# SEQUENTIALLY LINEARLY LINDELÖF SPACES

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Abstract. A topological Hausdorff space $X$ is *sequentially linearly Lindelöf* if for every uncountable regular cardinal $\kappa \leq w(X)$ and every $A \subseteq X$ of cardinality $\kappa$ there exists $B \subseteq A$ of cardinality $\kappa$ which converges to a point. We prove that the existence of a *good* $(\mu, \lambda)$*-scale* for a singular cardinal $\mu$ of countable cofinality and a regular $\lambda > \mu$ implies the existence of a sequentially linearly Lindelöf space of cardinality $\lambda$ and weight $\mu$ which is not Lindelöf.

Corollaries of the main result are: (1) it is consistent to have linearly Lindelöf non Lindelöf spaces below the continuum; (2) it is consistent to have a realcompact linearly Lindelöf non Lindelöf space below $2^{\aleph_\omega}$; (3) it is consistent to have a Dowker topology on $\aleph_{\omega+1}$ in which every subset of cardinality $\aleph_n$, $n > 0$, has a converging subset of the same cardinality; (4) the nonexistence of sequentially linearly Lindelöf non-Lindelöf spaces implies the consistency of large cardinals.

## 1. Introduction

It is well-known that *compactness* of a topological space $X$ is equivalent to:

$$CAP_\kappa(X) \text{ for all infinite regular } \kappa \tag{1}$$

where $CAP_\kappa(X)$ is the statement that every subset of $X$ of cardinality $\kappa$ has a point of complete accumulation. Omitting $\kappa = \aleph_0$ from compactness one gets the following weaker property

$$CAP_\kappa(X) \text{ for all regular } \kappa > \aleph_0 \tag{2}$$

known as *linear Lindelöfness*, because it is equivalent to the property that every open cover of $X$ *which is linearly ordered by inclusion* has a countable subcover. The property of being linearly Lindelöf but not Lindelöf will be abbreviated by LLnL and LL will abbreviate linear Lindelöfness. Three LLnL spaces were shown to exist in ZFC and a fourth, realcompact space was constructed from an additional assumption [14, 2, 1, 12].

$CAP_\kappa(X)$ holds trivially for all regular $\kappa > w(X)$, thus (2) is equivalent to

$$CAP_\kappa(X) \text{ for all regular } \aleph_0 < \kappa \leq w(X) \tag{2$'$}$$

If one strengthens compactness by replacing $CAP_\kappa(X)$ with $SCAP_\kappa(X)$, which means: "for every $A \subseteq X$ of cardinality $\kappa$ there exists $B \subseteq A$ of cardinality $\kappa$ that

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*converges* to a point $x \in X$", one gets the following property of "chain compactness":

$$SCAP_\kappa(X) \text{ for all infinite regular } \kappa \tag{3}$$

A Hausdorff space $X$ is chain-compact if and only if it is *scattered compact* by [15]. A set $A \subseteq X$ *converges* to $x \in X$ if $|A \setminus u| < |A|$ for all open $u \ni x$. For a regular $\kappa$, the property $SCAP_\kappa(X)$ is equivalent to "every sequence $f : \kappa \to X$ has a converging cofinal subsequence".

Spaces in which $SCAP_\kappa(X)$ holds for all $k \leq \lambda$ are called *initially $\lambda$-chain compact* and were considered in [23, 24].

In the present paper we consider the property which is obtained from compactness by applying *both* modifications above, or, equivalently, which is obtained from initial $w(X)$-chain compactness by omitting $\kappa = \aleph_0$:

$$SCAP_\kappa(X) \text{ for all regular } \aleph_0 < \kappa \leq w(X) \tag{4}$$

Let us call a space $X$ *sequentially*[1] *Linearly Lindelöf* iff it satisfies (4).

**Proposition 1.** (1) *for all regular $\kappa$,*

$$SCAP_\kappa(X) \wedge CAP_\kappa(Y) \Rightarrow CAP_\kappa(X \times Y)$$

(2) *Suppose $X$ is SLL and $Y$ is LL with $w(Y) \leq w(X)$. Then $X \times Y$ is LL.*

*Proof.* Let $\kappa$ be a regular cardinal and assume $SCAP_\kappa(X)$, $CAP_\kappa(Y)$. Let $A \subseteq X \times Y$ be given with $|A| = \kappa$. Since $\kappa$ is regular, there exists $B \subseteq A$ with $|B| = \kappa$ so that $B$ is a constant function from $X$ to $Y$, or $B^{-1} := \{(y, x) : (x, y) \in B\}$ is a constant function from $Y$ to $X$ or $B$ is a 1-1 function from $X$ to $Y$. In either of the first two cases $B$ has trivially a point of complete accumulation in $X \times Y$. In the third case assume, by thinning $B$ out and using $SCAP_\kappa(X)$, that $\Pr_X[B] := \{x : (\exists y)[(x, y) \in B]\}$ converges to $x_0 \in X$, namely, $|\Pr_X[B] \setminus u| < \kappa$ for all open $u \ni x_0$. By $CAP_\kappa(Y)$, fix $y_0 \in Y$, a point of complete accumulation of $Pr_Y[B]$. Now $(x_0, y_0)$ is clearly a point of complete accumulation of $B$, hence of $A$. This proves (1).

(2) follows immediately from (1). ☐

The Sorgenfrey line $K$ is Lindelöf, hence LL, and the diagonal of $K^2$ is a closed discrete uncountable subset of $K^2$. Hence $CAP_{\aleph_1}(K^2)$ fails and $K^2$ is not LL. From (2) above, the Sorgenfrey line is not SLL.

We shall need the following very simple

**Fact 2.** *Suppose $\kappa_0, \kappa_1, \ldots, \kappa_m$ are cardinals, $\kappa$ is a regular cardinal and $\bar{t} = \langle t_\alpha : \alpha < \kappa \rangle$ is a sequence in $\prod_{n \leq m}(\kappa_n + 1)$. Then there exists $t \in \prod_{n \leq m}(\kappa_n + 1)$ and a (cofinal) subsequence of $\bar{t}$ that converges to $t$.*

*Proof.* Successively thin out $\bar{t}$ so that for each $n \leq m$ the sequence $\langle t_\alpha(n) : \alpha < \kappa \rangle$ is either constant or strictly increasing of order-type $\kappa$. ☐

[1] It would have been more appropriate to use the name *chain linearly Lindelöf*, but since linearly Lindelöf spaces are sometimes called "chain Lindelöf" in the literature, this would be confusing.

1.1. **The results.** The property of being SLL but not Lidelöf is abbreviated by SLLnL. We shall construct SLLnL spaces from PCF-theory principles called "good scales". A slight variation in the construction provides a Dowker space $X$ of cardinality $\aleph_{\omega+1}$ which satisfies $SCAP_\kappa(X)$ for all $\aleph_0 < \kappa < \aleph_\omega$.

The first corollary of the construction is the consistent existence of SLLnL spaces below the continuum, which in particular settles the question of whether LLnL spaces can exist below the continuum. If one assumes the consistency of large cardinals, then the consistency of infinitely many different SLLnL spaces below the continuum follows. Finally, SLLnL spaces serve in proving the consistency of a realcompact LLnL space below $2^{\aleph_\omega}$. A realcompact LLnL topology on $2^{\aleph_\omega}$ itself is known to follow from the assumption $2^{\aleph_0} = 2^{\aleph_\omega}$ [1].

At the moment we do not know if SLLnL spaces exists just in ZFC. But we have meta-mathematical consequences of the *nonexistence* of SLLnL spaces. If there is no SLLnL space of cardinality $\aleph_{\omega+1}$ then there exists a *strong cardinal* in an inner model; if there is no SLLnL space at all, then for every $n > 0$ there is an inner model with $n$ *Woodin cardinals*. These results indicate that the consistency of not having SLLnL spaces would necessarily require stronger assumptions than the consistency of ZFC. At the moment it is not known whether it is consistent (from any assumptions) that no SLLnL topology exists on $\aleph_{\omega+1}$.

## 2. SLLnL spaces $X$ from good pcf-scales

Let $\langle \kappa_n : n < \omega \rangle$ be an increasing sequence of uncountable regular cardinals. Their limit, $\mu$, is a singular cardinal of countable cofinality. Let $\prod_n \kappa_n = \{f \mid f : \omega \to \mu \text{ and } (\forall n)(f(n) < \kappa_n)\}$. For $f, g \in \prod_n \kappa_n$ and $m \in \omega$ write $f <_m g$ iff $f(n) < g(n)$ for all $n > m$ and write $f =_m g$ if $f(n) = g(n)$ for all $n > m$. Let $f <^* g$ iff $f <_m g$ for some $m$; let $f =^* g$ iff $f =_m g$ for some $m$. The structure $(\prod_n \kappa_n, <^*)$ is a quasi-ordered set. A sequence $\overline{f} = \langle f_\alpha : \alpha < \lambda \rangle \subseteq \prod_n \kappa_n$ is called $<^*$-*increasing* if for all $\alpha < \beta < \lambda$ it holds that $f_\alpha <^* f_\beta$ and a sequence $\overline{f} = \langle f_\alpha : \alpha < \lambda \rangle \subseteq \prod_n \kappa_n$ is called $<^*$-*cofinal in* $\prod_n \kappa_n$ if for all $f \in \prod_n \kappa_n$ there exists $\alpha < \lambda$ so that $f <^* f_\alpha$.

Let $\mu$ be singular of countable cofinality and let $\lambda > \mu$ be regular. A $(\mu, \lambda)$-*scale* is a pair $(\overline{\kappa}, \overline{f})$ where $\overline{\kappa} = \langle \kappa_n : n < \omega \rangle$ is a strictly increasing sequence of uncountable regular cardinals with $\sup\{\kappa_n : n < \omega\} = \mu$ and $\overline{f} = \langle f_\alpha : \alpha < \lambda \rangle \subseteq \prod_n \kappa_n$ is $<^*$-increasing and $<^*$-cofinal in $\prod_n \kappa_n$.

A function $g : \omega \to \mathrm{On}$ is an *exact upper bound* (eub) of a $<^*$-increasing sequence $\langle f_\alpha : \alpha < \theta \rangle \subseteq \mathrm{On}^\omega$, where $\theta$ is a limit ordinal, if $f_\alpha <^* g$ for all $\alpha < \theta$ and for all $g' <^* g$ there is some $\alpha < \theta$ so that $g' <^* f_\alpha$. For example, if $(\overline{\kappa}, \overline{f})$ is a $(\kappa, \lambda)$-scale then the function $g$ with $g(n) = \kappa_n$ is an exact upper bound of $\overline{f}$. If $g_1$ and $g_2$ are exact upper bounds of a $<^*$-increasing $\overline{f}$ then $g_1 =^* g_2$.

We write $f < g$ for $f, g : \omega \to \mathrm{On}$ if $f(n) < g(n)$ for all $n$. A sequence $\langle h_\alpha : \alpha < \delta \rangle$ is $<$-increasing if $\alpha < \beta < \delta \Rightarrow h_\alpha < h_\beta$.

**Lemma 3.** *Suppose $\langle f_\alpha : \alpha < \delta \rangle \subseteq On^\omega$ is a $<^*$-increasing sequence, $\delta$ is a limit ordinal and $\mathrm{cf}\delta = \theta > \aleph_0$. Then the following conditions are equivalent:*

(1) *there exists an eub $g$ of $\overline{f}$ so that $\mathrm{cf} g(n) = \theta$ for all $n$;*
(2) *there exists a $<$-increasing sequence $\overline{h} = \langle h_i : i < \theta \rangle \subseteq On^\omega$ so that for all $i < \theta$ there is $\alpha < \delta$ with $h_i <^* f_\alpha$ and for all $\alpha < \delta$ there is $i < \theta$ with $f_\alpha <^* h_i$;*

(3) *For every unbounded set* $C \subseteq \alpha$ *there is some* $m_0 \in \omega$ *and an unbounded set* $A \subseteq C$ *with* $\mathrm{otp} A = \theta$ *so that* $f_\alpha(n) <_{m_0} f_\beta(n)$ *for all* $\alpha < \beta$ *in* $A$.

*Proof.* (1)⇒(2): Suppose $\overline{f}$ and $\theta$ are as above and $g$ is an eub of $\overline{f}$ with $\mathrm{cf} g(n) = \theta$ for all $n$. For each $n$ fix a sequence $\langle \beta_i^n : i < \theta \rangle$, strictly increasing with supremum $g(n)$. For $i < \theta$ define $h_i : \omega \to \mathrm{On}$ by $h_i(n) = \beta_i^n$. Thus $i < j < \theta$ implies $h_i < h_j$, and $\sup\{h_i : i < \theta\} = g$.

For every $f$ that satisfies $f <^* g$ the set $\{n \in \omega : f(n) < g(n)\}$ is co-finite in $\omega$, and for every $n$ in this set there is some $i < \theta$ so that $f(n) < \beta_i$. Since $\theta$ is regular uncountable and $\overline{h}$ increases in $<$, there is some fixed $i < \theta$ for which $f_\alpha <^* h_i$. Conversely, since $g$ is an eub of $\overline{f}$ and $h_i < g$ for $i < \theta$, there is some $\alpha < \delta$ for which $h_i <^* f_\alpha$. This proves (2).

(2)⇒(3): Suppose $\overline{h}$ is given as in (2) and that $C \subseteq \alpha$ is unbounded. For each $i < \theta$ let $\alpha(i) \in C$ be chosen such that $h_i <^* f_{\alpha(i)}$ and so that $i < j \Rightarrow \alpha(i) < \alpha(j)$. Let $A = \{\alpha(i) : i < \theta\}$. So $\mathrm{otp} A = \theta$ and also $A$ is unbounded in $\alpha$. By thinning out $\overline{h}$ we may assume that $f_{\alpha(i)} <^* h_{i+1}$. Let $m(i)$ be so that $h_i <_{m(i)} f_{\alpha(i)} <_{m(i)} h_{i+1}$. Since $\theta > \aleph_0$ is regular, we may assume, by shrinking $A$ and re-enumerating it increasingly, that there is some fixed $m_0 \in \omega$ for which $m(i) = m_0$ for all $i < \theta$. Suppose that $\alpha(i) < \alpha(j)$ are in $A$. Then

$$f_{\alpha(i)} <_{m_0} h_{i+1} \leq h_j <_{m_0} f_{\alpha(j)}$$

as required.

(3)⇒(1): Suppose $A \subseteq \delta$ is unbounded of order type $\theta$ and $m_0 \in \omega$ satisfies $\alpha < \beta$ in $A$ implies that $f_\alpha <_{m_0} f_\beta$. Let $g(n) = \sup\{f_\alpha(n) : \alpha \in A\}$. Now $g$ is clearly an eub of $\overline{f}$ and $\mathrm{cf} g(n) = \theta$ for all $n > m_0$. To obtain this for all $n$, the values $g(0), \ldots, g(m_0)$ can be re-defined as $\theta$. □

A $<^*$-increasing $\overline{f} = \langle f_\alpha : \alpha < \delta \rangle$ with $\mathrm{cf}\delta = \theta > \aleph_0$ that satisfies one (equivalently, all) of the conditions in Lemma 3 is called *flat*. Suppose $(\overline{k}, \overline{f})$ is a $(\mu, \lambda)$-scale for a singular $\mu$ of countable cofinality and a regular $\lambda > \mu$. An ordinal $\alpha < \lambda$ is called a "flat point in $\overline{f}$" if $\mathrm{cf}\alpha > \aleph_0$ and $\overline{f} \restriction \alpha$ is flat. A $(\mu, \lambda)$-scale is called *good* if for all $\alpha < \lambda$ with $\mu > \mathrm{cf}\alpha > \aleph_0$, $\alpha$ is a flat point in $\overline{f}$ and $f_\alpha$ is an eub of $\overline{f} \restriction \alpha$.

**Theorem 4.** *Suppose* $\mu$ *is singular of cofinality* $\omega$ *and* $\lambda > \mu$ *is regular. If there exists a good* $(\mu, \lambda)$*-scale there exists a Tychonov, sequentially linearly Lindelöf non-Lindelöf space of cardinality* $\lambda$ *and weight* $\mu$.

*Proof.* Suppose $\overline{f} = \langle f_\alpha : \alpha < \lambda \rangle \subseteq \prod_n \kappa_n$ is a good $(\mu, \lambda)$-scale. Define

$$X = \left\{ f \in \prod_n (\kappa_n + 1) : (\exists \alpha < \lambda)(f =^* f_\alpha) \right\} \tag{5}$$

The topology on $X$ is the topology inherited from the usual product topology on $\prod_n (\kappa_n + 1)$.

The cardinality of $X$ is clearly $\lambda$ and the weight of $X$ is $\mu$. The space $X$ is also clearly Tychonov.

**Lemma 5.** $X$ *is not Lindelöf.*

*Proof.* Let $\mathcal{U} = \{u_{n,\alpha} : \alpha < \kappa_n\}$ where $u_{n,\alpha} := \{f \in X : f(n) < \alpha\}$. Clearly, $|\mathcal{U}| = \mu$. For every $f \in X$ there exists $f_\alpha \in \overline{f}$ so that $f =^* f_\alpha$, thus for some $n$ it holds that $f(n) = f_\alpha(n) < \kappa_n$. Thus $f \in u_{n,f(n)+1}$. This shows that $\mathcal{U}$ is a cover of $X$.

To see that $\mathcal{U}$ has no smaller subcover, fix $\mathcal{V} \subseteq \mathcal{U}$ with $|\mathcal{V}| = \theta$ for some $\theta < \mu$. Let a function $g \in \prod_n \kappa_n$ be defined as follows:

$$g(n) = \begin{cases} 0 & \text{if } \kappa_n \leq \theta \\ \sup\{\alpha : u_{n,\alpha} \in \mathcal{V}\} & \text{if } \kappa_n > \theta \end{cases}$$

Since $\sup_n \kappa_n = \mu$ and $\theta < \mu$, there is some $m$ so that for all $n > m$ it holds that $\theta < \kappa_n$. For $n > m$, since $\kappa_n$ is regular and $|\mathcal{V}| = \theta < \kappa_n$, it holds that $g(n) < \kappa_n$. Since $\overline{f}$ is a scale, there exists some $\alpha < \lambda$ and $m_0$ so that $g(n) < f_\alpha(n)$ for all $n > m_0$. Let $f \in X$ be defined by

$$f(n) = \begin{cases} \kappa_n & \text{if } n \leq \max\{k, m_0\} \\ f_\alpha(n) & \text{if } n > k \end{cases}$$

The function $f$ is indeed an element of $X$ since $f \in \prod_n(\kappa_n + 1)$ and $f =^* f_\alpha$. Now one can check that $f \notin \bigcup \mathcal{V}$: Suppose $u_{n,\alpha} \in \mathcal{V}$. If $n \leq \max\{k, m_0\}$ then $f(n) = \kappa_n > \alpha$; and if $n > \max\{k, m_0\}$ then $f(n) = f_\alpha(n) > g(n) = \sup\{\beta : u_{n,\beta} \in \mathcal{V}\} \geq \alpha$. So $f \notin u_{\alpha,n}$. This shows that $\mathcal{V}$ is not a cover, and therefore that $X$ is not Lindelöf. □

**Lemma 6.** *$X$ is sequentially linearly lindelöf. In fact, for every regular $\aleph_0 < \kappa < \mu$ and $A \subseteq X$ with $|A| = \kappa$ there exists $B \subseteq A$ with $|B| = \kappa$ which converges in the* box topology *on $X$.*

*Proof.* Let $A \in [X]^\kappa$ be given for some regular uncountable $\kappa < \mu$. For every $g \in A$ fix $\alpha(g) < \lambda$ and $m(g) < \omega$ so that $g =_{m(g)} f_{\alpha(f)}$. By shrinking $A$ we may assume that $m(g)$ is some fixed $m_0$ for all $g \in A$, and that $g \mapsto \alpha(g)$ is either constant or 1-1.

In the first case there is some fixed $\alpha_0 < \lambda$ so that $g =_{m_0} f_{\alpha_0}$ for all $g \in A$. Thus, $|\{g \restriction \{0, 1, \ldots, m_0\} : g \in A\}| = \kappa$ and by Fact 2 there exists some $t \in \prod_{i \leq m_0}(\kappa_n + 1)$ and $B \in [A]^\kappa$ so that $\{g \restriction \{0, 1, \ldots, m_0\} : g \in B\}$ converges to $t$. Now define

$$f(n) = \begin{cases} t(n) & \text{if } 1 < n \leq m_0 \\ f_{\alpha_0}(n) & \text{if } n > m_0 \end{cases}$$

The function $f$ satisfies $f =_{m_0} f_{\alpha_0}$, hence $f \in X$. Also, $B$ converges to $f$ in the box topology on $X$.

In the second case $g \mapsto \alpha(g)$ is 1-1, and, by shrinking $A$ further, it may be assumed that the set $\{\alpha(g) : g \in A\}$ has order-type $\kappa$. Let $\langle \alpha_i : i < \kappa \rangle$ be the increasing enumeration of $\{\alpha(g) : g \in A\}$ and let us denote by $g_i$ the unique $g \in A$ for which $\alpha(g) = \alpha_i$.

Letting $\delta = \sup\{\alpha_i : i < \kappa\}$ we have that $\mathrm{cf}\delta = \kappa$ and $\delta < \mu$. Since $(\overline{\kappa}, \overline{f})$ is good, $\delta$ is a flat point in $\overline{f}$. Therefore, $f_\delta$ is an exact upper bound of $\overline{f} \restriction \delta$ and for all sufficiently large $n$ it holds that $\mathrm{cf} f_\delta(n) = \kappa$. Also, since $\{\alpha_i : i < \kappa\}$ is unbounded in $\delta$, clause (3) in Lemma 3 implies that by shrinking $A$ further, we can find some $m_1$ so that for all $n > m_1$ the sequence $f_{\alpha_i}(n)$ is strictly increasing with limit $f_\delta(n)$. Let $m = \max\{m_0, m_1\}$.

Consider the sequence $\langle g_i \restriction \{0,1,\dots,m\} : i < \kappa\}$. Further shrinking gives, by Fact 2, that this sequence converges to some $t \in \prod_{n\le m}(\kappa_n+1)$. Let $f \in \prod_n \kappa_n$ be defined by:

$$f(n) = \begin{cases} t(n) & \text{if } n \le m \\ f_\delta(n) & \text{if } n > m \end{cases}$$

Now$\{g_i : i < \kappa\}$ converges to $f$ in the box topology on $X$, and $f =_m f_\delta$. Thus $f \in X$ and the proof is complete. □

□

2.1. **A Dowker space $X$ with $SCAP_\kappa(X)$ for all $\aleph_0 < \kappa < w(X)$.** In [7] it was shows that the M. E. Rudin Dowker space $X^R$ [16] contains a closed and cofinal Dowker subspace $X^D$ of cardinality $\aleph_{\omega+1}$ which is defined inside $X^R$ by means of an $(\aleph_\omega, \aleph_{\omega+1})$-scale $(\overline{\kappa}, \overline{f}) = \langle f_\alpha : \alpha < \aleph_{\omega+1}\rangle$. Recall that

$$X^R = \left\{ g \in \prod_n (\kappa_n+1) : (\exists m)(\forall n)[\aleph_0 < \mathrm{cf} g(n) < \aleph_m] \right\} \tag{6}$$

and is Dowker under the box topology. The space $X^D$ is defined by:

$$X^D = \left\{ g \in X^R : (\exists \alpha < \aleph_{\omega+1})(g =^* f_\alpha) \right\} \tag{7}$$

If one uses a *good* $(\aleph_\omega, \aleph_{\omega+1})$-scale in this construction, than the resulting space satisfies additional properties:

**Theorem 7.** *Suppose that the $(\overline{\kappa}, \overline{f})$ is a good $(\aleph_\omega, \aleph_{\omega+1})$-scale. Then $X^D$ in (7) is Dowker and satisfies $SCAP_\kappa(X)$ for all $\aleph_0 < \kappa < \aleph_\omega$.*

*Proof.* The space $X^D$ is Dowker by [7]. Let $X$ be the space defined in (5) from $(\overline{\kappa}, \overline{f})$ and observe that $X^D = X^R \cap X$. Therefore, By Lemma 6, for every $A \subseteq X^D$ with $|A| = \kappa$, $\aleph_0 < \kappa < \aleph_\omega$, there exists $B \subseteq A$ of cardinality $\kappa$ which converges to some $g \in X$ in the box topology. The function $g \in X$ is found in the proof of Lemma 6 so that $g = t \cup g\restriction(m_0, \omega)$, where $t \in \prod_{n \le m_0}(\kappa_n+1)$ is chosen by applying Fact 2. By the proof of Fact 2, since $\mathrm{cf} g(n) > \aleph_0$ for all $g \in A \subseteq X^D$, $\mathrm{cf} t(n) > \aleph_0$ for all $n \le m_0$. The function $g\restriction(m_0,\omega)$ is either equal to $f\restriction(m_0,\omega)$ for some $f \in A \subseteq X^D$ or else has values of cofinality $\kappa$ only — according to whether $g$ is constructed in the first or the second case in the proof. Therefore there exists some $m$ so that $\aleph_0 < \mathrm{cf} g(n) < \aleph_m$ for all $n < \omega$, hence $g \in X^R$. Since $g \in X$, it follows that $g \in X^D$. □

## 3. The consistency of SLLnL spaces

Suppose that $\mu$ is a singular cardinal of countable cofinality. A famous theorem of PCF-theory is the existence of a $(\mu, \mu^+)$-scale. The existence of a *good* $(\mu, \mu^+)$-scale, however, is not a theorem of ZFC — it is consistent, from a supercompact cardinal, that there is no good $(\aleph_\omega, \aleph_{\omega+1})$-scale [22, 5] (for a proof from a larger large cardinal the model in [13] suffices, since Chang's conjecture for $(\aleph_{\omega+1}, \aleph_\omega)$ easily contradicts a good $(\aleph_\omega, \aleph_{\omega+1})$-scale).

**Definition 8.** *Suppose that $\mu$ is a singular cardinal of countable cofinality. Let $\mathrm{pp}_{J_{bd}}(\mu)$ denote the set of all regular $\lambda > \mu$ for which there exists a $(\mu, \lambda)$-scale*[2]*.*

The following summarizes the relevant facts about $\mathrm{pp}_{J_{bd}} \mu$:

**Proposition 9.** *For every singular $\mu$ of countable cofinality:*

1. $\mathrm{pp}_{J_{bd}} \mu$ *is an interval of regular cardinals which contains* $\mu^+$ [19, 3, 9].
2. *For every element $\lambda \in \mathrm{pp}_{J_{bd}} \mu$ except, maybe, the largest element, there exists a good $(\mu, \lambda)$-scale* [19].
3. *For every $\alpha < \omega_1$ it is consistent, relative to large cardinal axioms, that $\mathrm{pp}_{J_{bd}} \aleph_\omega$ contains all regular cardinals $\lambda \in [\aleph_{\omega+1}, \aleph_{\omega+\alpha})$* [21].
4. *it is consistent (if a supercompact cardinal is consistent) that there is no good $(\aleph_\omega, \aleph_{\omega+1})$-scale* [22, 4].
5. *$\square_\mu$, and even the weaker principle $\mu^+ \in I[\mu^+]$, implies the existence of a good $(\mu, \mu^+)$ scale* [19, 4].

By (1),(2) and (5) above, a necessary condition for the nonexistence of a good $(\mu, \mu^+)$-scale is that there is no $(\mu, \mu^{++})$-scale and that $\square_\mu$ fails.

**Theorem 10.**
1. *(Jensen) If $\square_\mu$ fails for some singular $\mu$ with countable cofinality, then there is an inner model with a strong cardinal* [10, 11].
2. *(Schimmerling, Steel, Zeeman) If $\square_\mu$ fails for a singular $\mu$ so that $(\forall \kappa < \mu)(\kappa^{\aleph_0} < \mu)$, then for each $n > 0$ there exists an inner model with $n$ Woodin cardinals (Corollary 5 in* [18]*; see also the references therein).*

It follows that the statement "there is no good $(\aleph_\omega, \aleph_{\omega+1})$-scale" has consistency strength of at least a strong cardinal, and that the statement "there are no good $(\mu, \mu^+)$-scales for any singular $\mu$ of cofinality $\aleph_0$" has consistency strength of at least $n$ Woodin cardinals for each $n$ (consider the first $\mu$ of cofinality $\omega$ which satisfies $(\forall \kappa < \mu)(\kappa^{\aleph_0} < \mu)$).

By Theorem 4 we have:

**Theorem 11.** *If there is no SLLnL space then for all $n$ there is an inner model with $n$ Woodin cardinals. If there is no SLLnL space of cardinality $\aleph_{\omega+1}$ then there is an inner model with a strong cardinal.*

**Theorem 12.** *For every ordinal $\alpha < \omega_1$ it is consistent, relative to large cardinal axioms, that:*

1. *For every $\beta < \alpha$ there is SLLnL topology on $\aleph_{\omega+\beta+1}$.*
2. *For every $\beta < \alpha$ there is SLLnL topology on $\aleph_{\omega+\beta+1}$ and $\aleph_{\omega+\alpha} < 2^{\aleph_0}$.*

*Proof.* Start with a model in which $\max \mathrm{pp}_{J_{bd}} \aleph_\omega > \aleph_{\omega+\alpha}$ [21]. By Proposition 9 there is a good $\aleph_{\omega+\beta+1}$-scale for each $\beta < \alpha$. By Theorem 4 there exists a SLLnL space of cardinality $\aleph_{\omega+\beta+1}$. This proves (1)

To prove (2) add $\aleph_{\omega+\alpha+1}$ Cohen reals to obtain $\aleph_{\omega+\alpha} < 2^{\aleph_0}$. Since CCC forcing does not change $\mathrm{pp}_{J_{bd}} \aleph_\omega$, the spaces constructed above exist also in the forcing extension. □

[2] The notation $\mathrm{pp}_{J_{bd}} \mu$ is not self-explanatory in this context, but it is a standard PCF-theory notation that we do not wish to change.

## 4. Realcompact LLnL spaces below $2^{\aleph_\omega}$

Every Lindelöf space is realcompact ([8], Theorem 3.11.12), so one may ask if the requirement of realcompactness makes a LL space Lindelöf. In [1] a realcompact LLnL space was constructed on $2^{\aleph_\omega}$ from the assumption $2^{\aleph_\omega} = 2^{\aleph_0}$. In this section we use SLLnL spaces to construct a realcompact LLnL space below $2^{\aleph_\omega}$.

Recall that a *Bernstein set* is a subset of $\mathbb{R}$ which meets every closed uncountable subset of $\mathbb{R}$. It is well known or else it is an easy exercise in diagonalization of length $2^{\aleph_0}$ that:

**Fact 13.** *$\mathbb{R}$ is the union of $2^{\aleph_0}$ pairwise disjoint Bernstein sets.*

**Theorem 14.** *If there exists a SLLnL topology on $2^{\aleph_0}$ there exists a realcompact LLnL topology which extends the usual topology on $\mathbb{R}$.*

*Proof.* Suppose that $X$ is SLLnL of cardinality $2^{\aleph_0}$. Fix, by Fact 13, a partition $\mathbb{R} = \bigcup_{x\in X} S(x)$ of $\mathbb{R}$ in which each $S(x)$ is a Bernstein set. Let $F : \mathbb{R} \to X$ be defined by $F(r) = x \iff r \in S(x)$. Now let $H \subseteq \mathbb{R} \times X$ be the graph of $F$ with the induced topology from the product topology on $\mathbb{R} \times X$:

$$H = \{\langle r, F(r)\rangle : r \in \mathbb{R}\}$$

Clearly, $\mathrm{Pr}_R \restriction H$ is a continuous 1-1 function from $H$ onto $\mathbb{R}$ and $\mathrm{Pr}_X \restriction X$ is a continuous surjection. Since there exists a continuous 1-1 function from $H$ onto $\mathbb{R}$, $H$ is hereditarily realcompact (by, e.g., [8] Theorem 3.11.14). Since $X$ is a continuous non-Lindelöf image of $H$, $H$ is not Lindelöf. The topology on $H$ can be regarded as an extension of the usual topology on $\mathbb{R}$ via $\mathrm{Pr}_R \restriction H$.

Let us see that $H$ is linearly Lindelöf. Let $\kappa > \aleph_0$ be regular and assume that $C \subseteq H$ and $|C| = \kappa$. $C = F\restriction A$ for some $A \subseteq \mathbb{R}$ with $|A| = \kappa$. Let a point $x_0 \in X$ be chosen as follows: By shrinking $A$ we may assume that $F\restriction A$ is either 1-1 or constant. If $F\restriction A$ is 1-1 we may assume by shrinking $A$ further that $\{F(r) : r \in A\}$ converges to some $x \in X$ and we let $x_0 = x$. In the other case let $x_0$ be the constant value of $F\restriction A$. Let $D \subseteq R$ be the set of all complete accumulation points of $A$. Since $|A| = \kappa$ and $\kappa > \aleph_0$ is regular, $D$ is an uncountable closed subset of $\mathbb{R}$. Therefore, since $S(x_0)$ is a Bernstein set, there exists some $r_0 \in D \cap S(x_0)$. Now $(r_0, x_0) \in H$ and is a point of complete accumulation of $C$. □

**Corollary 15.** *It is consistent that $2^{\aleph_\omega} > \aleph_{\omega+1}$ and that there exists a realcompact LLnL space of cardinality $\aleph_{\omega+1}$.*

*Proof.* Start with a model of $V = L$ and add $\lambda$ Cohen subsets to $\omega_1$ for some $\lambda > \aleph_{\omega+1}$. Since this forcing is $\omega_1$-complete, no new countable subsets are added, and the good $(\aleph_\omega, \aleph_{\omega+1})$-scale from $L$ is preserved. Then add $\aleph_{\omega+1}$ Cohen subsets to $\omega_0$. In the resulting model $2^{\aleph_0} = \aleph_{\omega+1} < 2^{\aleph_1} = \lambda \le 2^{\aleph_\omega}$, and a good $(\aleph_\omega, \aleph_{\omega+1})$-scale exists. By Theorem 4 there exists a SLLnL topology on $\aleph_{\omega+1}$ and by Theorem 14 there exists a realcompact LLnL topology on $\aleph_{\omega+1}$ in this model. □

Similarly one can get the consistency of a realcompact LLnL topology on $\aleph_{\omega+\alpha+1} = 2^{\aleph_0} < 2^{\aleph_\omega}$ for an arbitrary $0 < \alpha < \omega_1$ — from a large cardinal assumption.

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