A SHORT PROOF OF THE PCF THEOREM

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Abstract. It is shown that every proper ideal over a small set of regular cardinals has an immediate successor (Theorem 19 below). From this, a short proof of Shelah's pcf theorem follows.

1. PRELIMINARIES

We shall need to know that for every regular κ and regular $\lambda > \kappa^+ +$ there is a stationary set in $I[\lambda]$. We give below the definition of $I[\lambda]$ and the proof of this fact. The proof uses club-guessing, which should be well known.

Theorem 1 (Club guessing)**.** *If k and λ are regular uncountable cardinals and $\kappa^+ < \lambda$ then there is a* club guessing sequence *on $S^\lambda_\kappa := \{\delta < \lambda : \mathrm{cf}\delta = \kappa\}$. That is, there is a sequence $\langle c_\delta : \delta \in S^\lambda_\delta\rangle$ so that:*

1. *$c_\delta \subseteq \delta = \sup c_\delta$ is club of δ and $\mathrm{otp} c_\delta = \kappa$.*
2. *for every club $E \subseteq \lambda$ the set $\{\delta \in S^\lambda_\kappa : c_\delta \subseteq E\}$ is stationary in λ.*

Proof. Suppose that $\langle c_\delta : \delta \in S^\lambda_\kappa\rangle$ is any sequence that satisfies condition (1) and we shall find a club $E \subseteq \lambda$ so that $\langle c_\delta \cap E : \delta \in S^\lambda_\kappa \cap \lim E\rangle$ satisfies conditions (1) and (2). By induction on $i < \kappa^+$ define a club E_i.Let $E_0 = \lambda$ and for limit $i < \kappa^+$ let $E_i = \bigcap_{j<i} E_j$. Suppose E_i is define. The sequence $\langle c_\delta \cap E : \delta \in S^\lambda_\kappa \cap \lim E\rangle$ must satisfy condition (1) since $E \cap \delta$ and c_δ are clubs of δ for δ $in \lim E$ and hence their intersection is a club of δ of ordertype κ. Assume then that it does not satisfy condition (2). This means there is a club $C^i_1 \subseteq \lambda$ such that $\{\delta \in S^\lambda_\kappa \cap E : c_\delta \cap E \subseteq C^i_1\}$ is not stationary; namely, there is a club $C^i_2 \subseteq \lambda$ which avoids this set. Let $E_{i+1} = E_i \cap C^i_1 \cap C^i_2$.

Suppose the induction goes on for all $i < \kappa^+$ and let $E^* = \bigcap_{i<\kappa^+} E_i$. Since $\kappa^+ < \lambda$, E^* is a club of λ. Fix $\delta \in \lim E^* \cap S^\lambda_\kappa$. For each $i < \kappa^+$ it holds that (a): $i \in \lim E_i \cap C^i_2$ hence that $c_\delta \cap E_i \not\subseteq C^i_1$. However, since $E_{i+1} = E_i \cap C^i_1 \cap C^i_2$, it holds that (b): $c_\delta \cap E_{i+1} \subseteq C^i_2$. From (a) and (b) we get that $c_\delta \cap E_{i+1} \subsetneqq c_\delta \cap E_i$ for all $i < \kappa^+$. This makes the sequence $\langle c_\delta \cap E_i : k < \kappa^+\rangle$ a strictly decreasing sequence of subsets of c_δ of length κ^+ which is of course absurd, since $\mathrm{otp} c_\delta = \kappa$.

We conclude that for some $i < \kappa^+$ condition (2) holds for $\langle c_\delta \cap E_i : \delta \in \lim E_i\rangle$. □

Definition 2. *Suppose λ is regular. We define an ideal $I[\lambda]$ over λ.*

For $S \subseteq \lambda$, $S \in I[\lambda]$ if and only if there exists a sequence $\bar{P} = \langle P_\alpha : \alpha < \lambda\rangle$ and a closed and unbounded set $E \subseteq \lambda$ such that:

1. $P_\alpha \subseteq \mathcal{P}(\alpha)$ *and* $|P_\alpha| < \lambda$
2. *If $\delta \in E \cap S$ then there exists a set $c \subseteq \delta$ such that $\delta = \sup c$, $\mathrm{otp} c = \mathrm{cf}(\delta) < \delta$ and $(\forall \beta \in c) c \cap \beta \in \bigcup_{\gamma<\delta} P_\gamma$*

Theorem 3. *If κ, λ are regular cardinals and $\kappa^{++} < \lambda$ then there is a stationary set $S \subseteq S^\lambda_\kappa$ in $I[\lambda]$.*

Proof. Fix a club guessing sequence $\overline{C} = \langle c_\alpha : \alpha \in S^{\kappa^{++}}_\kappa\rangle$ as in Theorem 1 in the previous section.

Description of the proof.

The proof will involve two elementary chains of models of $H(\chi)$, for some large enough regular χ. The first chain will be used to define $\langle P_i : i < \lambda\rangle$; the other, to prove that this choice works. The first chain is going to be an element of every member in the second. At some point of the proof, though, we shall need some set which is definable in the *second* chain to belong to some member of the *first* chain. We use the prediction, or "guessing", property of a club guessing sequence to obtain this.

Fix an elementary chain $\overline{M} := \langle M_i : i \le \lambda\rangle$ of submodels of $\langle H(\chi), \in\rangle$ (for a large enough regular χ) satisfying:

- $||M_i|| < \lambda$
- $\langle M_j : j \le i\rangle \in M_{i+1}$, $i \in M_i$ and $i \subseteq M_i$
- $\overline{C}, \lambda \in M_0$ and $\kappa^{++} + 1 \subseteq M_0$

Let us define $P_i := M_i \cap \mathcal{P}(i)$. By condition (1) we see that $|P_i| < \lambda$. Let $S \subseteq \lambda$ be the set

$$S := \left\{ \alpha < \lambda : \mathrm{cf}\alpha = \kappa \wedge \exists c \,\text{club of}\, \alpha \big[(\forall \gamma < \alpha)(c \cap \gamma \in \bigcup_{i<\alpha} P_i)\big] \right\}$$

The sequence $\langle P_i : i < \lambda\rangle$ witnesses that $S \in I[\lambda]$ according to the definition of $I[\lambda]$. All that is left to be shown is that S is stationary.

Explanation If indeed there is a stationary $S \subseteq S^{\lambda}_{\kappa}$ in $I[\lambda]$ them M_0 should know about it and about some witness $\overline{P}$, by elementarity, and therefore P_i as chosen here must suffice.

We prove now that S is stationary. Fix a club $E \subseteq \lambda$ and we will show that E meets S. By shrinking E we may assume that $M_i \cap \lambda = i$ for all $i \in E$; this follows from the fact that $\{i < \lambda : M_i \cap \lambda = i\}$ is a club of λ.

Define a second elementary chain of models of $H(\chi)$, $\overline{N} := \langle N_\zeta : \zeta \leq \kappa^{++}\rangle$ satisfying:

- $\overline{M}, E, \lambda, \overline{C} \in N_0$ and $\kappa^{++} + 1 \subseteq N_0$
- $||N_\zeta|| = \kappa^{++}$
- $\langle N_\xi : \xi \leq \zeta\rangle \in N_{\zeta+1}$ for $\zeta < \kappa^{++}$

Define $f(\zeta) := \sup N_\zeta \cap \lambda$. The function $f : (\kappa^{++} + 1) \to \theta$ is increasing and continuous. Denote $\theta = f(\kappa^{++}$. By condition 3 in the choice of $\overline{N}$, we see that $f{\restriction}\zeta \in N_{\zeta+1}$ for every $\zeta < \kappa^{++}$.

We observe that for every $\zeta \leq \kappa^{++}$ the ordinal $f(\zeta)$ belongs to E. This is true by elementarity and the fact that $E \in N_\zeta$ for all ζ: if $\beta \in N_\zeta \cap \lambda$ is arbitrary, then $N_\zeta \models (\exists\gamma \in \lambda)\,[\gamma \in E \wedge \gamma > \beta]$. Therefore there exists $\beta < \gamma \in E \cap N$ is and thus E unbounded below $f(\zeta)$, implying $f(\zeta) \in E$. Thus ran$f \subseteq E$. Turn now to the chain $\overline{M}$ and work in $M_{\theta+1}$. Use the fact that $\theta \in M_{\theta+1}$ (condition (2) in the choice of $\overline{M}$) and the elementarity of $M_{\theta+1}$ and choose a function $g \in M_{\theta+1}$ such that $g : \kappa^{++} \to \theta$ increasing and continuous with $\theta = \sup \mathrm{ran} g$.

Since both f and g are increasing continuous on κ^{++} with ranges cofinally contained in θ, a standard argument allows us to fix a club$E \subseteq \kappa^{++}$ such that $f{\restriction}E = g{\restriction}E$.

Use the club guessing property of $\overline{C}$ to fix $\delta < S^{\kappa^{++}}_{\kappa}$ such that $c_\delta \subseteq E$, $\delta = \sup c_\delta$ and otp$c_\delta = \kappa$. Define $c := f[c_\delta] = g[c_\delta]$.

Thus $c \subseteq f(\delta)$ is a club of $f(\delta)$ of order type κ. We already know that $f(\delta) \in E$; we will show that $f(\delta) \in S$ by showing that $c \cap \gamma \in \bigcup_{i<\theta_\delta} P_i$ for every $\gamma < f(\delta)$.

Let, then, $X = c_\delta \cap \zeta$ for some $\zeta \in c_\delta$ be an initial segment of c_δ and let $Y := f[X]$ be the corresponding initial segment of c. As c_δ and ζ belong to N_0, we have $X \in N_0 \subseteq N_{\zeta+1}$; and since $f{\restriction}\zeta \in N_{\zeta+1}$ by condition 3 in the choice of $\overline{N}$ we conclude that $Y = f[X] \in N_{\zeta+1}$.

If we knew that $Y \in M_i$ for *some* $i < \lambda$ we would be done: Suppose that there were some $i < \lambda$ such that

$$H(\chi) \models \exists i < \lambda\,[Y \in M_i)]$$

Since $Y, \overline{M}, \lambda \in M_{\zeta+1}$ and $N_{\zeta+1} \prec H(\chi)$

$$N_{\zeta+1} \models \exists i < \lambda[Y \in M_i]$$

by elementarity. Thus there is some $i < f(\zeta+1) < f(\delta)$ such that $Y \in P_i$, as required.

But why is there such $i < \lambda$ at all? The reason is the club guessing sequence, which enables $\overline{M}$ to "predict" the set Y. Recall that $Y = f[X]$. The parameter X in this definition belongs to M_0. But f may not belong to any $M_i \in \overline{M}$. However, $f \restriction c_\delta = g \restriction c_\delta$ and so we can define Y using g instead. The function g belongs to $M_{\theta+1}$, and the proof is now complete, as $Y = g[X]$ is definable in $M_{\theta+1}$ and hence $Y \in M_{\theta+1}$. □

2. Exact upper bounds modulo an ideal

The basic object we are examining is the following: let A be some infinite set and I be an ideal over A. Let $\overline{f} = \langle f_\alpha : \alpha < \delta \rangle$ be a sequence of functions from A to the ordinals which is increasing modulo I, where δ is some limit ordinal. The question we address is the existence and structure of an exact upper bound modulo I for this sequence. In this Section we establish notation and prove a few basic facts about exact upper bounds modulo an ideal which are needed to facilitate the rest of the discussion.

2.1. Basics.

Let A be a fixed infinite set. By On^A we denote the class of all functions from A to the ordinal numbers. Given an ideal I over A, we quasi order On^A by defining $f \leq_I g$ for $f, g \in \mathrm{On}^A$ iff $\{a \in A : f(a) > g(a)\} \in I$. Similarly, $=_I$ and $<_I$ are defined.

In the special case that $I = \{\emptyset\}$, the relation $<_I$ is the relation of domination everywhere and is denoted by $<$.

For subsets $F_1, F_2 \subseteq \mathrm{On}^A$ write $F_1 \equiv_I F_2$ if for every $f \in F_1$ there is $g \in F_2$ such that $f \leq_I g$ and for every $g \in F_2$ there is $f \in F_1$ such that $g \leq_I F$. The relation $\equiv_I$ is an equivalence relation.

We shall investigate the relation between the following two properties of subsets of On^A:

Definition 4. (1) $F \subseteq On^A$ has an exact upper bound *iff there exists $g \in On^A$ such that:*
(a) $(\forall f \in F)(f \leq_I g)$
(b) $g' <_I g \Rightarrow (\exists f \in F)(g' <_I f)$
A function g which satisfies (a) and (b) is an exact upper bound *(an eub) of F.*

(2) $F \subseteq On^A$ has true cofinality *iff there exists some* $F' \subseteq On^A$ *such that* $F' \equiv_I F$ *and* F' *is linearly ordered by* $<_I$ *and has no last element.*

If F *has true cofinality then the* true cofinality *of* F *is denoted by* $\operatorname{tcf} F$ *and is the cofinality of the order type of some (of every) linearly ordered* F' *that is equivalent to* F.

The following points should be noticed: first, each of the properties defined below is invariant under $\equiv_I$. Second, neither property implies the other. Third, eubs are also least upper bounds, except in the trivial case where F has an upper bound which assumes the value 0 on a positive set, and is therefore an eub vacuously.

Fact 5. (1) *A set* $F \subseteq On^A$ *without a maximum with respect to* $\leq_I$ *has an eub* f *iff* F *is equivalent to a copy of a product of regular cardinals, namely there exists sets* $S(a) \subseteq f(a)$ *for* $a \in A$ *such that* $\operatorname{otp} S(a) = \operatorname{cf} f(a)$ *is and* $F \equiv_I \prod_{a\in A} S(a)$.

(2) *A set* $F \subseteq On^A$ *has true cofinality* λ *iff it is equivalent to a* $<_I$ *increasing sequence* $\langle f_\alpha : \alpha < \lambda\rangle$.

Both properties above are preserved when the ideal I is extended. We shall be using the following fact freely:

Fact 6. *Suppose* $I_1 \subseteq I_2$ *are ideals over an infinite set* A *and* $F \subseteq On^A$. *Then:*

(1) *If* g *is an eub of* F *modulo* I_1 *and* $0 <_{I_1} g$ *then* g *is also an eub of* F *modulo* I_2.

(2) *If* F *has true cofinality* λ *modulo* I_1 *than* F *has true cofinality* λ *modulo* I_2.

A particular instance of this fact is when $I_2 = I_1 \restriction B := \langle I \cup (A \setminus B)\rangle$, the ideal generated by joining $A \setminus B$ to I, for some $B \in I_1^+$.

The following is a simple, yet important example of a set of functions which has both an eub and true cofinality regardless to which ideal I over A is involved:

Fact 7. *Suppose* λ *is regular and* $\lambda > |A|$. *Then* $\operatorname{tcf}(\lambda^A, <) = \lambda$. *Consequently,* $\operatorname{tcf}(\lambda^A, <_I) = \lambda$ *for every ideal* I *over* A, *by Fact 6.*

Proof. Let $g_i(a) = i$ for $\gamma < \lambda$ and $a \in A$. The sequence $\overline{g} = \langle g_\gamma : \gamma < \lambda\rangle$ is $<$-increasing. It is also cofinal in $(\lambda^A, <)$ by the following "rectangle argument": Let $g \in \lambda^A$ be arbitrary. Since $\lambda > |A|$ is regular, $\gamma := \sup\{g(a) : a \in A\} < \lambda$ and therefore $g \leq g_\gamma$. $\square$

Claim 8. *Suppose that $\lambda > |A|$ is regular and $\overline{f} = \langle f_\alpha : \alpha < \delta \rangle \subseteq On^A$ is $<_I$-increasing. The following are equivalent:*

1. *There is an eub f of $\overline{f}$ such that $\mathrm{cf} f(a) = \lambda$ for all $a \in A$.*
2. *There are sets $S(a)$ for $a \in A$ with $\mathrm{otp} S(a) = \lambda$ such that $\overline{f} \equiv_I \prod S(a)$.*
3. *There is some $<$-increasing $\overline{g} = \langle g_\gamma : \gamma < \lambda \rangle$ and some increasing, continuous and cofinal subsequence $\langle \alpha(\gamma) : \gamma < \lambda \rangle \subseteq \lambda$ such that $f_{\alpha(\gamma)} <_I g_{\gamma+1} <_I f_{\alpha(\gamma+1)}$.*
4. *There is some $\leq$-increasing $\overline{g} = \langle g_\gamma : \gamma < \lambda \rangle$ and some increasing, continuous and cofinal subsequence $\langle \alpha(\gamma) : \gamma < \lambda \rangle \subseteq \lambda$ such that $f_{\alpha(\gamma)} <_I g_{\gamma+1} <_I f_{\alpha(\gamma+1)}$.*

Definition 9.
1. *A $<_I$-increasing $\overline{f} = \langle f_\alpha : \alpha < \delta \rangle \subseteq On^A$ is* flat *(of cofinality λ) mod I if and only if one of the equivalent conditions in 8 holds for $\overline{f}$.*
2. *$\alpha < \delta$ is a* flat point *in a $<_I$-increasing $\overline{f} = \langle f_\beta : \beta < \delta \rangle$ if and only if $\overline{f} \restriction \alpha$ is flat.*

The next theorem gives a necessary and sufficient condition for the existence of an eub of a sequence with large cofinalities, in terms of flatness of initial segments of the sequence.

Theorem 10. *Suppose $\overline{f} = \langle f_\alpha : \alpha < \lambda \rangle \subseteq On^A$ is a $<_i$-increasing and $\lambda = \mathrm{cf}\lambda > |A|^+$. For every regular $\kappa \in (|A|, \lambda)$ the following are equivalent:*

- *$\overline{f}$ has an eub g so that $\{a : \mathrm{cf} g(a) \leq \kappa\} \in I$*
- *The set $\{\alpha <\ \mathrm{cf}\alpha = \kappa$ and $\overline{f} \restriction \alpha$ is flat $\}$ is stationary in λ.*

Proof. Suppose $|A| < \kappa < \lambda$ and κ is regular. Suppose first that $\overline{f}$ has an eub g with $\mathrm{cf} g(a) > \kappa$ for all $a \in A$ except for a set in I. By changing g on a set in I we may assume that $\mathrm{cf} g(a) > \kappa$ for all $a \in A$.

Let $C \subseteq \lambda$ be a given club. By induction on $i < \kappa$ find $\alpha(i) \in C$ as follows. At stage i let $g_i(a) = \sup_{j<i} f'_{\alpha(j)}(a)$, where $f'_{\alpha(j)}(a) = f_{\alpha(j)}$ if $f_{\alpha(j)}(a) < g(a)$ and is 0 otherwise. Since $g_i(a)$ is a supremum of $< \kappa$ ordinals below $g(a)$ and $\mathrm{cf} g(a) > \kappa$, $g_i < g$. Hence there is some $\alpha < \lambda$ so that $g <_I f_\alpha$. Let $\alpha(i) = \min(C \setminus \{\alpha + 1\} \cup \{\alpha(j) + 1 : j < i\}$.

Since $\langle \alpha(i) : i < \kappa \rangle$ is astrictly increasing sequence of points in C, $\alpha = \sup\{\alpha(i) : i < \kappa\}$ belongs to C and has cofinality κ. We argue that $\overline{f} \restriction \alpha$ is flat. Indeed, $\overline{f} \restriction \alpha$ is equivalent to $\langle g_i : i < \kappa \rangle$ which is $\leq$-increasing, and is hence flat.

Now for the other direction, suppose that $S = \{\alpha < \lambda : \mathrm{cf}\alpha = \kappa\}$ is stationary in λ. The trichotomy theorem applies to $\overline{f}$, thus to show the existence of an eub g it suffices to rule out Bad and Ugly. If either condition o the two holds for $\overline{f}$, then the set of $\alpha < \lambda$ such that the same condition holds for $\overline{f}\restriction\alpha$ is a club of λ, and hence meets S. This leeds to a contradition.

Fix an eub g of $\overline{f}$. The fact that $\{a \in A : \mathrm{cf} g(a) < \kappa\} \in I$ is obvious. □

3. The possible true cofinalities of small sets of regular cardinals

Definition 11. *Suppose $(Q, \leq)$ is a quasi-ordered set with no maximal elements. Let $\mathfrak{b}(Q, \leq)$ be the lest cardinality of an unbounded subset of Q. If $(Q, \leq)$ has a maximum put $\mathfrak{b}(Q, \leq) = \infty$, where ∞ is greater than all cardinals.*

Let $\mathfrak{d}(Q, \leq)$ be the smallest cardinality of a dominating subset of Q.

Claim 12. *Suppose that $(Q, \leq)$ is a quasi-ordered set with no maximum. Then $\mathfrak{b}(Q, \leq)$ is either 2 or an infinite regular cardinal.*

Proof. Suppose first that $\mathfrak{b}(Q, \leq) = n$ is finite, and fix an unbounded set $\{q_i : i < n\}$ of cardinality n. Since all singletons in Q are bounded subsets of Q, $n > 1$.

By minimality of n, the set $\{q_i : i < n - 1\}$ is bounded by some $p \in Q$. Now the set $\{p, q_{n-1}\}$ is unbounded, since $\{q_i : i < n\}$ is unbounded, and is of cardinality 2.

Suppose now $\mathfrak{b}(Q, \leq) = \lambda$ is infinite, and fix an unbounded set $\{q_\alpha : \alpha < \lambda\} \subseteq Q$ of cardinality λ. Let $\langle p_\alpha : \alpha < \lambda \rangle$ be chosen recursively as follows: p_α is an upper bound of $\{q_\beta : \beta < \alpha\} \cup \{p_\beta : \beta < \alpha\}$. Since every subset of Q of cardinality smaller than λ is bounded, this recursive definition is good.

Now $\{p_\alpha : \alpha < \lambda\}$ is a well-ordered chain in Q or ordertype λ, and for every $\alpha < \lambda$, q_α is dominated by some member of the chain (e.g. $p_{\alpha+1}$). Thus, $\{p_\alpha : \alpha < \lambda\}$ is not bounded.

If λ were singular with $\mathrm{cf}\lambda = \kappa < \lambda$ there would be a cofinal $A \subseteq \lambda$ with $\mathrm{otp} A = \kappa$. By minimality of λ, there would be some $p \in Q$ which bounds A, and therefore bounds all of $\{p_\alpha : \alpha < \lambda\}$, contrary to $\{p_\alpha : \alpha < \lambda\}$ being unbounded in Q. We conclude that λ is regular. □

Claim 13. *Suppose $(Q, \leq)$ is a quasi-ordered set with no maximum. Then $\mathfrak{b}(Q, \leq) \leq \mathfrak{d}(Q, \leq)$.*

Proof. Since $(Q, \leq)$ has no maximum, $\mathfrak{d}(Q, \leq) \geq 2$. So in the case $\mathfrak{b}(Q, \leq) = 2$ it is hods that $\mathfrak{b}(Q, \leq) \leq \mathfrak{d}(Q, \leq)$ (but it could be that $\mathfrak{b}(Q, \leq) = 2$ and $\mathfrak{d}(Q, \leq)$ is infinite).

Suppose then that $\mathfrak{b}(Q, \leq) = \lambda$ is an infinite regular cardinal, and fix an unbounded wellordered chain $B \subseteq Q$ of ordertype λ. If $D \subseteq Q$ is dominating, then every element of D dominates a proper initial segment of B and those initial segments are unbounded in B. Thus, by regularity of λ, $|D| \geq \lambda$. thus, $\mathfrak{b}(Q, \leq) \leq \mathfrak{d}(Q, \leq)$. □

The cardinal $\mathfrak{d}(Q, \leq)$ may be singular.

Definition 14. *A quasi-ordered set $(Q, \leq)$ with no maximum has* true cofinality λ *if $\lambda = \mathfrak{b}(Q, \leq) = \mathfrak{d}(Q, \leq)$ and λ is infinite.*

Definition 15. *A set A of cardinals is* small *if $|A| < \min A$.*

Suppose that A is a small set of regular cardinals and that $I \subseteq \mathcal{P}(S)$ is a proper ideal. The quasi ordered set $(\prod A, <_I)$ has no maximum and therefore $\mathfrak{b}(\prod A, <_I) < \infty$. We denote this number by $\mathfrak{b}(I)$ for short.

Definition 16. *For a small set A of regular cardinals let*

$$\mathrm{PCF}\, A = \{\mathfrak{b}(I) : I \subseteq \mathcal{P}(A) \text{ is a proper ideal } \}$$

We show that in a product of a small set of regular cardinals modulo an ideal I, above every unbounded $<_I$-increasin sequence there is a $<_I$-increasing sequence with an exact upper bound.

Theorem 17. *Suppose A is a small set of regular cardinals. Let I be an ideal over A and assume that $\mathfrak{b}(I) = \lambda$. Then for every sequence $\overline{h} = \langle h_\alpha : \alpha < \lambda \rangle \subseteq \prod A$ there exists a $<_I$-increasing sequence $\overline{f} = \langle f_\alpha : \alpha < \lambda \rangle$ so that*

- *$h_\alpha \leq f_\alpha$ (everywhere) for all $\alpha < \lambda$*
- *$\overline{f}$ has an eub g with $\mathrm{cf}\, g(a) > |A|$ for all $a \in A$.*

Proof. Suppose that $\overline{h} = \langle h_\alpha : \alpha < \lambda \rangle \subseteq \prod A$ is given.

If $A \cap \lambda \in I$, set $B = A \setminus (\lambda \cap A)$ and by a straightforward induction on $\alpha < \lambda$ construct a sequence $\overline{f} = \langle f_\alpha : \alpha < \kappa \rangle$ so that $h_\alpha \leq f_\alpha$ and $\langle f_\alpha \restriction B : \alpha < \lambda \rangle$ is $<$-increasing. Such a sequence is also $<_I$-increasing and has an eub g mod I with $\mathrm{cf}\, g(a) = \lambda$ for all $a \in B$ (for $a \notin B$ $g(a)$ can be defined to be λ).

We assume then that $A \cap \lambda \notin I$. The set $A \cap \lambda$ cannot be finite, for then $\lambda = \mathfrak{b}(I) \leq \mathfrak{b}(I \restriction (A \cap \lambda)) \leq \max\{A \cap \lambda\} < \lambda$ which is impossible. We are left with the case that $A \cap \lambda \notin I$ and $\lambda > |A|^{+\omega}$. This is the interesting case, which we have if, e.g., $A = \{\omega_n : 0 < n < \omega\rangle$ and I contains all finite subsets of A.

Let $\kappa = |A|^+$. Since $\lambda > \kappa^{++}$, there is a stationary $S \subseteq S^\lambda_\kappa$ in $I[\lambda]$ by Theorem 3. Fix a stationary $S \subseteq S^\lambda_\kappa$ in $I[\lambda]$ together with $\langle P_\alpha : \alpha < \lambda\rangle$ which demonstrates that $S \in I[\lambda]$, that is, it satisfies condition (1) in Definition 2 and condition (2) holds for all $\delta \in S$. We may assume that P_α is $\subseteq$-increasing with α.

Define $f_\alpha \geq h_\alpha$ by induction on $\alpha < \lambda$ as follows. At step $\alpha < \lambda$ define, for each $c \in P_\alpha$ with $\mathrm{otp} c < \kappa$, a function g_c by $g_c(a) = \sup\{f_\beta(a) : \beta \in c\}$. Since $a > \kappa^+$ is regular and $\mathrm{otp} c < \kappa$, $g_c \in \prod A$. The set $\{g_c : c \in P_\alpha\} \cup \{f_\beta : \beta < \alpha\}$ has cardinality $< \lambda$ and is therefore bounded in $(\prod A, <_I)$. Let $f'_\alpha \in \prod A$ be a bound of this set and now set $f_\alpha = \max\{f'_\alpha, h_\alpha\}$.

Clearly, $h_\alpha \leq f_\alpha$ for all $\alpha < \lambda$ and $\overline{f}$ is $<_I$-increasing.

Suppose $\delta \in S \in I[\lambda]$ and fix $c \subseteq \delta = \sup c$ with $\mathrm{otp} c = \kappa$ so that $(\forall \beta \in c)\ c \cap \beta \in \bigcup_{\gamma \in c} P_\gamma$.

For each $\beta \in c$ let $g_\beta = \sup\{f_\xi : \xi \in c_\delta \cap \beta\}$. Clearly, $\overline{g} = \langle g_\beta : \beta \in c\rangle$ is $\leq$-increasing of length κ and $\overline{f} \restriction \delta \leq_I \overline{g}$.

For every $\beta \in c$ there is $\gamma \in c$ such that $c \cap \beta \in P_\gamma$, so by the construction of $\overline{f}$ it holds that $g_\beta <_I f_\gamma$. This shows that also $\overline{f} \leq_I \overline{g}$ holds, and thus $\overline{f} \restriction \delta \sim_I \overline{g}$, that is, $\overline{f} \restriction \alpha$ is equivalent to a $\leq$-increasing sequence of length κ, and is therefore flat.

We have thus shown that $\overline{f}$ has a stationary set $S \subseteq S^\lambda_\kappa$ of flat points. By Theorem 10, $\overline{f}$ has an eub g with $\mathrm{cf} g(a) > |A|^+$ for all $a \in A$, as required. □

Corollary 18. *If A is a small set of regular cardinals then*

$$\mathrm{PCF}\, A = \{\mathrm{tcf}(I) : I \subseteq \mathcal{P}(A) \text{ is a proper ideal and } \mathrm{tcf}(I) \text{ exists}\}$$

Proof. Suppose $\lambda = \mathfrak{b}(I) \in \mathrm{PCF}\, A$. Let $\overline{h} = \langle h_\alpha : \alpha < \lambda\rangle$ be $<_I$-increasing and unbounded in $(\prod A, <_I)$. By the previous theorem there is $\overline{f} = \langle f_\alpha : \alpha < \lambda\rangle$ with $h_\alpha \leq f_\alpha$ for all $\alpha < \lambda$ so that $\overline{f}$ has an eub g. We may assume that $g(a) \leq a$ for all $a \in A$. Let $B = \{a \in A : g(a) = a\}$. If $B \in I$ then $\overline{f}$ is bounded in $(\prod A, <_I)$ by g' where $g'(a) = 0$ if $a \in B$ and $g'(a) = g(a)$ otherwise. Since $\overline{h}$ and hence also $\overline{f}$ is unbounded, $B \notin I$. Thus, $\overline{f}$ is both $<_{I \restriction B}$-increasing and $<_{I \restriction B}$-dominating, and thus demonstrates that $\mathrm{tcf}(\prod A, <_{I \restriction B}) = \lambda$. This shows the inclusion $\subseteq$. The converse inclusion is immediate. □

Theorem 19. *Suppose A is a small set of regular cardinals. Then for every proper ideal I over A there is an ideal $I^s \supseteq I$ over A so that:*

(1) $\mathfrak{b}(I) < \mathfrak{b}(I^s)$
(2) *$I^s \subseteq J$ for every $J \supseteq I$ that satisfies $\mathfrak{b}(J) > \mathfrak{b}(I)$.*
(3) *There is a single set $B_I \in I^+$ so that $I^s = \langle I \cup \{B_I\}\rangle$, that is, I^s is generated from I by a single set.*

Proof. Let $I^+ = \langle I \cup \{B \in I^+ : \mathfrak{b}(I) = \mathrm{tcf}(\prod B, <_I)\}\rangle$.

Suppose $F \subseteq \prod A$ has cardinality $\mathfrak{b}(I)$ and we shall show that F is bounded mod I^s. We may assume F is a $<_I$-increasing sequence $\overline{f} = \langle f_\alpha : \alpha < \lambda\rangle$ where $\lambda = \mathfrak{b}(I)$. If $\overline{f}$ is bounded mod I, it is also bounded mod I^s. So assume $\overline{f}$ is not bounded mod I. By Theorem 17 we may assume that $\overline{f}$ has an eub g. If $B = \{a \in a : g(a) \geq a\} \in I$ then $\overline{f}$ is bounded mod I. Otherwise, $B \in I^s$. In each case, $\overline{f}$ is bounded mod I^s.

The fact that $\mathfrak{b}(I^s) > \mathfrak{b}(I)$ implies in particular that $I^s \setminus I \neq \emptyset$.

Suppose now that $I \subseteq J$ and that $\mathfrak{b}(J) > \lambda$. If $B \notin J$ then $\mathrm{tcf}(\prod A, <_{J\restriction B}) = \lambda$, in particular $\mathfrak{b}(\prod A, <_{J\restriction B}) = \lambda \geq (\prod A, <_J)$ — contradiction.

Now we prove that I^s is generated from I by a single set $B \in I^s \setminus I$. Suppose that for some $B \in I^s \setminus I$, $\langle I \cup \{B\}\rangle \neq I^s$; this means that thare is some $B' \in I^2$ so that $B' \not\subseteq_I B$ and putting $B'' = B \cup B'$ we see that this is equivalent to the existence of a $B'' \in I^s$ so that $B \subseteq_I B''$ but $B'' \not\subseteq_I B$. (Recall that $B \subseteq_I B''$ if $B \setminus B'' \in I$.)

Assume now that for all $B \in I^s \setminus I$ it holds that $I^s \neq \langle I \cup \{B\}\rangle$ and we shall reach a contradiction. By induction on $i < |A|^+$ we define a $\subseteq_I$-increasing sequence $\langle B_i : i < |A|^+\rangle$, and for each i we define a $<_{I\restriction B}$-increasing and $<_{I\restriction B}$ cofinal $\overline{f}^i = \langle f^i_i : i < (I)\rangle \subseteq \prod B_i$. We agree to view a function $f \in \prod B_i$, for $B \subseteq A$, as a member of $\prod A$ by putting $f(a) = 0$ for all $a \in A \setminus B$.

We require that:

- $f^j_\alpha \leq f^i_\alpha$ (everywhere) for all $i < j < |A|^+$ and all $\alpha < \lambda$.

At successor stage $i+1 < |A|^+$ find $B_{i+1} \in I^+$ so that $B_i \subseteq_I B_{i+1}$ and $B_{i+1} \not\subseteq_I B_i$ and, using the fact that $\mathrm{tcf}(\prod B_{i+1}, <_I) = \lambda$, define $f^{i+1}_\alpha \geq f^i_\alpha$ by induction on $\alpha < \lambda$ so that $\overline{f}^{i+1} \subseteq \prod B_{i+1}$ is $<_{I\restriction B_{i+1}}$-increasing and $<_{I\restriction B_{i+1}}$-cofinal in $\prod B_{i+1}$.

If $i < |A|^+$ is limit, let $h_\alpha = \sup\{j < i : f^j_\alpha\}$ for each $\alpha < \lambda$. Since $a > |A|$ is regular for each $a \in A$, this supremum belongs to $\prod A$. By Theorem 17 there is a $<_I$-increasing sequence $\overline{f} = \langle f_\alpha : \alpha < \lambda\rangle$ with an eub g that satisfies $\mathrm{cf} g(a) > |A|$ for all $a \in A$ and so that $h_\alpha \leq f_\alpha$ for all $\alpha < \lambda$. Define this sequence to be $\overline{f}^i$ and Let $B_i := \{a \in A : g(a) \geq a\}$.

We verify now that $B_j \subseteq_I B_i$ and $B_i \not\subseteq B_j$ for all $i < j$. Suppose $j < i$ and let $C := B_j \setminus B_i = \{a \in B_j : g(a) < a\}$. If $C \notin I$ then for some

$\alpha < \lambda$ it holds that $g\restriction C <_I f^j_\alpha$ — since $\overline{f}^j$ is in this case $<_{I\restriction C}$-increasing and $<_{I\restriction C}$-cofinal. But $f^j_\alpha \le f^i_\alpha$, so $g\restriction C <_I f^j_\alpha\restriction C \le f^i_\alpha\restriction C <_I g\restriction C$, which is impossible; thus $B_j \subseteq_I B_I$ for all $j < I$ when i is limit. Since $B_{j+1} \not\subseteq_I B_j$ and $B_{j+1} \subseteq B_i$, we also get that $B_i \not\subseteq_I B_j$ for all $j < i$. This completes the induction.

Let now $C_i := \{a \in A : f^i_1(a) > 0\}$. Since the sequence $\langle f^i_1 : i < |A|^+\rangle$ is $\le$-increasing by the construction, $i < i < |A|^+$ implies that $C_i \subseteq C_j$. Given $i < j < |A|^+$ we have $B_j \setminus B_i \notin I$, and since $f^i_1\restriction(B_j \setminus B_i) = 0$ while $0 <_{I\restriction B_j} f^j_1$, there exists some $a \in C_j \setminus C_i$. We conclude that $\langle C_i : i < |A|\rangle$ is a strictly increasing sequence of subsets of A of length $|A|^+$ — a contradiction. □

3.1. **The inductive definiton of PCF.** Let A be a small set of regular cardinals. We define recursively a strictly increasing and continuous sequence of proper ideals $\langle I_\alpha : \alpha \le \alpha(*)\rangle$ with a strictly increasing sequence of bounding numbers $\lambda_\alpha = \mathfrak{b}(I_\alpha)$. Let $I_0 = \{\emptyset\}$ (so $\lambda_0 = \min A$). For limit α let $I_\alpha = \bigcup_{\beta<\alpha} I_\beta$. Since an increasing union of proper ideals over A is a proper ideal over A, I_α is proper, and clearly $\lambda_\alpha = \mathfrak{b}(I_\alpha) > \lambda_\beta$ for all $\beta < \alpha$.

If I_α is defined and is proper, let $I_{\alpha+1} = I^s_\alpha$. Let $B_\alpha \in I_{\alpha+1} \setminus I_\alpha$ be so that of $I_{\alpha+1} = \langle I_\alpha \cup \{B_\alpha\}\rangle$ from I_α.

Since a strictly increasing sequence of proper ideals over A has to be of bounded length, there is some ordinal $\alpha(*)$ so that $I_{\alpha(*)}$ is proper but $I^s_{\alpha(*)} = \langle I_{\alpha(*)} \cup \{B_{\alpha(*)}\}\rangle = \mathcal{P}(A)$.

Claim 20. *For every proper ideal $I \subseteq \mathcal{P}(A)$,*

$$\mathfrak{b}(I) = \min\{\lambda_\alpha : \alpha \le \alpha(*) \wedge B_\alpha \notin I\}$$

Proof. Given a proper ideal I, clearly $I^s_{\alpha(*)} = \mathcal{P}(A) \not\subseteq I$. By the continuity of $\langle I_\alpha : \alpha \le \alpha(*)\rangle$ thus there is a largest $\alpha \le \alpha(*)$ so that $I_\alpha \subseteq I$. Therefore $\lambda_\alpha = \mathfrak{b}(I_\alpha) \le \mathfrak{b}(I)$. Conversely, $\mathfrak{b}(I) \le \mathfrak{b}(I\restriction B_\alpha)$ and sicne $\operatorname{tcf}(\prod A, <_{\lambda\restriction B_\alpha}) = \lambda_\alpha$ and $B_\alpha \notin I$, $\operatorname{tcf}(\prod A, <_{I\restriction B_\alpha}) = \lambda_\alpha$ as well. □

Corollary 21. $\mathrm{PCF}\, A = \{\lambda_\alpha : \alpha \le \alpha(*)\}$

Theorem 22 (The pcf theorem)**.** *For every small set A of regular cardinal there is a sequence of sets $\langle B_\lambda : \lambda \in \mathrm{PCF}\, A\rangle \subseteq \mathcal{P}(A)$ so that letting $I_{<\lambda} = \langle\{B_\theta : \theta < \lambda \wedge \theta \in \mathrm{PCF}\, A\}\rangle$, the following holds for all $\lambda \in \mathrm{PCF}\, A$:*

- $\mathrm{PCF}\, A$ *has a maximum and* $B_{\max \mathrm{PCF}\, A} =_{I_{<\max \mathrm{PCF}\, A}} A$.
- $\mathfrak{b}(\prod A, <_{I_{<\lambda}}) = \operatorname{tcf}(\prod A, <_{I_{<\lambda}\restriction B_\lambda}) = \lambda$.

- *For every proper ideal I over A*
  (1) *$\mathfrak{b}(I) = \lambda$ for the first λ so that $B_\lambda \notin I$.*
  (2) *$\{\mathfrak{b}(J) : I \subseteq J$ and J is a proper ideal over $A\} = \{\lambda \in \mathrm{PCF}\, A : B_\lambda \notin I\}$.*
  (3) *$\prod A, <_I$ has true cofinality if and only if I omits exactly one B_λ;*

Corollary 23. *For every $T \subseteq \mathrm{PCF}\, A$ the set $\{U : U$ is an ultrafilter over A and $(\prod A/U) \in T\}$ is a closed subset of the space of ultrafilters over A.*

$\mathrm{cf}(\prod A/U) = \lambda$ iff $B_\lambda \in U$. Unique λ such.

Claim 24. *Suppose that μ is singular of cofinality $\kappa > \aleph_0$ and that $C \subseteq \mu$ is closed unbounded with $\mathrm{otp} C = \kappa$. The for some club $C' \subseteq C$ of μ it holds that $\max \mathrm{PCF}\{\lambda^+ : \lambda \in C'\} = \mu^+$.*

Proof. First assume, by thinning C out, that every $\lambda \in C$ is singular and that $\min C > \kappa$.

Show that for every stationary $S \subseteq C$ there is stationary $S' \subseteq S$ so that $\mu^+ \in \mathrm{PCF}\{\lambda_i^+ : i \in S\}$. This shows:

(1) $\mu^+ \in \mathrm{PCF}\{\lambda^+ : \lambda \in C\}$, hence $B_{\mu^+}[C^+]$ exists.
(2) $B_{\mu^+}[C^+]$ contains C'^+ for some club $C' \subseteq C$.

And (2) is certainly enough for the Lemma, as $\mu^+ \le \mathfrak{b}(\prod C'^+, <_{J_{bd}}) \le \max \mathrm{PCF}\, C'^+ \le \max \mathrm{PCF}\, B_{\mu^+} = \mu^+$.

Given a stationary $S \subseteq C$ construct a $<_{J_{bd}}$ increasing $\overline{f} = f_\alpha : \alpha < \mu^+\rangle \subseteq \prod S^+$ with an eub g for which $\liminf_{\mathrm{bd}} \mathrm{cf} g(a) = \mu$. Now argue that except for a non-stationary subset of C, it holds that $\mathrm{cf} g(\lambda) = \lambda^+$; for else by Fodor's Lemma on an unbounded set cofinality is small! $\square$

Theorem 25. *For each $\lambda \in N$, $\{a : \chi_n(a) = f^\lambda_{\chi_n(\lambda)}(a)\} =_{J_{<\lambda}} B_\lambda$.*

Theorem 26. *If $|A| < \theta \le \min A$ is regular, then for every $D \in [\mathrm{PCF}\, A]^\theta$ there is a sequence $\langle B_\lambda : \lambda \in D\rangle$ of PCF generators which satisfies:*

(i) $\mathrm{PCF}\, B_\lambda \cap D = B_\lambda$ *(B_λ is "closed" in D)*

(ii) $\mu \in B_\lambda \Rightarrow B_\mu \subseteq B_\lambda$ *("transitivity")*

for all $\lambda, \mu \in D$.

Proof. First let us see that from any sequence $\langle B_\lambda : \lambda \in D\rangle$ that satisfies the transitivity condition (2) above a sequence $\langle B'_\lambda : \lambda \in D\rangle$ can be gotten that satisfies (1) and (2). Suppose that $\langle B_\lambda : \lambda \in D\rangle$ satisfies (2).

Now let N be a θ-i.a. model.

$\square$

Suppose
$\mathrm{cf}(\prod A, <) = \max pcf$. Even better:
Normal scales. The characteristic function theorem.
$N \cap \mu$ determined by $\chi_N \restriction (|N|, \mu)$.

Theorem 27 (The characteristic function theorem)**.** *Suppose that $A \subseteq Reg$, and $\langle B_\lambda : \lambda \in A\rangle$ is a sequence of PCF generators for A. Fix some system $\langle f^\lambda_\alpha : \alpha < \lambda \in A\rangle$ of normal scales. Suppose $|A| < \kappa < \min A$ and κ regular. Suppose $N = N_\kappa$ is a κ-i.a. model with $\langle f^\lambda_\alpha : \alpha < \lambda \in A\rangle, A \in N_0$, $A \subseteq N_0$. Then:*

(1) *for every $\lambda \in A$ there is some club $E_\lambda \subseteq \kappa$ so that for all $i < j$ in E_k it holds that*

$$\{\theta \in A : \chi_{N_i}(\lambda)(\theta) < f^\lambda_{\chi_{N_j}(\lambda)}(\theta)\} = \{\theta \in A : \theta \leq \lambda \wedge \chi_{N_\kappa}(\theta) = f^\lambda_{\chi_{N_\kappa}}(\lambda)(\theta)\}$$

4. Transitive generators

Theorem 28. *Suppose $A \subseteq Reg$ and $|A|^+ < \min A$. Then there exists a sequence $\langle B_\lambda : \lambda \in A\rangle$ of pcf generators which is:*

(1) $J_{\leq}[A] = J_{<\lambda}[A] + B_\lambda$
(i) *B_λ is closed in A:* $\mathrm{PCF}\, B_\lambda \cap A = B_\lambda$
(ii) *"Smoothness", or "Transitivity": $\theta \in B_\lambda \Rightarrow B_\theta \subseteq B_\lambda$ for all $\lambda \in B$.*

Proof. We first prove the existece of a sequence of generators which satisfies (ii). Then it will be fairly easy to obtain from a sequence which satisfies (ii) a sequence that satisfies (i) and (ii).

Let N be a κ-i.a. model for regular $\kappa \in (|A|, \min A)$ so that $A \in N_0$ and $A \subseteq N_0$. Let $\langle f^\lambda_\alpha : \lambda \in A, \alpha < \lambda\rangle$ be a system of normal scales for all $\lambda \in A$.

For every $\lambda \in A$ there is a closed unbounded $E_\lambda \subseteq \kappa$ with the property that for every $i < j$ in E_λ it holds that

$$\begin{aligned} B^{\lambda,i,j} :=& \{\theta \in A \cap (\lambda+1) : \chi_{N_i}(\theta) < f^\lambda_{\chi_{N_j}}(\theta)\} \\ =& \{\theta \in A : \chi_{N_\kappa}(\theta) = \chi_{N_\kappa}(\theta)\} \\ =&_{J_{<\lambda}} B^*_\lambda \end{aligned}$$

$E = \bigcap_{\lambda \in A} E_\lambda$ is a club of κ. Fix $i < j$ to be the first (or just any) two members of E.

For each $\lambda \in A$ define inductively $B^{\lambda,n}$, $n < \omega$ as follows: $B^{\lambda,0} = B^{\lambda,i,j}$ and $B^{\lambda,n+1} = B^{\lambda,n} \cup \bigcup\{B^{\mu,n} : \mu \in B^{\lambda,n}\}$. Let

$$B_\lambda = \bigcup_n B^{\lambda,n}.$$

Simultaneously, for every $\lambda \in A$ and every $\alpha < \lambda$, define $f^{\lambda,n}_\alpha$ with domain $B^{\lambda,n}$ and so that $f^{\lambda,n}_\alpha \subseteq f^{\lambda,n+1}_\alpha$ for all n, by: $f^{\lambda,0}_\alpha = f^\lambda_\alpha \restriction B^{\lambda,0}$; for $n+1$ and $\theta \in B^{\lambda,n+1} \setminus B^{\lambda,n}$ let $\mu \in B^{\lambda,n}$ be the least for which $\theta \in B^{\lambda,n}$ and put $f^{\lambda,n+1}_\alpha(\theta) = f^{\mu,n}_\beta(\theta)$ where $\beta = f^{\lambda,n}_\alpha(\mu)$. Let

$$f^{\lambda,\omega}_a = \bigcup_n f^{\lambda,n}_\alpha.$$

Clearly, the sequence $\langle B_\lambda : \lambda \in A\rangle$ satisfies (ii) (if m is the least such that $\mu \in B^{\lambda,n}$ then for all $n > m$ it holds that $B^{\mu,n} \subseteq B^{\lambda,n+1}$). Since $B^{\lambda,0} \subseteq B^\lambda$ and $B^{\lambda,0} =_{J_{<\lambda}} B^*_\lambda$ it is clear that $J_{\leq\lambda} \subseteq J_{<\lambda} + B_\lambda$. It remains only to check that B_λ is not "too large", that is, to verify that $B_\lambda \in J_{\leq\lambda}$. For this we claim that:

$$\theta \in B_\lambda \Rightarrow f^{\lambda,\omega}_{\chi_{N_\kappa}(\lambda)}(\theta) = \chi_{N_\kappa}(\theta) \tag{1}$$

Given $\theta \in B_n$ let n be the least for which $\theta \in B^{\lambda,n}$. If $n = 0$ then then (1) holds by the definition of $B^{\lambda,0}$. If $n = m+1$ let $\mu \in B^{\lambda,m}$ be the least for which $\theta \in B^{\mu,m}$. Let $\beta = f^{\lambda,m}_{\chi_{N_\kappa}(\lambda)}(\mu)$. By the induction hypothesis, $f^{\mu,m}_{\chi_{N_\kappa(\mu)}}(\theta) = \chi_{N_\kappa}(\theta)$. By the inductive definition,

$$f^{\lambda,m+1}_{\chi_{N_\kappa}(\lambda)}(\theta) = f^{\mu,m}_\beta(\theta) = \chi_{N_\kappa}(\theta) \tag{2}$$

and (1) is proved.

Now observe that since i, j, A and $\{f^\lambda_\alpha : \lambda \in A, \alpha < \lambda\}$ belong to N_{j+1}, also $\{f^{\lambda,\omega}_\alpha : \lambda \in A, \alpha < \lambda\} \in N_{j+1}$. Thus, $\{f^{\lambda,\omega}_\alpha : \alpha < \lambda\}$ has an upper bound g modulo $J_{\leq\lambda}$ (since $<_{J_{\leq\lambda}}$ is λ^+-directed), with $g \in N_{j+1}$. However, $g < \chi_{N_\kappa}$, thus, by (1),

$$B_\lambda \subseteq \{\theta \in A : f^{\lambda,\omega}_{\chi_{N_\kappa}(\lambda)}(\theta) > g(\theta)\} \in J_{\leq\lambda}$$

and we are done.

Now we need to show that from $\langle B_\lambda : \lambda \in A\rangle$ which satisfies (ii) we can obtain $\langle B'_\lambda : \lambda \in A\rangle$ which satisfies (ii) and (i). Let $B'_\lambda = B_\lambda$ for $\lambda = \min A$ and define B'_λ by induction on $\lambda \in A$. For $\lambda = \min A$ let $B'_\lambda = B_\lambda$. Suppose that B'_θ is defined for all $\theta < \lambda$ in A. Since $\max \mathrm{PCF}\, B_\lambda = \lambda$, it holds that for finitely many $\theta_1, \theta_2, \ldots, \theta_n \in \mathrm{PCF}\, B_\lambda \cap A$ we have $\mathrm{PCF}\, B_\lambda \subseteq B_\lambda \cup \bigcup_{i\leq n} B'_{\theta_i}$. Let $B'_\lambda = B_\lambda \cup \bigcup_{i\leq n} B'_{\theta_i}$. Clearly, $B'_\lambda = J_{<\lambda} B_\lambda$. Also, $\mathrm{PCF}\, B'_\lambda = \mathrm{PCF}\, B_\lambda \cup \mathrm{PCF}(\bigcup_{i\leq n} B'_{\theta_i})$.

To show transitivity, argue by double induction. Suppose that $\mu \in B'_\lambda = B_\lambda \cup \bigcup_{i\leq n} B'_{\theta_i}$. If $\mu \in B'_{\theta_i}$ for some $i \leq n$ then by the induction hypothesis (for θ_i) $B'_\mu \subseteq B'_\theta$. Else, $\mu \in B_\lambda$ and by transitivity of the original sequence, $B_\mu \subseteq B_\lambda$, which implies that $\operatorname{PCF} B_\mu \subseteq \operatorname{PCF} B_\lambda \subseteq B'_\lambda$. Since $B'_\mu = B_\mu \cup \bigcup_{j\leq k} B'_{\sigma_j}$ for $\sigma_j \in \operatorname{PCF} B_\mu \cap A$ for $j \leq k$, it follows that $\sigma_j \in B'_\lambda$ and by the induction hypothesis (for μ) it follows that $B'_{\sigma_i} \subseteq B'_\sigma$ and hence $B'_\mu \subseteq B'_\lambda$. $\square$

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