ON TWO PROBLEMS OF ERDŐS AND HECHLER: NEW METHODS IN SINGULAR MADNESS

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Abstract. For an infinite cardinal μ, $\mathrm{MAD}(\mu)$ denotes the set of all cardinalities of *nontrivial maximal almost disjoint families* over μ.

Erdős and Hechler proved in [7] the consistency of $\mu \in \mathrm{MAD}(\mu)$ for a singular cardinal μ and asked if it was ever possible for a singular μ that $\mu \notin \mathrm{MAD}(\mu)$, and also whether $2^{\mathrm{cf}\,\mu} < \mu \implies \mu \in \mathrm{MAD}(\mu)$ for every singular cardinal μ.

We introduce a new method for controlling $\mathrm{MAD}(\mu)$ for a singular μ and, among other new results about the structure of $\mathrm{MAD}(\mu)$ for singular μ, settle both problems affirmatively.

1. Introduction

1.1. **Background.** Let μ be an infinite cardinal. A family of sets $\mathcal{A}$ is *μ-almost disjoint* (*μ-ad* for short) if $|A| = \mu = |\bigcup \mathcal{A}|$ for every $A \in \mathcal{A}$ and $|A \cap B| < \mu$ for every distinct $A, B \in \mathcal{A}$. $\mathcal{A}$ is *maximal μ-almost disjoint* (*μ-mad*) if there is no $C \subseteq \bigcup \mathcal{A}$ such that $\mathcal{A} \cup \{C\}$ is μ-almost disjoint; in this case we also say that $\mathcal{A}$ is *mad in μ*. It is clear that every μ-almost disjoint family consisting of fewer than $\mathrm{cf}\,\mu$ sets is mad in μ; such a family will be called *trivial*. We denote by $\mathrm{MAD}(\mu)$ the set of all cardinalities of nontrivial mad families in μ. A standard diagonalization argument shows that $\mathrm{cf}\,\mu \notin \mathrm{MAD}(\mu)$, therefore $\mathrm{MAD}(\mu)$ is contained in the interval of cardinals $[\mathrm{cf}(\mu)^+, 2^\mu]$.

W. W. Comfort asked (see [7]) under what conditions it holds that $\mu \in \mathrm{MAD}(\mu)$ for a singular cardinal μ. P. Erdős and S. Hechler [7] proved that $\mu \in \mathrm{MAD}(\mu)$ if $\lambda^{\mathrm{cf}\,\mu} < \mu$ for every $\lambda < \mu$. Thus, if $2^{\aleph_0} < \aleph_\omega$ then the interval $[2^{\aleph_0}, \aleph_\omega]$ of cardinals is contained in $\mathrm{MAD}(\aleph_\omega)$.

Erdős and Hechler asked in [7] whether it is consistent that $\mu \notin \mathrm{MAD}(\mu)$ for some singular cardinal μ and, more concretely, whether Martin's axiom together with $2^{\aleph_0} > \aleph_\omega$ implies that $\aleph_\omega \notin \mathrm{MAD}(\aleph_\omega)$. They also asked whether $2^{\mathrm{cf}\,\mu} < \mu$ implies $\mu \in \mathrm{MAD}(\mu)$ for singular cardinals μ other than $\aleph_\omega$.

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Both problems are settled affirmatively by the general results below on $\mathrm{MAD}(\mu)$ for a singular μ.

1.2. **Notation.** Let $\mathfrak{a}_\mu = \min \mathrm{MAD}(\mu)$ and let $\mathfrak{a} = \mathfrak{a}_{\aleph_0}$. For a singular μ it holds that $\mathrm{MAD}(\operatorname{cf}\mu) \subseteq \mathrm{MAD}(\mu)$, therefore $\mathfrak{a}_\mu \leqslant \mathfrak{a}_{\operatorname{cf}\mu}$.

A crucial role in the results is played by two *bounding numbers*: $\mathfrak{b}_\mu$ and $\mathfrak{b}_{\operatorname{cf}\mu}$.

For every quasi-ordering $(P, \leqslant)$ with no maximum, the *bounding number* $\mathfrak{b}(P, \leqslant)$ is the least cardinality of a subset of P with no upper bound. For a regular cardinal κ, let $\mathfrak{b}_\kappa$ denote the bounding number of $(\kappa^\kappa, \leqslant^*)$, where $f \leqslant^* g$ means that $|\{i < \kappa \colon f(i) > g(i)\}| < \kappa$; let $\mathfrak{b} = \mathfrak{b}_{\aleph_0}$. It is well known that $\kappa < \mathfrak{b}_\kappa \leqslant \mathfrak{a}_\kappa$ for a regular cardinal κ (for $\kappa = \aleph_0$ see [6]; the general case is similar) and that under Martin's axiom $\mathfrak{b} = 2^{\aleph_0}$.

Suppose that μ is a singular cardinal of cofinality κ and that $\langle \mu_i \colon i < \kappa\rangle$ is a strictly increasing sequence of regular cardinals with supremum μ. Standard diagonalization shows that $\mathfrak{b}(\prod_{i<\kappa} \mu_i, \leqslant^*) > \mu$. Denote by $\mathfrak{b}_\mu$ the *supremum of* $\mathfrak{b}(\prod \mu_i, \leqslant^*)$ *over all strictly increasing sequences of regular cardinals* $\langle \mu_i \colon i < \kappa\rangle$ *with supremum* μ.

Each of the following three relations is consistent with ZFC: $\mathfrak{b} < \mathfrak{b}_{\aleph_\omega}$, $\mathfrak{b} = \mathfrak{b}_{\aleph_\omega}$ and $\mathfrak{b} > \mathfrak{b}_{\aleph_\omega}$.

1.3. **The results.** We prove that for every singular cardinal μ:

(1) $\mathfrak{a}_\mu \geqslant \min\{\mathfrak{b}_\mu, \mathfrak{b}_{\operatorname{cf}\mu}\}$.
(2) $\mathfrak{a}_\mu \leqslant \lambda < \mathfrak{b}_\mu \Longrightarrow \lambda \in \mathrm{MAD}(\mu)$.

Thus, if $\mathfrak{b}_{\operatorname{cf}\mu} > \mu$ it follows from (1) that $\mathfrak{a}_\mu > \mu$, hence $\mu \notin \mathrm{MAD}(\mu)$; and if $\mathfrak{a}_{\operatorname{cf}\mu} < \mu$ it follows from (2) that $\mu \in \mathrm{MAD}(\mu)$. In particular:

(b) $MA + 2^{\aleph_0} > \aleph_\omega \Longrightarrow \aleph_\omega \notin \mathrm{MAD}(\aleph_\omega)$.
(a) $2^{\operatorname{cf}\mu} < \mu \Longrightarrow \mu \in \mathrm{MAD}(\mu)$ for every singular μ.

which, respectively, settle in the affirmative both problems of Erdős and Hechler from [7].

If one assumes the consistency of large cardinals, $\mathfrak{b}_{\aleph_\omega}$ can be shifted up arbitrarily high below $\aleph_{\omega_1}$. Following this with a ccc forcing for controlling $\mathfrak{b}$ proves the following:

(3) For every regular $\lambda \in (\aleph_\omega, \aleph_{\omega_1})$ and regular uncountable $\theta \leqslant \lambda^+$ it is consistent that

$$\mathrm{MAD}(\aleph_\omega) = [\theta, \lambda^+].$$

So, e.g. the following are consistent:

- $\mathrm{MAD}(\aleph_\omega) = \{\aleph_1, \aleph_2, \ldots, \aleph_{\omega+\beta+2} = 2^{\aleph_\omega}\}$ for an arbitrary $\beta < \omega_1$.
- $\mathrm{MAD}(\aleph_\omega) = \{\aleph_{\omega+\beta+2}\}$ for an arbitrary $\beta < \omega_1$.
- $\mathrm{MAD}(\aleph_\omega) = [\aleph_{\omega+\alpha+1}, \aleph_{\omega+\beta+2}]$ for arbitrary $\alpha \leqslant \beta < \omega_1$.

And so on.

We refer the reader to the comprehensive list of references in D. Monk's recent [12], in which maximal almost disjoint families are viewed as partitions of unity in the Boolean algebra $\mathcal{P}(\mu)/_{[\mu]^{<\mu}}$.

1.4. **Preliminary facts.** We will use the following facts from [7]:

(1) $\mathrm{MAD}(\mathrm{cf}\,\mu) \subseteq \mathrm{MAD}(\mu)$ and
(2) $\mathrm{MAD}(\mu)$ is closed under singular suprema.

The latter fact is stated in [7] in a less general form, so we give a proof here:

Lemma 1.1. *Assume that* $\lambda = \sup_{i<\theta} \lambda_i$, *where* $\{\lambda_i \colon i < \theta\} \subseteq \mathrm{MAD}(\mu)$ *and* $\theta < \lambda$. *Then* $\lambda \in \mathrm{MAD}(\mu)$.

Proof. We may assume that $\theta \leqslant \lambda_0$. Let $\mathcal{A}$ be a mad family in μ with $|\mathcal{A}| = \lambda_0$. Write $\mathcal{A} = \{A_i \colon i < \lambda_0\}$ and for each $i < \theta$ choose a mad family $\mathcal{B}_i$ with $\bigcup \mathcal{B}_i = A_i$ and $|\mathcal{B}_i| = \lambda_i$. Set

$$\mathcal{C} = \bigcup_{i<\theta} \mathcal{B}_i \cup \{A_j \colon \theta \leqslant j < \lambda_0\}.$$

Then $|\mathcal{C}| = \lambda$ and $\mathcal{C}$ is mad in μ. □

The following fact will also be used in some proofs.

Lemma 1.2. *Let* $\kappa = \mathrm{cf}\,\mu$ *and let* $\mathcal{A}$ *be a* μ*-almost disjoint family of size* κ. *Then there exists a mad family* $\mathcal{A}' \supseteq \mathcal{A}$ *such that* $|\mathcal{A}'| = \mathfrak{a}_\mu$ *and* $\bigcup \mathcal{A}' = \bigcup \mathcal{A}$.

Proof. Fix a μ-mad family $\mathcal{B}$ with $|\mathcal{B}| = \mathfrak{a}_\mu$. Choose $\mathcal{B}_0 = \{B_i \colon i < \kappa\} \subseteq \mathcal{B}$. Let $X = \bigcup_{i<\kappa} B_i$ and define

$$\mathcal{B}' = \{B \cap X \colon X \in \mathcal{B} \setminus \mathcal{B}_0 \;\&\; |X \cap B| = \mu\}.$$

Let $\langle A_i \colon i < \kappa \rangle$ be a one-to-one enumeration of $\mathcal{A}$. Define a bijection $f \colon \bigcup \mathcal{A} \to X$ so that $f[A_i \setminus \bigcup_{j<i} A_j] = B_i \setminus \bigcup_{j<i} B_j$. Finally, set $\mathcal{A}' = \mathcal{A} \cup \{f^{-1}[B] \colon B \in \mathcal{B}'\}$. Observe that $\mathcal{A}'$ is mad and $|\mathcal{A}'| = \mathfrak{a}_\mu$. □

2. Inequalities

From now on, μ will always denote a singular cardinal whose cofinality is denoted by κ.

2.1. **Bounding numbers and madness in singular cardinals.**

Theorem 2.1. *For every singular cardinal* μ,

$$\mathfrak{a}_\mu \geqslant \min\{\mathfrak{b}_\mu, \mathfrak{b}_{\mathrm{cf}\,\mu}\}. \tag{1}$$

Proof. Let $\kappa = \mathrm{cf}\,\mu$. Suppose to the contrary that $\mathfrak{a}_\mu < \min\{\mathfrak{b}_\mu, \mathfrak{b}_\kappa\}$ and fix a strictly increasing sequence of regular cardinals $\langle \mu_i \colon i < \kappa \rangle$ with supremum μ such that $\mathfrak{b}(\prod_{i<\kappa} \mu_i, \leqslant^*) > \mathfrak{a}_\mu$.

Let $\mathcal{A} = \big\{\{i\} \times \mu : i < \kappa\big\}$. By Lemma 1.2, there exists a family $\mathcal{B} \subseteq [\kappa \times \mu]^\mu$ such that $\mathcal{B} \cup \mathcal{A}$ is mad in μ, $\mathcal{B} \cap \mathcal{A} = \emptyset$ and $|\mathcal{B}| = \mathfrak{a}_\mu$.

For each $B \in \mathcal{B}$, define a function $f_B : \kappa \to \kappa$ by $f_B(i) = \min\{j < \kappa : |B \cap (\{i\} \times \mu)| < \mu_j\}$. This function is well defined, since $|B \cap (\{i\} \times \mu)| < \mu$ for each $i < \kappa$.

Since $|\mathcal{B}| = \mathfrak{a}_\mu < \mathfrak{b}_\kappa$, there exists a function $f : \kappa \to \kappa$ so that $f_B <^* f$ for all $B \in \mathcal{B}$. Without loss of generality we may assume that f is strictly increasing.

For each $B \in \mathcal{B}$, for all but boundedly many $i < \kappa$ it holds that $\sup\{\alpha < \mu_{f(i)} : (i, \alpha) \in B\} < \mu_{f(i)}$. Let $g_B(i)$ be defined by

$$g_B(i) = \begin{cases} 0 & \text{if } \sup\{\alpha < \mu_{f(i)} : (i,\alpha) \in B\} = \mu_{f(i)} \\ \sup\{\alpha < \mu_{f(i)} : (i,\alpha) \in B\} & \text{otherwise} \end{cases}$$

For each $B \in \mathcal{B}$ the function g_B belongs to $\prod_{i<\kappa} \mu_{f(i)}$. Since

$$\mathfrak{b}(\prod_{i<\kappa} \mu_{f(i)}, \leqslant^*) \geqslant \mathfrak{b}(\prod_{i<\kappa} \mu_i) > \mathfrak{a}_\mu$$

we can fix a function $g \in \prod_{i<\kappa} \mu_{f(i)}$ so that $g_B <^* g$ for all $B \in \mathcal{B}$.

Define

$$C = \bigcup_{i<\kappa} \{i\} \times [g(i), \mu_{f(i)}).$$

Clearly, $|C| = \mu$. For each $B \in \mathcal{B}$ there exists $j_B < \kappa$ such that $g_B(i) < g(i)$ for all $i > j_B$. This implies that $\{i\} \times [g(i), \mu_{f(i)})$ is disjoint from B for all $i > j_B$. Hence $|B \cap C| \leqslant \mu_{f(j_B)} < \mu$. Clearly, $|C \cap (\{i\} \times \mu)| \leqslant \mu_{f(i)} < \mu$ for all $i < \kappa$; so $\mathcal{A} \cup \mathcal{B} \cup \{C\}$ is μ-almost disjoint, contrary to the maximality of $\mathcal{A} \cup \mathcal{B}$. □

A positive answer to the first question of Comfort, Erdős and Hechler follows now as a corollary:

Corollary 2.2. *If Martin's Axiom holds and $2^{\aleph_0} > \mu > \operatorname{cf} \mu = \aleph_0$ then $\mu \notin \mathrm{MAD}(\mu)$.*

2.2. **Between $\mathfrak{a}_\mu$ and $\mathfrak{b}_\mu$.** In this section we shall show that $\mathrm{MAD}(\mu)$ contains the interval of cardinals $[\mathfrak{a}_\mu, \mathfrak{b}_\mu)$ and even $[\mathfrak{a}_\mu, \mathfrak{b}_\mu]$ in the case $\mathfrak{b}_\mu$ is a successor of a regular cardinal.

Theorem 2.3. *For every singular cardinal μ and every cardinal λ,*

$$\mathfrak{a}_\mu \leqslant \lambda < \mathfrak{b}_\mu \Longrightarrow \lambda \in \mathrm{MAD}(\mu). \tag{2}$$

If $\mathfrak{b}_\mu$ is a successor of a regular cardinal, then $\mathfrak{a}_\mu \leqslant \mathfrak{b}_\mu \implies \mathfrak{b}_\mu \in \mathrm{MAD}(\mu)$.

To prove the Theorem it suffices, by Lemma 1.1, to show that every regular $\lambda \in [\mathfrak{a}_\mu, \mathfrak{b}_\mu)$ belongs to $\mathrm{MAD}(\mu)$.

The proof of this will now be divided to two cases. First we prove that every regular $\mathfrak{a}_\mu < \lambda < \mu$ belongs to $\mathrm{MAD}(\mu)$. The proof in this case does not require any specialized techniques. Then we prove the same for regular $\mu < \lambda < \mathfrak{b}_\mu$ and for $\mathfrak{b}_\mu$ itself when it is the successor of a regular cardinal. In this case the proof requires some machinery from pcf theory.

Despite of the technical differences between both proofs, they are similar, and could, in fact, be combined to a single proof. Both follow the same scheme of gluing together λ different μ-mad families, each of size $\mathfrak{a}_\mu$, to a single μ-mad family of size λ. In the case $\lambda < \mu$, a simple presentation of μ as a disjoint union of λ parts works; in the second part we need to rely on smooth pcf scales to get a presentation of μ as an *almost increasing* and *continuous* union of length λ of sets of size μ.

2.2.1. *The case* $\lambda < \mu$.

Lemma 2.4. *Suppose* $\mu > \operatorname{cf}\mu = \kappa$. *Then for every regular cardinal* λ,

$$\mathfrak{a}_\mu \leqslant \lambda < \mu \implies \lambda \in \mathrm{MAD}(\mu).$$

Proof. Suppose λ is regular and $\mathfrak{a}_\mu \leqslant \lambda < \mu$. Since $\mathfrak{a}_\mu > \kappa = \operatorname{cf}\mu$, $\lambda > \kappa$.

Fix a strictly increasing sequence of regular cardinals $\langle \mu_i \colon i < \kappa\rangle$ such that $\sup_{i<\kappa}\mu_i = \mu$ and $\lambda < \mu_0$. We will work in $\mu \times \lambda$ instead of μ. Let $S = \{\delta < \lambda \colon \operatorname{cf}\delta = \kappa\}$. For each $\delta \in S$ fix a strictly increasing, continuous sequence $D_\delta = \langle \gamma^\delta_i \colon i < \kappa\rangle$ with limit δ such that $\gamma^\delta_0 = 0$. Define

$$F^\delta_j = \bigcup\{\mu \times \{\beta\} \colon \gamma^\delta_j \leqslant \beta < \gamma^\delta_{j+1}\}.$$

Thus $\mathcal{F}_\delta = \{F^\delta_j \colon j < \kappa\}$ is a disjoint family of sets, each set of size μ, which covers $\mu \times \delta$. Let $\mathcal{A}_\delta \subseteq [\mu \times \delta]^\mu$ be such that $\mathcal{A}_\delta \cup \mathcal{F}_\delta$ is mad in $\mu \times \delta$, $\mathcal{A}_\delta \cap \mathcal{F}_\delta = \emptyset$ and $|\mathcal{A}_\delta| = \mathfrak{a}_\mu$.

Define

$$\mathcal{B} = \{\mu \times \{\alpha\} \colon \alpha < \lambda\} \cup \bigcup_{\delta \in S} \mathcal{A}_\delta.$$

Then $|\mathcal{B}| = \lambda$ and $\mathcal{B} \subseteq [\mu \times \lambda]^\mu$. We will show that $\mathcal{B}$ is μ-mad.

First, observe that $\mathcal{B}$ is almost disjoint: clearly each element of $\mathcal{A}_\delta$ is almost disjoint from any set of the form $\mu \times \{\alpha\}$, because if $\alpha < \delta$ then $\mu \times \{\alpha\} \subseteq F^\delta_j$ for $j < \kappa$ such that $\gamma^\delta_j \leqslant \alpha < \gamma^\delta_{j+1}$. Finally, consider $A_i \in \mathcal{A}_{\delta_i}$, $i < 2$, with $\delta_0 < \delta_1$. Then $A_0 \subseteq \bigcup_{j<j_0} F^{\delta_1}_j$, where $j_0 < \kappa$ is such that $\delta_0 < \gamma^{\delta_1}_{j_0}$. Thus $|A_0 \cap A_1| < \mu$.

To see that $\mathcal{B}$ is mad fix an arbitrary $Z \in [\mu \times \lambda]^\mu$. There exists a sequence $\langle \alpha_i \colon i < \kappa\rangle$ in λ such that

$$|Z \cap (\mu \times \{\alpha_i\})| \geqslant \mu_i.$$

If $|\{\alpha_i \colon i < \kappa\}| < \kappa$ then $|Z \cap (\mu \times \{\alpha\})| = \mu$ for some α. So suppose that $|Z \cap (\mu \times \{\alpha_i\})| < \mu$ for every $i < \kappa$. Taking a subsequence, we may assume that $\langle \alpha_i \colon i < \kappa\rangle$ is strictly increasing. Let δ be its supremum. By regularity of λ, $\delta \in S$ and therefore $Z \in [\mu \times \delta]^\mu$. Shrinking Z if necessary, assume that $Z \subseteq \bigcup_{i<\kappa} \mu \times \{\alpha_i\}$. Then $|Z \cap F^\delta_j| < \mu$ for every $j < \kappa$. Thus, $|Z \cap A| = \mu$ for some $A \in \mathcal{A}_\delta$. This completes the proof. □

Corollary 2.5. *Let* $\mu > \operatorname{cf}\mu = \kappa$. *If* $\mathfrak{a}_\kappa \leqslant \mu$ *then* $[\mathfrak{a}_\kappa, \mu] \subseteq \mathrm{MAD}(\mu)$. *In particular, if* $2^\kappa < \mu$ *then* $\mu \in \mathrm{MAD}(\mu)$.

Corollary 2.5 anwers affirmatively the second question of Erdős and Hechler in [7].

2.2.2. *The case* $\lambda > \mu$. A (μ, λ)-*scale* is a sequence $\overline{f} = \langle f_\alpha \colon \alpha < \lambda \rangle \subseteq \prod_{i<\kappa} \mu_i$ such that $\langle \mu_i \colon i < \kappa \rangle$ is a strictly increasing sequence of regular cardinals with limit μ, and so that $\alpha < \beta < \lambda \implies f_\alpha <^* f_\beta$ and for every $g \in \prod_{i<\kappa} \mu_i$ there is $\alpha < \lambda$ with $g <^* f_\alpha$. The relation $f <^* g$ means that the set $\{i < \kappa \colon f(i) \geqslant g(i)\}$ is bounded in κ. If a (μ, λ)-scale exists, then λ must be a regular cardinal $> \mu$. When μ is fixed, "(μ, λ)-scale" will be abbreviated by "λ-scale". A λ-scale $\overline{f}$ is *smooth* if for every $\delta < \lambda$ with $\operatorname{cf} \delta > \kappa$ the sequence $\overline{f} \restriction \delta = \langle f_\alpha \colon \alpha < \delta \rangle$ is cofinal in $(\prod_{i<\kappa} f_\delta(i), <^*)$. In this case we say that f_δ is an *exact upper bound* of $\overline{f} \restriction \delta)$. We will denote by $[f, g)$ the set $\{(i, \alpha) \colon i < \kappa \wedge f(i) \leqslant \alpha < g(i)\}$.

The proof in the present case goes through two steps. First, it is shown that whenever a smooth (μ, λ)-scale exists and $\mathfrak{a}_\mu < \lambda$, it holds that $\lambda \in \operatorname{MAD}(\mu)$. Then it is shown that for every $\mu < \lambda < \mathfrak{b}_\mu$ there is a smooth (μ, λ)-scale and that in case $\mathfrak{b}_\mu$ is a successor of a regular cardinal there is also a smooth $(\mu, \mathfrak{b}_\mu)$-scale.

Lemma 2.6. *Assume* $\lambda > \mu > \operatorname{cf} \mu = \kappa$ *and there exists a smooth* (μ, λ)-*scale. If* $\mathfrak{a}_\mu \leqslant \lambda$ *then* $\lambda \in \operatorname{MAD}(\mu)$.

Proof. Suppose there exists a smooth λ-scale $\langle g_\xi \colon \xi < \lambda \rangle \subseteq \prod_{i<\kappa} \mu_i$. Let $S = \{\delta < \lambda \colon \operatorname{cf} \delta = \kappa\}$, and for each $\delta \in S$ fix a strictly increasing, continuous, sequence $\langle \gamma_i^\delta \colon i < \kappa \rangle$ with limit δ such that $\gamma_0^\delta = 0$ and put $D_\delta = \{\gamma_i^\delta \colon i < \kappa\}$.

By induction on $\xi < \lambda$ we construct a smooth λ-scale $\overline{f} = \langle f_\xi \colon \xi < \lambda \rangle \subseteq \prod_{i<\kappa} \mu_i$ which satisfies the following two conditions:

1. If $\delta < \lambda$ is a limit and $\operatorname{cf} \delta \leqslant \kappa$ then $f_\delta(i) = \sup_{\xi \in D_\delta} f_\xi(i)$.
2. For each $\xi < \lambda$ the set $[f_\xi, f_{\xi+1}) = \{(i, \alpha) \colon f_\xi(i) \leqslant \alpha < f_{\xi+1}(i)\}$ has cardinality μ.

By induction on $\xi < \lambda$ we define an increasing and continuous sequence of ordinals $\zeta(\xi) < \lambda$ and a $<^*$-increasing sequence of functions $f_\xi \in \prod_{i<\kappa} \mu_i$ so that $f_\xi = g_{\zeta(\xi)}$ for all $\xi < \lambda$ *except* when ξ is limit of cofinality $\leqslant \kappa$. Then $\overline{f} := \langle f_\xi \colon \xi < \lambda \rangle$ will be a smooth λ-scale as required.

At a limit stage ξ of cofinality $\leqslant \kappa$ let $\zeta(\xi) = \bigcup_{\xi' < \xi} \zeta(\xi')$ and use condition (1) to define f_ξ; at successor $\xi + 1$ choose $\zeta(\xi + 1)$ so that $\max\{f_\xi, g_{\zeta(\xi)}\} <^* g_{\zeta(\xi+1)}$ and (2) holds, and let $f_{\xi+1} = g_{\zeta(\xi+1)}$. Suppose now that ξ is a limit of cofinality $> \kappa$. By the smoothness of $\overline{g}$, and since $\langle g_{\zeta(\xi')} \colon \xi' < \xi \rangle$ is $<^*$-increasing, after defining $\zeta(\xi) = \bigcup_{\xi' < \xi} \zeta(\xi')$ we get that $g_{\zeta(\xi)}$ is an exact upper bound of $\langle g_{\zeta(\xi')} \colon \xi' < \xi \rangle$. But then $g_{\zeta(\xi)}$ is also an exact upper bound of $\langle f_{\zeta(\xi')} \colon \xi' < \xi \rangle$, and we let $f_\xi = g_{\zeta(\xi)}$.

Let f_λ be defined on κ by $f_\lambda(i) = \mu_i$.

Claim 2.7. *Suppose* $\delta \leqslant \lambda$ *and* $A \subseteq [0, f_\delta)$ *has cardinality* μ. *If* $\operatorname{cf} \delta > \kappa$ *there is some* $\delta' < \delta$ *so that* $|A \cap [0, f_{\delta'})| = \mu$.

Proof. Find $g < f_\delta$ so that $\sum_{i<\kappa} |A \cap (i \times g(i))| = \mu$. By smoothness there exists some $\delta' < \delta$ so that $g <^* g_{\delta'}$. □

For every $\xi < \lambda$ let $A_\xi = [f_\xi, f_{\xi+1})$ and let $\mathcal{A} = \{A_\xi : \xi < \lambda\}$. Then $\mathcal{A} \subseteq \mathcal{P}\big([0, f_\lambda)\big)$ is μ-almost disjoint and $|\mathcal{A}| = \lambda$.

For each $\delta \in S$ and $i < \kappa$ let $F_i^\delta = [f_{\gamma_i^\delta}, f_{\gamma_{i+1}^\delta})$. Then $\mathcal{F}_\delta = \{F_i^\delta : i < \kappa\}$ is a μ-almost disjoint family whose union is, by condition (1) on $\overline{f}$, equal to $[0, f_\delta)$. Fix a μ-ad family $\mathcal{B}_\delta \subseteq \mathcal{P}([0, f_\delta))$ such that $|\mathcal{B}_\delta| = \mathfrak{a}_\mu$, $\mathcal{B}_\delta \cup \mathcal{F}_\delta$ is μ-mad and $\mathcal{B}_\delta \cap \mathcal{F}_\delta = \emptyset$ (by Lemma 1.2).

Claim 2.8. *If $\delta \in S$ and $B \in \mathcal{B}_\delta$ then for all $i < \kappa$ it holds that $|B \cap [0, f_{\gamma_i^\delta})| < \mu$.*

Proof. If not so, let $i_0 < \kappa$ be the largest so that $|B \cap [0, f_{\gamma_{i_0}^\delta})| < \mu$; i_0 exists because D_δ is closed. Now $|B \cap F_{i_0}^\delta| = \mu$ — a contradiction. □

Let $\mathcal{B} = \bigcup_{\delta \in S} \mathcal{B}_\delta$. Then $|\mathcal{B}| = \mathfrak{a}_\mu \cdot \lambda = \lambda$ and therefore $|\mathcal{A} \cup \mathcal{B}| = \lambda$. We will show now that $\mathcal{A} \cup \mathcal{B}$ is μ-mad.

Suppose that $A = A_\xi \in \mathcal{A}$ and $B \in \mathcal{B}_\delta$ for some $\delta \in S$. If $\xi \geqslant \delta$ then clearly $|A \cap B| < \mu$ and if $\xi < \delta$, there is some $i < \kappa$ so that $A_\xi \subseteq^* F_i^\delta$ and $|A \cap B| < \mu$ follows from Claim 2.8.

If $B_1 \in \mathcal{B}_{\delta_1}$ and $B_2 \in \mathcal{B}_{\delta_2}$ with $\delta_1 < \delta_2$ in S, then there is some $i < \kappa$ so that $f_{\delta_1} <^* f_{\gamma_i^{\delta_2}}$ and Claim 2.8 gives $|B_1 \cap B_2| < \mu$.

This establishes that $\mathcal{A} \cup \mathcal{B}$ is μ-ad. To verify maximality, let $Z \subseteq [0, f_\lambda)$ be arbitrary of size μ. By Claim 2.7 the first $\xi \leqslant \lambda$ for which $|Z \cap [0, f_\xi)| = \mu$ is either a successor or of cofinality $\leqslant \kappa$. Cofinality $< \kappa$ is ruled-out by condition (1) on $\overline{f}$. The case ξ successor implies that $|Z \cap A_\xi| = \mu$. Finally, in the remaining case $\xi = \delta \in S$, there is some $B \in \mathcal{B}_\delta$ so that $|Z \cap B| = \mu$ □

Now the proof of Theorem 2.3 will be completed by the following Lemma, whose proof is actually found implicitly in [15]. We shall sketch a proof here too.

Lemma 2.9. *Suppose μ is singular and $\mu < \lambda < \mathfrak{b}_\mu$, λ regular. Then there is a smooth (μ, λ)-scale. If $\mathfrak{b}_\mu$ is a successor of a regular cardinal, there is also a smooth $(\mu, \mathfrak{b}_\mu)$-scale.*

Proof. Since $\lambda < \mathfrak{b}_\mu$, there exists a product $\prod_{i<\kappa} \mu_i$, where $\kappa = \operatorname{cf} \mu$, so that $\mathfrak{b}(\prod_{i<\kappa} \mu_i, <^*) > \lambda$.

By Claim 1.3 in [15] there exists a λ-scale $\overline{f} = \langle f_\alpha : \alpha < \lambda \rangle$ in some $\prod_{i<\kappa} \mu_i'$ such that for all regular $\theta \in (\kappa, \mu)$ every $\alpha < \lambda$ with $\operatorname{cf} \alpha = \theta$ satisfies that $\overline{f} \restriction \alpha$ is *flat*, that is, is equivalent modulo the bounded ideal on κ to a strictly increasing sequence of ordinal functions on κ.

By Lemma 15 in [10], *every* $\alpha < \lambda$ with $\operatorname{cf} \alpha > \kappa$ satisfies that $\overline{f} \restriction \alpha$ has an exact upper bound. Now it is clear how to replace $\overline{f}$ by a smooth λ-scale.

Suppose now that $\mathfrak{b}_\mu = \lambda^+$, $\lambda = \operatorname{cf} \lambda$. By [14], 4.1, the set $S^{\lambda^+}_{<\lambda} := \{\alpha : \alpha < \lambda^+ \wedge \operatorname{cf} \alpha < \lambda\}$ is a union of λ sets, each of which carries a square sequence. Therefore, $S^{\lambda^+}_{<\lambda} \in I[\lambda]$. By 2.5 in chapter 1 of [15], there exists a $(\mu, \mathfrak{b}_\mu)$-scale in which all points of cofinality $< \mu$ are flat and therefore a smooth $(\mu, \mathfrak{b}_\mu)$-scale. □

In contrast to the case of singular μ, let us mention the following result of A. Blass [4], which generalizes Hechler's [8]: it is consistent that $\mathrm{MAD}(\aleph_0) = C$, for any prescribed closed set of uncountable cardinals C which satisfies that $[\aleph_1, \aleph_1 + |C|] \subseteq C$ and $\lambda^+ \in C$ whenever $\lambda \in C$ has countable cofinality. For example, by Blass' or by Hechler's results there are universes of set theory in which $\mathrm{MAD}(\aleph_0) = \{\aleph_1, \aleph_{\omega+1}\}$. By Corollary 2.5, in any universe that satisfies this it holds that $[\aleph_1, \aleph_{\omega+1}] \subseteq \mathrm{MAD}(\aleph_\omega)$.

Recently Brendle [5], using techniques from [16], proved the consistency of $\mathfrak{a} = \aleph_\omega$.

Problem 2.10. *Is it consistent that* $\mathfrak{a}_{\aleph_\omega} = \aleph_\omega$*?*

3. Consistency results on $\mathrm{MAD}(\aleph_\omega)$ from large cardinal axioms

The inequality (1) can be used to control $\mathrm{MAD}(\aleph_\omega)$ by first increasing $\mathfrak{b}_{\aleph_\omega}$ and then increasing $\mathfrak{b}$. PCF theory implies that whenever the SCH fails at a singular cardinal μ, it holds that $\mathfrak{b}_\mu > \mu^+$. On the other hand, $\mathfrak{b}_\mu$ cannot be changed by a ccc forcing.

Before we state the result, let us recall some pcf terminology.

$$\mathrm{pcf}\{\aleph_n : n < \omega\} = \Big\{ \mathfrak{b}\big(\prod_n \aleph_n, \leqslant_I\big) : I \subseteq \mathcal{P}(\omega) \text{ is a proper ideal } \Big\}$$

The relation $<_I$ is defined by $f <_I g \Leftrightarrow \{n : f(n) \geqslant g(n)\} \in I$.

$\mathrm{pcf}\{\aleph_n : n < \omega\}$ is an interval of regular cardinals and has a maximum. For every $\lambda \in \mathrm{pcf}\{\aleph_n : n < \omega\}$ there exists a *pcf generator* $B_\lambda \subseteq \omega$ so that the following holds: denote by $J_{<\lambda}$ the ideal which is generated by $\{B_\theta : \theta \in \mathrm{pcf}\{\aleph_n : n < \omega\} \wedge \theta < \lambda\}$; then

$$\lambda = \mathfrak{b}\big(\prod_n \aleph_n, \leqslant_{J_{<\lambda}}\big)$$

Finally, $(\aleph_\omega)^{\aleph_0} = \max \mathrm{pcf}\{\aleph_n : n < \omega\} \times 2^{\aleph_0}$. Therefore, if $\aleph_\omega$ is a strong limit, $2^{\aleph_\omega} = \max \mathrm{pcf}\{\aleph_n : n < \omega\}$.

Fact 3.1. For every $\beta < \omega_1$ it is consistent (from large cardinal axioms) that $2^{\aleph_\omega} = \mathfrak{b}_\mu = \aleph_{\omega+\beta+1}$.

Proof. Let V be any universe of set theory in which $\aleph_\omega$ is a strong limit cardinal and $2^{\aleph_\omega} = \max \mathrm{pcf}\{\aleph_n : n \in \omega\} = \aleph_{\omega+\beta+1}$ [13, 9].

In V, the ideal $J_{<\max \mathrm{pcf}\{\aleph_n : n<\omega\}}$ is proper and is generated by countably many sets, therefore by simple diagonalization there exists an infinite $B \subseteq \omega$ so that $J_{<\max \mathrm{pcf}\{\aleph_n : n<\omega\}} \restriction B$ is contained in the ideal of finite subsets of B. Since $\mathfrak{b}(\prod_n \aleph_n, \leqslant_{J_{<\max \mathrm{pcf}\{\aleph_n : n<\omega\}}}) = \aleph_{\omega+\beta+1}$, it follows that $\mathfrak{b}(\prod_{n\in B} \aleph_n, \leqslant^*) = \aleph_{\omega+\beta+1}$, hence $\mathfrak{b}_{\aleph_\omega} = \aleph_{\omega+\beta+1}$. □

Theorem 3.2. *For every $\beta < \omega_1$ and $\alpha \leqslant \omega + \beta + 2$ it is consistent (from large cardinals) that $2^{\aleph_\omega} = \aleph_{\omega+\beta+2}$ and $\mathrm{MAD}(\aleph_\omega) = [\aleph_\alpha, \aleph_{\omega+\beta+2}]$.*

Proof. Start from a model V in which $2^{\aleph_0} = \aleph_1$, $\aleph_\omega$ is strong limit and $2^{\aleph_\omega} = \aleph_{\omega+\beta+2}$. Such a model exists by the previous Fact.

For every regular $\aleph_\omega < \lambda \leqslant \aleph_{\omega+\beta+2}$ there is a smooth λ-scale by Lemma 2.9. Consequently, there is also a smooth $\aleph_{\omega+\beta+2}$-scale.

Now apply Theorem 2.3 to finish the proof. □

By Theorem 5.4(b) in [3], after adding many Cohen subsets to ω_1, max $\mathrm{MAD}(\aleph_\omega)$ does not increase by much. Therefore it is consistent to have $\mathrm{MAD}(\aleph_\omega) = [\aleph_1, \aleph_{\omega+\beta+2}]$ as above, and to have $2^{\aleph_\omega}$ arbitrary large.

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