A SYMMETRIZED METRIC RAMSEY THEOREM

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Abstract. For every finite metric space A there exists a finite metric space B and a real number $r > 0$ such that for every coloring of B by two colors there exists a monochromatic $A' \subseteq B$ such that every isometry between two subsets of A' extends to a full autoisometry of B and A' is either isometric to A or is r-homothetic to A.

In analogy to Hrushovski's theorem for graphs [2], Solecki [3] proved, using Herwig's and Lascar's theorem [1] the following:

Theorem (Solecki 2005)**.** *For every finite metric space A there is a finite metric space B containing A as a subspace so that every isometry between subspaces of A extends to an autoisometry of B. Moreover, B can be found so that the distances between points in B belong to the additive semigroup generated by the distances between points in A.*

A metric space B is *homogeneous* if for every $x, y \in B$ there exists an autoisometry g of B such that $g(x) = y$.

Lemma 1. *Solecki's theorem above holds with the additional condition that B is homogeneous.*

Proof. Given a finite metric space A, fix, by Solecki's theorem, a finite metric space B such that A is a subspace of B and every isometry between subspaces of A extends to an autoisometry of B. Let U be the set of all isometries between subspaces of A and for every $f \in U$ fix $f^* \supseteq f$, an autoisometry of B which extends f. Let G be the group of isometries of B which is generated by $\{f^* : f \in U\}$.

Let $C = \{g(x) : g \in G,\ x \in A\}$. Clearly, C is a subspace of B which is invariant under every member of G and $A \subseteq C$. Hence, G acts on C as isometries. Suppose $f \in U$ is an isometry between subspaces of A. Then $f^* \in G$ extends f and is an isometry of C. Thus, also C satisfies the conclusion of Solecki's theorem.

To see that C is homogeneous, let $g_1(x)$, $g_2(y)$ be arbitrary elements of C, where $g_1, g_2 \in G$ and $x, y \in A$. The map $f = \{\langle x, y\rangle\}$ (the map from $\{x\}$ to $\{y\}$) belongs to U, hence $f^* \in G$. Now $(g_2 f^* g_1^{-1})(g_1(x)) = g_2(f^*(g_1^{-1}(g_1(x)))) = g_2(f(x)) = g_2(y)$, which proves that C is homogeneous. □

Let $r > 0$ be a real number. A metric space (B, d_2) is *r-homothetic* to a metric space (A, d_1) if there exists a bijection $f : A \to B$ such that $d_2(f(x), f(y)) = r d_1(x, y)$ for all $x, y \in A$.

Theorem 1. *For every finite metric space A there exists a finite metric space B such that for every coloring of the points of B by two colors there exists a monochromatic subspace $A' \subseteq B$ such that every isometry between subspaces of A' extends to an autoisometry of B and A' is homothetic to A. In fact, there exists a real number $r > 0$ such that for every coloring of B by two colors there exists A' as above which is either monochromatic red and isometric to A or is monochromatic blue and r-homothetic to A.*

Proof. Fix a homogeneous metric space C which contains A as a subspace and satisfies that every isometry between subspaces of A extends to an autoisometry of C. Let $d_0 = \min\{d(x,y) : x, y \in C,\ x \neq y\}$ — the smallest positive distance in A — and let $r > 0$ be chosen so that $rd_0 > \operatorname{diam} C$.

Let $B = C \times C$ and define a metric d^* on B as follows:

$$d^*(\langle x, y\rangle, \langle w, z\rangle) = \begin{cases} rd(x, w) & \text{if } x \neq w \\ d(y, z) & \text{otherwise} \end{cases}$$

The function $d^* : B^2 \to \mathbb{R}$ is clearly symmetric, positive and $d^*(p_1, p_2) = 0 \implies p_1 = p_2$. To verify that d^* satisfies the triangle inequality, let $p_1 = \langle x_1, y_1\rangle$, $p_2 = \langle x_2, y_2\rangle$, $p_3 = \langle x_3, y_3\rangle$ be arbitrary distinct points from B.

If x_1, x_2, x_3 are distinct then $d^*(p_i, p_j) = rd(x_i, x_j)$ for $1 \leq i \leq j \leq 3$, hence $d^*(p_1, p_2) + d^*(p_2, p_3) \geq d^*(p_1, p_2)$. If $x_1 = x_2 = x_3$, then $d^*(p_i, p_j) = d(y_i, y_j)$ and the triangle inequality holds similarly.

In the case $x_1 = x_2 \neq x_3$ it holds that $d^*(p_2, p_3) = d^*(p_1, p_3)$, so clearly $d^*(p_1, p_2) + d^*(p_2, p_3) \geq d^*(p_1, p_3)$. The case $x_1 \neq x_2 = x_3$ is similar. The remaining case is $x_1 = x_3 \neq x_2$. In this case $d(p_1, p_2) = rd(x_1, x_2) \geq rd_0 > \operatorname{diam} C$. As $d^*(p_1, p_3) = d(y_1, y_3) \leq \operatorname{diam} C$, the inequality holds.

Suppose the points of B are colored by two colors. If there exists $x_0 \in C$ such that $\langle x_0, y\rangle$ is red for all $y \in C$, then $C' = \{\langle x_0, y\rangle : y \in C\} \subseteq B$ is an isometric, red copy of C, which contains an isometric copy A' of A, each of whose partial isometries extends to an autoisometry of C'. For every autoisometry of C', its union with the identity function on $B \setminus C'$ extends it to an autoisometry of B, as for all $p \in B \setminus C'$ and $q, r \in C'$ it holds that $d^*(p, q) = d^*(p, r)$. Thus, every isometry between subspaces of A' extends to an autoisometry of B.

If no x_0 as in the previous paragraph exists, then for every $x \in C$ there exists $y(x) \in C$ so that $\langle x, y(x)\rangle$ is blue. The subspace $C' = \{\langle x, y(x)\rangle : x \in C\} \subseteq B$ is r-homothetic to C with the constant r chosen above, so there exists an r-homothetic copy $A' \subseteq C'$ of A such that every isometry between subspaces of A' extends to an autoisometry of C'.

It suffices now to argue that every autoisometry of C' extends to an autoisometry of B. Let f be a given autoisometry of C' and let f' denote the unique autoisometry of C which satisfies $f(\langle x, y(x)\rangle) = \langle f'(x), y(f'(x))\rangle$ for all $x, \in C$.

The assumption that C' is homogeneous implies that for every $x \in C$ an autoisometry g_x of C can be fixed so that $g_x(y(x)) = y(f'(x))$. Let

$f^* : B \to B$ be defined by $f^*(\langle x, y\rangle) = \langle f'(x), g_x(y)\rangle$ for $\langle x, y\rangle \in B$. Clearly, f^* is an autoisometry of B which extends f. □

A simple inductive argument, based on the previous theorem, gives the following:

Theorem 2. *Suppose A is a finite metric space. For every $k \geq 2$ there exists a finite metric space B and real numbers $r_0 > r_1 > \dots, r_{k-1} = 1$ such that for every coloring of B by k colors there exists $A' \subseteq B$ and $i < k$ such that every isometry between two subspaces of A' extends to an autoisometry of B, A' is monochromatic with color i and A' is r_i-homothetic to A.*

References

[1] B. Herwig and D. Lascar. Extending partial automorphisms and the profinite topology on free groups, Trans. AMS 352 (1999) 1985–2021.

[2] Ehud Hrushovski. *Extending partial isomorphisms of graphs*, Combinatorica 12 (4) (1992) 411–416.

[3] S. Solecki. *Extending Partial Isometries*, Israel J. Math. 150 (2005), 315–332.

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