \mathcal{A} is an antichain in $[\mu]^\theta$ as well.

For every $X \in \mathcal{A}$ fix a dense \mathcal{D}_X in $\langle [X]^\theta, \subseteq \rangle$ of cardinality $\mathcal{D}(\theta, \theta)$ and let $\mathcal{D} = \bigcup \{\mathcal{D}_X : X \in \mathcal{A}\}$. The cardinality of \mathcal{D} is $|\mathcal{A}| + \mathcal{D}(\theta, \theta)$ and as every $Z \in \mathcal{D}_X$ for $X \in \mathcal{A}$ belongs to $\overline{[\mu]^\theta}$, we have that $\mathcal{D} \subseteq \overline{[\mu]^\theta}$. Given any $Y \in \overline{[\mu]^\theta}$, there exists some $X \in \mathcal{A}$ such that $Y \cap X \in [X]^\theta$ and therefore there is some $Z \in \mathcal{D}_X$ such that $Z \subseteq Y$. This establishes that \mathcal{D} is dense in $\overline{[\mu]^\theta}$.

Conversely, let $\mathcal{D} \subseteq \overline{[\mu]^\theta}$ be dense in $\overline{[\mu]^\theta}$ and let $\mathcal{A} \subseteq \overline{[\mu]^\theta}$ be an antichain in $\overline{[\mu]^\theta}$. Let $f : \mathcal{A} \rightarrow \mathcal{D}$ be such that $f(X) \subseteq X$ for all $X \in \mathcal{A}$. As \mathcal{A} is an antichain, f is injective and hence $|\mathcal{A}| \leq |\mathcal{D}|$. If $X \in \overline{[\mu]^\theta}$ then $\mathcal{D} \cap [X]^\theta$ is dense in $[X]^\theta$ and hence $\mathcal{D}(\theta, \theta) \leq |\mathcal{D}|$. \square

Corollary 2.5. *If $\theta = \text{cf}\mu < \mu$ and $\mathcal{D}(\theta, \theta) \leq \mu^+$ then every maximal antichain in $\overline{[\mu]^\theta}$ has cardinality $\overline{\mathcal{D}}(\mu, \theta)$.*

We are ready to prove now two Silver-type theorem for density. The first is tailored after the simplest (and most well known) version of Silver's theorem and the second follows the general version.

The proofs of Theorems 2.6 and 2.8 below follow in their outline two elementary proofs by Baumgartner and Prikry: [1], for the central case, and [2], for the general theorem. The following modifications were required. First, one has to use almost disjoint families of sets instead of general families. The reason is that in the pressing down argument with the density function is not injective in general, but is so with the additional condition of almost

disjointness. Second, a use of a pcf scale in the proof of Theorem 2.6 replaces an indirect argument in [1]. This is not strictly necessary, but makes the proof clearer. Finally, the density of stationary subsets with inclusion replaces the stronger hypothesis about cardinal arithmetic in the general case.

Theorem 2.6. *Suppose $\kappa = \text{cf}\kappa < \theta = \text{cf}\mu < \mu$. If $\underline{\mathcal{D}}(\mu, \theta) \leq \mu^+$ and the set $\{\lambda : \lambda \in S_\kappa^\mu \text{ is a cardinal and } \overline{\mathcal{D}}(\lambda, \kappa) = \lambda^+\}$ is a stationary subset of μ , then $\mathcal{D}(\mu, \theta) = \mu^+$.*

Proof. Since $\underline{\mathcal{D}}(\mu, \theta) \leq \mu^+ \leq \overline{\mathcal{D}}(\mu, \theta)$ and $\mathcal{D}(\mu, \theta) = \underline{\mathcal{D}}(\mu, \theta) + \overline{\mathcal{D}}(\mu, \theta)$, it holds that $\mathcal{D}(\mu, \theta) = \overline{\mathcal{D}}(\mu, \theta)$. By Corollary 2.5, $\mathcal{D}(\mu, \theta)$ is equal to the cardinality of any maximal antichain in $\langle \overline{[\mu]^\theta}, \subseteq \rangle$. So it suffices to prove that $|\mathcal{A}| \leq \mu^+$ for any antichain $\mathcal{A} \subseteq \overline{[\mu]^\theta}$. Fix such an antichain \mathcal{A} .

Let $\langle \kappa_i : i < \theta \rangle$ be a strictly increasing and continuous sequence of cardinals converging to μ with $\theta < \kappa_0$. Let $S = \{i \in S_\kappa^\theta : \overline{\mathcal{D}}(\kappa_i, \kappa) = \kappa_i^+\}$. By the assumptions, S is a stationary subset of θ .

By a basic pcf theorem (e.g. by [16], Claim 2.1, p. 55) $\text{tcf} \langle \prod_{i \in S} \kappa_i^+, <_{NS \upharpoonright S} \rangle = \mu^+$. We fix, then, a pcf scale $\bar{f} = \langle f_\alpha : \alpha < \mu^+ \rangle \subseteq \prod_{i < \theta} \kappa_i^+$, that is, a sequence which is $<_{NS \upharpoonright S}$ -increasing — $\alpha < \beta < \mu^+ \Rightarrow \{i \in S : f_\alpha(i) \geq f_\beta(i)\}$ is non-stationary — and $<_{NS \upharpoonright S}$ -cofinal — for every $f \in \prod_{i \in S} \kappa_i^+$ there is some $\alpha < \mu^+$ such that $\{i \in S : f(i) > f_\alpha(i)\}$ is non-stationary. We shall only use the cofinality of the scale.

Now fix, for every $i \in S$ a dense set \mathcal{D}_i in $\langle \overline{[\kappa_i]^\kappa}, \subseteq \rangle$ of cardinality κ_i^+ and an enumeration $\mathcal{D}_i = \{Z_j^i : j < \kappa_i^+\}$, and, for each $i < \theta$, a dense set $\mathcal{D}'_i \subseteq [\theta \times \kappa_i]^\theta$ with $|\mathcal{D}'_i| \leq \mu^+$.

Given $X \in \mathcal{A}$ let $S_X^1 = \{i \in S : \kappa_i \in \text{acc}(X)\}$. As X is unbounded in μ , $\langle \kappa_i : i < \theta \rangle$ is increasing, continuous and unbounded in μ and $S \subseteq \theta$ is stationary, S_X^1 is stationary in θ .

For each $i \in S_X^1$ let $f_X(i) = j < \kappa_i^+$ be such that $Z_j^i \in [\kappa_i]^\kappa$ is a subset of $X \cap \kappa_i$ which is cofinal in κ_i and of order-type κ . Such j exists because $X \cap \kappa_i$ is cofinal, $\text{cf}\kappa_i = \kappa$ and \mathcal{D}_i is dense in $\langle \overline{[\kappa_i]^\kappa}, \subseteq \rangle$.

The ideal $NS \upharpoonright S$ is contained in the ideal $NS \upharpoonright S_X^1$, hence the relation $<_{NS \upharpoonright S}$ is contained in the relation $<_{NS \upharpoonright S_X^1}$ and thus \bar{f} is a scale also with respect to $NS \upharpoonright S_X^1$. Therefore, there is some $\alpha(X) < \mu^+$ such that $f_X <_{NS \upharpoonright S_X^1} f_{\alpha(X)}$.

Claim 2.7. *For every $\alpha < \mu^+$, at most μ^+ many $X \in \mathcal{A}$ satisfy that $\alpha = \alpha(X)$.*

Proof. Let $\alpha < \mu^+$ be fixed and for every $i \in S_\kappa^\theta$ let us fix an injection $g_i : f_\alpha(i) \rightarrow \kappa_i$. Suppose $X \in \mathcal{A}$ satisfies that $\alpha = \alpha(X)$, so $f_X <_{NS \upharpoonright S_X^1} f_\alpha$.

By shrinking S_X^1 to a smaller stationary set we may assume that $f_X(i) < f_\alpha(i)$ for all $i \in S_X^1$. For each $i \in S_X^1$ let $r(i) = \min\{j < i : g_i(f_X(i)) < \kappa_j\}$. Since κ_i is limit, r is well-defined and is a regressive function on S_X^1 .

By Fodor's lemma, there is some stationary $S_X^2 \subseteq S_X^1$ and some fixed $j(X) < \theta$ such that $r(i) = j(X)$ for all $i \in S_X^2$. Let $h_X(i) := g_i(f_X(i)) \in \kappa_{j(X)}$ for all $i \in S_X^2$. Now the function $h_X : S_X^2 \rightarrow \kappa_{j(X)}$ (which is a set of ordered pairs) is a subset of $\theta \times \kappa_{j(X)}$. Let $Z(X) \in \mathcal{D}'_{j(X)}$ be chosen such that $Z(X) \subseteq h_X$ (so $Z(X)$ is a partial function from θ to $\kappa_{j(X)}$).

Suppose $X, Y \in \mathcal{A}$ are distinct and suppose that $j(X) = j(Y)$. If $Z = Z(X) = Z(Y)$ then $\text{dom } Z$ is unbounded in θ and for every $i \in \text{dom } Z$ the set $Z_{f_X(i)}^i = Z_{f_Y(i)}^i$ is unbounded in κ_i and contained in $X \cap Y$. Hence $|X \cap Y| = \theta$ — a contradiction to $|Z \cap Y| < \theta$.

Thus, the mapping $X \mapsto \langle j(X), Z(X) \rangle$ is injective on the set of all $X \in \mathcal{A}$ such that $f_X <_{NS} f_\alpha$. As there are at most $|\mathcal{D}'_{j(X)}| + \theta \leq \mu^+$ such pairs, we are done. \square

We have shown that the antichain \mathcal{A} has size at most μ^+ . \square

2.1. The general version. Thorem 2.6 above is formulated after the most popular version of Silver's theorem. Silver's original paper as well as [2] included, however, a more general formulation, involving the γ -th successors of κ_i and of μ for arbitrary ordinals $\gamma < \theta$. The case $\gamma = 0$ in the general case is actually a theorem by Erdős, Hajnal and Milner from 1967 about almost disjoint families [4] (for more on the history see [8]).

Let $\mathcal{S}_\kappa^\theta$, for $\kappa = \text{cf } \kappa < \theta = \text{cf } \theta$, denote the family of all stationary subsets of $S_\kappa^\theta = \{\alpha < \theta : \text{cf } \alpha = \kappa\}$.

Theorem 2.8. *Suppose $\kappa = \text{cf } \kappa < \theta = \text{cf } \mu < \mu$ and that $\langle \kappa_i : i < \theta \rangle$ is an increasing and continuous sequence of cardinals with limit μ and $\theta < \kappa_0$.*

Let \mathcal{A} be an antichain in $\langle [\mu]^\theta, \subseteq \rangle$ and let $\gamma < \theta$ be an ordinal.

Suppose that there exists a sequence $\langle \mathcal{D}_i : i \in S_\kappa^\theta \rangle$ such that

- (1) $\mathcal{D}_i \subseteq \overline{[\kappa_i]^\kappa}$ and $|\mathcal{D}_i| \leq \kappa_i^{+\gamma}$ for all $i \in S_\kappa^\theta$;
- (2) for every $A \in \mathcal{A}$ the set $S_A := \{i \in S_\kappa^\theta : (\exists X \in \mathcal{D}_i)(X \subseteq A)\}$ is stationary.

Then $|\mathcal{A}| \leq \mu^{+\gamma} + \underline{\mathcal{D}}(\mu, \theta) + \mathcal{D}(\mathcal{S}_\kappa^\theta, \subseteq)$.

We remark that if $2^\theta < \mu$ then $\mathcal{D}(\mathcal{S}_\kappa^\theta, \subseteq)$ can be removed from the conclusion, giving $|\mathcal{A}| \leq \mu^{+\gamma} + \underline{\mathcal{D}}(\mu, \theta)$, and if $\mathcal{D}(\kappa_i, \theta) < \mu$ for all i then also $\underline{\mathcal{D}}(\mu, \theta)$ can be removed. In the latter case the theorem has a meaningful content also in the case $\gamma = 0$.

Proof. Suppose $\kappa, \theta, \mu, \mathcal{A}, \gamma$ and $\langle \mathcal{D}_i : i \in S_\kappa^\theta \rangle$ are as stated in the hypothesis of the theorem and fix in addition, for each $i \in S_\kappa^\theta$, an injection $t_i : \mathcal{D}_i \rightarrow \kappa_i^{+\gamma}$. Fix also a dense set $\mathcal{D}'_i \subseteq [\theta \times \kappa_i]^\theta$ of cardinality $\mathcal{D}(\kappa_i, \theta)$ and an enumeration $\{S_\zeta : \zeta < \zeta(*)\}$ of a dense subset of $\langle \mathcal{S}_\kappa^\theta, \subseteq \rangle$ for $\zeta(*) = \mathcal{D}(\mathcal{S}, \subseteq)$.

To save on notation let us abbreviate the term $\mu^{+\gamma} + \underline{\mathcal{D}}(\mu, \theta) + \mathcal{D}(\mathcal{S}_\kappa^\theta, \subseteq)$ by $\lambda(\gamma)$ for each $\gamma < \theta$.

For each $A \in \mathcal{A}$ let $g_A : S_A \rightarrow \kappa_i^{+\gamma}$ by letting $g_A(i) := t_i(X)$ be the least such that $X \subseteq A$

The proof proceeds now by induction on $\gamma < \theta$ to show that $|\mathcal{A}| \leq \lambda(\gamma)$.

Assume $\gamma = 0$. Then for each $A \in \mathcal{A}$ and $i \in S(A)$ it holds that $g_A(i) < \kappa_i$. By Fodor's lemma there is some $j(A) < \theta$ and a stationary $S_A^1 \subseteq S_A$ so that $\text{ran}(h_A \upharpoonright S_A^1) \subseteq \kappa_{j(A)}$. Let $Y(A) \in \mathcal{D}'_{j(A)}$ such that $Y(A) \subseteq g_A \upharpoonright S_A^1$. As in the previous proof, the mapping $A \mapsto \langle j(A), Y(A) \rangle$ is injective. The number of possible pairs $\langle j(A), Y(A) \rangle$ is at most $\mathcal{D}(\kappa_i, \theta) \times \mu$ so we have established $|\mathcal{A}| \leq \mu + \mathcal{D}(\kappa_i, \theta) \leq \lambda(0)$. Observe that $\mathcal{D}(\mathcal{S}_\kappa^\theta, \subseteq)$ was not used in this case!

Now assume $\gamma = \beta + 1$.

Claim 2.9. *For every $g \in \prod_{i \in S} \kappa_i^{+\gamma}$ there are at most $\lambda(\beta)$ members $A \in \mathcal{A}$ for which there exists some stationary $S' \subseteq S_A$ such that $g_A(i) < g(i)$ for all $i \in S'$.*

Proof. Let $g \in \prod_{i \in S} \kappa_i^{+\gamma}$ be given, and let $D_i^g := \{X \in \mathcal{D}_i : t_i(X) < g(i)\}$. Thus the injection $t_i \upharpoonright \mathcal{D}_i^g$ demonstrates that $|\mathcal{D}_i^g| \leq |g(i)| \leq \kappa_i^{+\beta}$. Finally, let $\mathcal{A}_g = \{A \in \mathcal{A} : (\exists S' \in \mathcal{S})(S' \subseteq S_A \wedge (g_A \upharpoonright S') < g)\}$. Now \mathcal{A}_g , β and $\langle \mathcal{D}_i^g : i \in S_\kappa^\theta \rangle$ satisfy the hypothesis of the theorem and the conclusion follows by the induction hypothesis. \square

For $\zeta < \zeta(*)$, let $\mathcal{A}_\zeta = \{A \in \mathcal{A} : \zeta(A) = \zeta\}$. This correspondence partitions \mathcal{A} to at most $\zeta(*)$ subfamilies.

Claim 2.10. $|\mathcal{A}_\zeta| \leq \lambda(\beta)$ for every $\zeta < \zeta(*)$.

Proof. Consider the relation R on \mathcal{A} given by:

$$A R B \iff \{i \in S_A^1 \cap S_B^1 : g_A(i) < g_B(i)\} \text{ is stationary.}$$

Observe that if $A, B \in \mathcal{A}$ are distinct then $\{i \in S_A \cap S_B : g_A(i) = g_B(i)\}$ is bounded in θ , in particular non-stationary. So if $\zeta(A) = \zeta(B)$ and $A \neq B$ we have

$$A R B \vee B R A. \tag{1}$$

(Both disjuncts may hold simultaneously).

Let $\zeta < \zeta(*)$ be given. By Claim 2.9, for every $A \in \mathcal{A}_\zeta$ there are no more than $\lambda(\beta)$ members B of \mathcal{A}_ζ for which $B R A$. Define inductively, as long as possible, an injective sequence $\langle A_\xi : \xi < \xi(*) \rangle$ such that $\neg(A_\xi R A_\varphi)$ for all $\varphi < \xi$. If $\xi(*) > \lambda(B)$, then as $\neg(A_{\lambda(B)} R A_\varphi)$ for all $\varphi < \lambda(\beta)$ it follows by (1) that $A_\varphi R A_{\lambda(\beta)}$ for all $\varphi < \lambda(\beta)$, and this contradicts Claim 2.9.

Necessarily, then, $\xi(*) \leq \lambda(\beta)$. Thus every $A \in \mathcal{A}_\zeta$ satisfies $A R A_\xi$ for some $\xi < \xi(*) \leq \lambda(\beta)$ and another use of Claim 2.9 gives the required $|\mathcal{A}_\zeta| \leq \lambda(\beta)$. \square

Now the inequality $|\mathcal{A}| \leq \lambda(\gamma)$ follows easily.

Suppose finally that $0 < \gamma < \theta$ is limit. Since $\gamma < \theta$ and the non-stationary ideal is θ -complete, for every $A \in \mathcal{A}$ there is some $\beta(A)$ such that $g_A(i) < \kappa_i^{+\beta(A)}$ stationarily often, so $|\mathcal{A}| \leq \lambda(\gamma)$ by the induction hypothesis. \square

3. THE EVENTUAL GDH FOLLOWS FROM THE EVENTUAL SDH

Let us now define the *Singular Density Hypothesis* and the *Generalized Density Hypothesis* by modifying the well known SCH and GCH:

Definition 3.1.

- (1) *The SDH at a singular μ with $\text{cf}\mu = \theta$, denoted SDH_μ , is the statement:*

$$\overline{\mathcal{D}}(\mu, \theta) = \mu^+. \quad (\oplus)$$

- (2) *The GDH at a pair of cardinals $\lambda \leq \mu$, denoted $GDH_{\lambda, \mu}$, is the statement:*

$$\mathcal{D}(\mu, \lambda) = \begin{cases} \mu & \text{if } \text{cf}\mu \neq \text{cf}\lambda \\ \mu^+ & \text{if } \text{cf}\mu = \text{cf}\lambda \end{cases} \quad (\otimes)$$

Similarly to what SCH and GCH say about cardinal exponentiation, the SDH says that the “essential part¹” of $\mathcal{D}(\mu, \theta)$ assumes the least possible value at a the singular μ of cofinality θ , and the GDH says that the λ -density of μ assumes its minimal possible value.

The SDH and GDH with no parameters mean, respectively, that the parametrized statements above hold at all possible instances.

Instances of the GDH do not necessarily imply the parallel instances of the GCH. However, the following theorem shows that SDH together with $GDH_{\lambda, \lambda}$ for all regular λ already imply the GCH.

Theorem 3.2. *If $\mathcal{D}(\lambda, \text{cf}\lambda) = \lambda^+$ for all cardinals λ then the GCH holds.*

Proof.

Claim 3.3. *For every cardinal λ , if $2^{<\lambda} = \lambda$ then $2^\lambda = \overline{\mathcal{D}}(\lambda, \text{cf}\lambda)$.*

Proof of Claim. Fix an bijection $f : \bigcup_{\alpha < \lambda} \mathcal{P}(\alpha) \rightarrow \lambda$, a sequence $\langle \alpha_i : i < \text{cf}\lambda \rangle$ increasing and cofinal in λ and a dense $\{A_\beta : \beta < \overline{\mathcal{D}}(\lambda, \text{cf}\lambda)\}$. Given $X \subseteq \lambda$ let $X(i) := X \cap \alpha_i$ and let $Y = \{f(X(i)) : i < \text{cf}\lambda\}$. Finally, let $F(X) = \min\{\beta < \overline{\mathcal{D}}(\lambda, \text{cf}\lambda) : A_\beta \subseteq Y\}$. This is an injection, as $X \neq Z$ for $X, Z \subseteq \lambda$ implies that for all sufficiently large $i < \text{cf}\lambda$ it holds that $X(i) \neq Z(i)$. \square

Assume that $\mathcal{D}(\lambda, \text{cf}\lambda) = \lambda^+$ for every cardinal λ . We prove the GCH by induction on the cardinals. Suppose the GCH holds below λ . Then $2^{<\lambda} = \lambda$. By $\mathcal{D}(\lambda, \text{cf}\lambda) = \lambda^+$ and Claim 3.3 it follows that $2^\lambda = \lambda^+$. \square

Corollary 3.4. $GDH \iff GCH$

Proof. The implication from right to left is trivial. The converse implication holds because the GDH implies the hypothesis of 3.2. \square

¹Compare this with the evolution of formulations of the SCH which is described in [9]. The most modern and most informative one is $\text{cov}(\mu, \theta) = \mu^+$. The role of $\text{cov}(\mu, \theta) = \mu^+$ for exponentiation is played by upper density for the density function.

Rinot [15] proved that the SDH implies Shelah's Strong Hypothesis and Matet characterized the SCH in terms of desnity ([14], 2.17) and proved that Shelah's Strong Hypothesis implies many instances of the GDH ([14], 2.19).

Let us define the *Eventual* GDH, EGDH, for short, as the statement: there exists κ such that for all λ with $\text{cf}\lambda \geq \kappa$ there is some μ_λ such that $\text{GDH}_{\lambda, \mu}$ holds for all $\mu \geq \mu_\lambda$.

Theorem 3.5. *If κ is regular and the SDH holds for all μ with cofinality κ in some end-segment of the cardinals, then the EGDH holds: for every λ with $\text{cf}\lambda \geq \kappa$, for all μ in some end-segment of the cardinals the GDH holds at μ with λ .*

Proof. Suppose κ is regular, μ_κ is a cardinal, and that $\mathcal{D}(\mu, \kappa) = \mu^+$ for all singular $\mu \geq \mu_\kappa$ with $\text{cf}\mu = \kappa$. By replacing μ_κ with $(\mu_\kappa)^\kappa$, if necessary, we assume that $(\mu_\kappa)^\kappa = \mu_\kappa$.

We need to show that for every cardinal λ with $\text{cf}\lambda \geq \kappa$ there is an end-segment of the cardinals in which

$$\mathcal{D}(\mu, \lambda) = \begin{cases} \mu & \text{if } \text{cf}\mu \neq \text{cf}\lambda \\ \mu^+ & \text{if } \text{cf}\mu = \text{cf}\lambda \end{cases} \quad (\otimes)$$

By induction on $\lambda \geq \kappa$ we define a cardinal μ_λ and for λ with $\text{cf}\lambda \geq \kappa$ prove by induction on $\mu \geq \mu_\lambda$ that (\otimes) holds.

The first case we consider is of a *regular* $\lambda \geq \kappa$. Let a regular $\lambda \geq \kappa$ be given. If $\lambda = \kappa$ then μ_λ is already defined. If $\lambda > \kappa$ let μ_λ be chosen so that $\mu_\lambda \geq \mu_\kappa$ and $(\mu_\lambda)^\lambda = \mu_\lambda$. Now let us show by induction on $\mu \geq \mu_\lambda$ that (\otimes) holds. If $\mu = \mu_\lambda$ then $\text{cf}\mu > \lambda$ and $\mu \leq \mathcal{D}(\mu, \lambda) \leq \mu^\lambda = \mu$, so (\otimes) indeed holds.

Assume next that $\text{cf}\mu \neq \lambda$. In this case for every $X \in [\mu]^\lambda$ there exists some $\alpha < \mu$ such that $X \cap \alpha \in [\alpha]^\lambda$. The induction hypothesis implies that $\mathcal{D}(\alpha, \lambda) \leq |\alpha|^+ \leq \mu$, so (\otimes) follows readily.

The remaining case is, then, $\text{cf}\mu = \lambda$. By the induction hypothesis, $\underline{\mathcal{D}}(\mu, \lambda) = \lambda$. As $\mu_\kappa \leq \mu_\lambda < \mu$, an end-segment of singulars μ' of cofinality κ below μ satisfy $\mathcal{D}(\mu', \kappa) = \mu'^+$. By Theorem 2.6, $\mathcal{D}(\mu, \lambda) = \mu^+$.

Assume now that λ is singular. If $\text{cf}\lambda < \kappa$ we are not really required to do anything, so let us define μ_λ as 0. If $\text{cf}\lambda = \theta \geq \kappa$ let μ_λ be chosen so that $\mu_\lambda > \mu_{\lambda'}$ for all $\lambda' < \lambda$ and $(\mu_\lambda)^\lambda = \mu_\lambda$. Now proceed to prove (\otimes) by induction on $\mu \geq \mu_\lambda$. The cases $\mu = \mu_\lambda$ and $\text{cf}\mu \neq \text{cf}\lambda$ follow in the same way as for regular λ .

We are left with the case $\text{cf}\mu = \text{cf}\lambda = \theta$ and $\lambda < \mu$. Fix an increasing sequence of regular cardinals $\langle \lambda_i : i < \theta \rangle$ that converges to λ , and such that $\theta < \lambda_0$. By the induction hypothesis on λ , (\otimes) holds for μ with each λ_i , so $\mathcal{D}(\mu, \lambda_i) = \mu$ for each i and we can fix a dense $\mathcal{D}_i \subseteq [\mu]^{\lambda_i}$ of cardinality $|\mathcal{D}_i| = \mu$. Fix an injection $f : \bigcup_{i < \theta} \mathcal{D}_i \rightarrow \mu$.

As $\theta < \lambda$ and $\mu_\theta < \mu$, the induction hypothesis (on λ) implies that $\mathcal{D}(\mu, \theta) = \mu^+$. Fix, then, a dense $\mathcal{D}_\theta \subseteq [\mu]^\theta$ of cardinality μ^+ . Let $A_i =$

$\text{ran}(f \upharpoonright [\mu]^{\lambda_i})$ for $i < \theta$. Clearly, $|A_i| = \mu$ for each $i < \theta$ and as $[\mu]^{\lambda_i} \cap [\mu]^{\lambda_j} = \emptyset$ for $i < j < \theta$ and f is injective, the A_i -s are pairwise disjoint.

Let $\mathcal{D} = \{\bigcup_{\alpha \in X} f^{-1}(\alpha) : X \in \mathcal{D}_\theta \wedge |A_i \cap X| \leq 1 \wedge |\bigcup_{\alpha \in X} f^{-1}(\alpha)| = \lambda\}$.

By the definition of \mathcal{D} it is a subset of $[\mu]^\lambda$ and since $|\mathcal{D}_\theta| = \mu^+$, the cardinality of \mathcal{D} does not exceed μ^+ . We prove next that \mathcal{D} is dense in $[\mu]^\lambda$ (so a priori $|\mathcal{D}| = \mu^+$) and with this finish the proof.

Let $Y \in [\mu]^\lambda$ be arbitrary. For each $i < \lambda$ choose a set $Y_i \in [Y]^{\lambda_i} \cap \mathcal{D}_i$. This is possible since \mathcal{D}_i is dense in $[\mu]^{\lambda_i}$. Let $Z = \{f(Y_i) : i < \theta\}$. Clearly, $Z \in [\mu]^\theta$ and $|Z \cap A_i| = 1$ for every $i < \theta$. By the density of \mathcal{D}_θ , there exists some $X \in \mathcal{D}_\theta$ such that $X \subseteq Z$. Thus, $|X \cap A_i| \leq 1$ for each $i < \theta$. As $|X| = \theta$, for arbitrarily large $i < \theta$ it holds that $|X \cap A_i| = 1$. It follows that $\bigcup_{\alpha \in X} f^{-1}(\alpha) \subseteq Y$ belongs to \mathcal{D} and is contained in Y . □

4. CONCLUDING REMARKS

The density function was not yet applied to topology, but it is reasonable to assume that applications will be found.

We do not know if the negation of the EGDH is consistent.

If the EGDH holds, then for any two regular cardinals θ_1, θ_2 above κ , for every sufficiently large μ

$$\mu = \min\{\mathcal{D}(\mu, \theta_1), \mathcal{D}(\mu, \theta_2)\}. \quad (2)$$

Is the negation of the following consistent?

- (a) For every cardinal κ there are $\theta_2 > \theta_1 > \kappa$ and some $\mu_0 > \theta_2$ such that for all $\mu \geq \mu_0$ (2) holds.

An even harder consistency will be the negation of the following:

- (b) For every κ there is a finite set F of cardinals above κ and some μ_0 such that for all $\mu \geq \mu_0$

$$\mu = \min\{\mathcal{D}(\mu, \theta) : \theta \in F\}.$$

Replacing “finite” with “countable” in this statement produces a ZFC theorem (see Corollary 2.8 in [11]).

Moti Gitik [6] proved recently the consistency of the negations of both (a) and (b) relative to a proper class of strong cardinals.

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