Symmetrized induced Ramsey Theory

Stefan Geschke[1], Menachem Kojman[2]

[1] e-mail: stefan.geschke@hcm.uni-bonn.de
Hausdorff Center for Mathematics
Villa Maria
Endenicher Allee 62
53115 Bonn, Germany

[2] e-mail: kojman@math.bgu.ac.il
Department of Mathematics
Ben Gurion University of the Negev
Beer Sheva, Israel

Abstract. We prove induced Ramsey theorems in which the monochromatic induced subgraph satisfies that all members of a prescribed set of its partial isomorphisms extend to automorphisms of the colored graph (without requirement of preservation of colors).

We consider vertex and edge colorings, and extensions of partial isomorphisms in the set of all partial isomorphisms between singletons as considered by Babai and Sós [3], the set of all finite partial isomorphisms as considered by Hrushovski [13], Herwig [9] and Herwig-Lascar [10], and the set of all total isomorphisms.

We observe that every finite graph embeds into a finite vertex transitive graph by a so called *bi-embedding*, an embedding that is compatible with a monomorphism between the corresponding automorphism groups. We also show that every countable graph bi-embeds into Rado's universal countable graph Γ.

1. Introduction

We prove results which extend two different lines of research in graph theory. The first is the problem of embedding graphs into graphs with prescribed symmetry properties. Babai and Sós [3] have shown how to embed a given finite graph into a finite transitive one and Hrushovski [13] has shown that every finite graph H can be embedded into a finite transitive graph G so that every partial isomorphism of H extends to a total automorphism of G. Hrushovski's theorem was needed in the model-theoretic proof of the *small index property* of the automorphism group of Rado's countable universal graph Γ [12]. Further research in this direction was done by Herwig [9] and Herwig and Lascar [10]. Symmetrized embedding relations between infinite structures were studied on families of sets by Kojman and Shelah in [15], where, among other results, the existence of a bi-universal Borel family of sets over a countable set was proved, and by Chatzidakis and Hrushovski [4] on differential fields.

The second line of research is induced Ramsey theory which investigates the existence of prescribed induced subgraphs that are monochromatic with respect to vertex or edge colorings (see [18] and [7] for the history of the subject).

In the present paper we study the Ramsey-theoretic properties of symmetrized graph extensions. A typical symmetrized induced partition relation with a target graph H and a source graph G asserts that not all induced copies of H in G which satisfy that each member in a prescribed subset of their partial isomorphisms extends to a total automorphim of G can be "killed" by vertex or by edge colorings with a fixed finite number r of colors.

Such results strengthen induced Ramsey theorems, as the collection of symmetrized copies of a graph H in a graph G is strictly narrower than the collection of all induced copies of H in G, and they strengthen the symmetrization results in the line of [3,13] by showing that not only a single symmetrized copy of H in G can be obtained, but many such copies.

We prove symmetrized finite Ramsey theorems for vertex and for edge colorings, and provide upper bounds. We also treat the infinite case and prove that for every vertex coloring of the countable random graph Γ by finitely many colors there is a color which for every countable graph G contains an induced copy of G all of whose total automorphisms extend to automorphisms of Γ. The fact that every countable graph G has at least *one* induced symmetrized copy in Γ resembles Uspenskij's treatment [21,22] of Uryson's universal separable metric space $\mathbb{U}$: Uspenskij proved that every separable metric space X isometrically embeds into $\mathbb{U}$ by an embedding that is compatible with a continuous embedding of the group of auto-isometries of X into the group of auto-isometries of $\mathbb{U}$. It follows that $\mathrm{Aut}(\mathbb{U})$ is a universal topological group of countable weight.

1.1. Symmetrized subgraph relations

By "graph" we always mean a simple graph $G = (V, E) = (V(G), E(G))$ where $V \neq \emptyset$ is the set of vertices and $E \subseteq [V]^2$ is the set of undirected edges.

A *partial isomorphism* of a graph H is an isomorphism $f : A \to B$ between two induced subgraphs of H. Suppose H is an induced subgraph of G. A partial isomorphism f of H is *implemented by* $\mathrm{Aut}(G)$ if there exists $f^* \in \mathrm{Aut}(G)$ such that $f \subseteq f^*$. For a set F of isomorphisms between subgraphs of H we write $H \subseteq^F G$ if H is an induced subgraph of G and every $f \in F$ is implemented by $\mathrm{Aut}(G)$. Let $H \leq^F G$ mean that H is isomorphic to some H' which satisfies $H' \subseteq^{F'} G$, where F' is the set of partial isomorphisms of H' that is induced by F via the isomorphism between H and H'.

Let $P = P(H)$ denote the set of all isomorphisms between *finite* induced subgraphs of H, let $P_1 = P_1(H)$ denote the set of all maps whose domain and range are singleton subsets of $V(H)$ and let $A = A(H)$ abbreviate $\mathrm{Aut}(H)$. Thus, $H \subseteq^P G$ means that every isomorphism between finite induced subgraphs of H extends to a total automorphism of G, $H \subseteq^{P_1} G$ means that every vertex of H can be moved to every other vertex of H by an automorphism of G, and $H \subseteq^A G$ means that every total automorphism of H extends to an automorphism of G. All three relations above are transitive.

More generally, as the sets $P_1(H)$ and $P(H)$ increase with H, if $H_1 \subseteq H_2 \subseteq^F G$ for $F \in \{P_1, P\}$, then also $H_1 \subseteq^F G$. That is, the relations $H \subseteq^{P_1} G$ and $H \subseteq^P G$ are downwards hereditary to induced subgraphs of H. If H_1 is an induced subgraph of H_2, then not every automorphism of H_1 needs to extend to an automorphism of H_2 and not every automorphism of H_2 necessarily restricts to an automorphism of H_1. There is indeed no implication between $H_1 \subseteq^A G$ and $H_2 \subseteq^A G$ when H_1 and H_2 are induced subgraphs of a graph G with $V(H_1) \subseteq V(H_2)$. All three relations $\subseteq^F$ are upwards hereditary on the right hand side via $\subseteq^A$ extensions, that is, $H_1 \subseteq^F H_2 \subseteq^A G$ implies that $H_1 \subseteq^F G$.

The relation $H \subseteq^P G$ always implies $H \subseteq^{P_1} G$, and, if G is finite, then $H \subseteq^P G$ implies $H \subseteq^A G$ since $A(H) \subseteq P(H)$. Neither of the relations $H \subseteq^{P_1} G$ and $H \subseteq^A G$ implies the other or the relation $H \subseteq^P G$.

For infinite H and G it can happen that $H \subseteq^P G$ without $H \subseteq^A G$. Let Γ denote Rado's countable and universal *random graph*. Every isomorphism between two finite induced subgraphs of Γ is implemented by $\mathrm{Aut}(\Gamma)$, that is, $\Gamma \subseteq^P \Gamma$, and hence every induced subgraph H of Γ satisfies $H \subseteq^P \Gamma$. However, not every induced subgraph H of Γ satisfies $H \subseteq^A \Gamma$. (An example: let $H = K \cup \{v\}$, an infinite clique K with one additional vertex v, with the neighborhood of v in K both infinite and co-infinite. For every copy H' of H in Γ the copy of K obtained by omitting the image of v from H' is not a $\subseteq^A$-subgraph of Γ. Also see Remark 2 below.)

As every countable graph embeds as an induced subgraph of Γ, it holds that for every countable graph G,

$$G \leq^P \Gamma. \tag{1}$$

A fundamental theorem of Hrushovski [13] says that to implement all partial isomorphisms of a *finite* graph H some finite G extending H suffices. Recall that a graph G is (*vertex*) *transitive* if for any two vertices $v, u \in V(G)$ there is an automorphism $\sigma \in \mathrm{Aut}(G)$ such that $\sigma(v) = u$. Equivalently, G is transitive iff $G \subseteq^{P_1} G$.

Theorem 1. (Hrushovski [13]) *For every finite graph H there exists a vertex transitive finite graph G such that*

$$H \subseteq^P G. \tag{2}$$

Denoting the number of vertices of H by n, the upper bound on the size of G provided by Hrushovski's proof was $(2n2^n)!$. A much better upper bound of $2^{2n \log n}$ follows from the simpler proof of Hrushovski's theorem due to Herwig and Lascar [10]. (Here and later in the paper log denotes logarithm with base 2). Denoting the maximal degree of a vertex in H by k, the size of G can be bounded by $(nk)^k \leq n^{2n}$. A slighly better bound of $(3en/4)^n$ is also provided in the paper by Herwig and Lascar. An exponential lower bound on the size of G for a general H was provided by Hrushovski.

There is a natural strengthening of the relation $\subseteq^A$ between graphs. In analogy to [15] we call an embedding e of a graph H as an induced subgraph into a graph G a *bi-embedding* if there is a group homomorphism $h : \mathrm{Aut}(H) \to \mathrm{Aut}(G)$ such that for all $\sigma \in \mathrm{Aut}(H)$, $e \circ \sigma = h(\sigma) \circ e$. Clearly, such a homomorphism h is necessarily injective. If H is bi-embeddable into G, we write $H \leq_{\mathrm{bi}} G$. If H is an induced subgraph of G, then H is a bi-embedded subgraph of G ($H \subseteq_{\mathrm{bi}} G$) if the inclusion map from H to G is a bi-embedding. The relations $H \leq_{\mathrm{bi}} G$ and $H \subseteq_{\mathrm{bi}} G$ are transitive.

1.2. Induced partition relations

For graphs H and G the symbol $G \rightarrowtail (H)^d_r$ with natural $r \geq 1$ and $d \in \{1, 2\}$ means that for every coloring of vertices of G, if $d = 1$, or of edges of G, if $d = 2$, by r colors, there exists a monochromatic induced copy of H in G. It is customary to omit $r = 2$ from this notation.

The induced Ramsey theorem [18] states that for every finite graph H, $d \in \{1,2\}$, and finite $r \geq 1$ there exists a finite graph G such that $G \rightarrowtail (H)^d_r$. For $r = 2$ the upper bound on $|G|$ for H with n vertices is $2^{cn(\log n)^2}$ (see [14] and [6]).

We strengthen this relation by requiring that some of the partial isomorphisms of the monochromatic induced copy of H are implemented by $\mathrm{Aut}(G)$. For a set F of partial isomorphisms of H the symbol

$$G \rightarrowtail^F (H)^d_r$$

means the same as above with the additional condition that when identifying H with its monochromatic copy in G, every partial isomorphism in F is implemented by $\mathrm{Aut}(G)$.

$$G \rightarrowtail_{\mathrm{bi}} (H)^d_r$$

means that the embedding of H into G as a monochromatic induced subgraph of G is a bi-embedding and

$$G \rightarrowtail^F_{\mathrm{bi}} (H)^d_r$$

means that the monochromatic copy H' of H in G satisfies both $H' \subseteq_{\mathrm{bi}} G$ and $H' \subseteq^F G$.

The properties of the relations $\subseteq^F$ which were discussed above give that $G \rightarrowtail^{P_1} (H)^d_r$ and $G \rightarrowtail^P (H)^d_r$ are downwards hereditary to induced subraphs of H and that for an arbitrary set F of partial isomorphisms of H the two relations $H \leq^F H'$ and $G \rightarrowtail^A (H')^d_r$ together imply $G \rightarrowtail^F (H)^d_r$. $G \leq^A G'$ and $G \rightarrowtail^F (H)^d_r$ together imply $G' \rightarrowtail^F (H)^d_r$. $G \rightarrowtail_{\mathrm{bi}} (H)^d_r$ implies $G \rightarrowtail^A (H)^d_r$. The relation $G \rightarrowtail_{\mathrm{bi}} (H)^d_r$ is upward hereditary with respect to extensions of G in which G is bi-embedded (*bi-extensions*) and downward hereditary with respect to bi-embedded subgraphs of H.

1.3. The results

We prove finite and infinite *symmetrized* induced Ramsey theorems for each choice of $d \in \{1,2\}$ and $F \in \{P_1, A, P\}$. We prove that for every finite graph H and natural number $r \geq 1$ there exists a finite graph G so that

$$G \rightarrowtail^F (H)^d_r. \tag{3}$$

Upper bounds on the number of vertices in G are given in the number of vertices of H and r in each case for finite H. In the case $d = 1$ and $F = P_1$, for example, the size of G is bounded asymptotically by $|H|^{(2+\varepsilon)r}$ for arbitrarily small $\varepsilon > 0$, whereas with $F = P$ and $d = 1$ the bound jumps up to the vicinity of the bound in the standard induced Ramsey theorem for edge colorings.

In the case of vertex colorings, we actually obtain induced Ramsey theorems for the partition relations of the form $G \rightarrowtail^P_{\mathrm{bi}} (H)^1_r$.

As mentioned above, not every induced subgraph G of the infinite random graph Γ satisfies $G \subseteq^A \Gamma$. However, we prove that for every countable graph G and $r > 0$,

$$\Gamma \rightarrowtail_{\mathrm{bi}} (G)^1_r. \tag{4}$$

In fact, for every coloring of the vertex set of Γ by finitely many colors, a single color contains bi-embedded copies of all countable graphs.

We also observe that the standard proof of the induced bipartite Ramsey theorem due to Nešetřil and Rödl actually gives its symmetrized form.

2. Vertex Colorings

Definition 1. *a) Suppose $G = (V, E)$ is a graph and for every $v \in V$, $H_v = (U_v, E_v)$ is a graph. $\sum_G H_v$ is the graph whose vertex set is $\{(v, u) : v \in V,\ u \in U_v\}$ and in which $\{(v_1, u_1), (v_2, u_2)\}$ is an edge iff $\{v_1, v_2\} \in E$ or ($v_1 = v_2$ and $\{u_1, u_2\} \in E_{v_1}$).*

b) If H_v is a fixed graph H for all $v \in V$, then $\sum_G H_v$ is denoted by $G \otimes H$ and called the wreath product *of G with H.*

c) The r-th wreath power *of a graph G, denoted by $G^{\otimes r}$, is defined inductively by $G^{\otimes 1} = G$ and $G^{\otimes (r+1)} = G \otimes G^{\otimes r}$.*

Observe that $|V(G^{\otimes r})| = |V(G)|^r$.

Lemma 1. *Suppose that G is vertex transitive. Then for all natural $r \geq 1$,*

$$G^{\otimes r} \text{ is vertex transitive and } G^{\otimes r} \rightarrowtail_{\text{bi}} (G)^1_r. \tag{5}$$

Proof. Suppose G is vertex transitive. We prove (5) by induction on $r \geq 1$. The case $r = 1$ is trivial.

Given a vertex coloring of $G^{\otimes (r+1)}$ by $r+1$ colors, assume first that for every $v \in V(G)$ there exists a vertex $u(v) \in V(G^{\otimes r})$ so that $(v, u(v))$ is red. The graph G' spanned in $G^{\otimes (r+1)}$ by $\{(v, u(v)) : v \in V(G)\}$ is isomorphic to G and monochromatic.

We show that

$$e : G \to G^{\otimes r+1}; v \mapsto (v, u(v))$$

is a bi-embedding. Choose a distinguished vertex $u_0 \in V(G^{\otimes r})$. By the induction hypothesis, $G^{\otimes r}$ is vertex transitive and hence for each $v \in G$ there is an automorphism τ_v of $G^{\otimes r}$ that maps $u(v)$ to u_0. Let τ be the automorphism of $G^{\otimes r+1}$ that is defined by letting $\tau(v, u) = (v, \tau_v(u))$ for all $v \in G$ and all $u \in G^{\otimes r}$.

We define a homomorphism $h : \text{Aut}(G) \to \text{Aut}(G^{\otimes r+1})$ as follows: Every $\sigma \in \text{Aut}(G)$ induces an automorphism $\overline{\sigma}$ of $G^{\otimes r+1}$ by the formula

$$\overline{\sigma}(v, u) := (\sigma(v), u).$$

Note that the map $\sigma \mapsto \overline{\sigma}$ is a homomorphism from $\text{Aut}(G)$ into $\text{Aut}(G^{\otimes r+1})$. For each $\sigma \in \text{Aut}(G)$ let

$$h(\sigma) := \tau^{-1} \circ \overline{\sigma} \circ \tau.$$

Then h is is a homomorphism and 1-1.

For each $v \in V(G)$ we have

$$\begin{aligned}(h(\sigma) \circ e)(v)) = (\tau^{-1} \circ \overline{\sigma} \circ \tau)(v, u(v)) &= (\tau^{-1} \circ \overline{\sigma})(v, u_0) \\ &= \tau^{-1}(\sigma(v), u_0) = (\sigma(v), u(\sigma(v)) = (e \circ \sigma)(v).\end{aligned}$$

A similar but easier argument establishes that $G^{\otimes (r+1)}$ is vertex transitive.

If the assumption above fails, then there is some $v \in V(G)$ so that (v, u) is not red for all $u \in V(G^{\otimes r})$. The graph H spanned by $\{(v, u) : u \in V(G^{\otimes r})\}$ is isomorphic to $G^{\otimes r}$ and is colored by the given coloring by at most r colors. By the induction hypothesis, there is a monochromatic $G' \subseteq_{\text{bi}} H$ which is isomorphic to G. Every automorphism of H extends to an automorphism of $G^{\otimes (r+1)}$ by making the extension act as the identity function outside H. Since this way of extending automorphisms gives a homomorphism from $\text{Aut}(H)$ into $\text{Aut}(G^{\otimes r+1})$, we have $H \subseteq_{\text{bi}} G^{\otimes r+1}$. Now transitivity of $\subseteq_{\text{bi}}$ implies $G' \subseteq_{\text{bi}} G^{\otimes r+1}$, finishing the proof of the lemma.

2.1. Finite graphs

Theorem 2. *There is a constant c such that for every graph H with n vertices and all $r \geq 1$ there is a vertex transitive graph G with no more than $c^r n^{2r}$ vertices so that*

$$G \rightarrowtail^{P_1} (H)^1_r.$$

Proof. Given a graph H with n vertices, find by [3] a vertex transitive graph H_1 with at most cn^2 vertices that contains H as an induced subgraph. The graph $G := H_1^{\otimes r}$ is vertex transitive, $|G| \leq c^r n^{2r}$ and $G \rightarrowtail^A (H_1)^1_r$ by Lemma 1. As $H \subseteq^{P_1} H_1$, $G \rightarrowtail^{P_1} (H)^1_r$ follows.

We now turn to symmetrized partition results involving the relations $\subseteq^P$ and $\subseteq_{\mathrm{bi}}$. Using a slight variation of the Herwig-Lascar proof of Hrushovski's theorem, we will see that every finite graph bi-embeds into a finite graph that is vertex transitive. Let us first observe that every graph bi-embeds into a small regular graph.

Lemma 2. *Every graph H of size n bi-embeds into an n-regular graph of size $2n$, i.e., into a graph of size $2n$ in which every vertex has exactly n neighbors.*

Proof. Consider the disjoint union $H_0 \cup H_1$ of two copies of H. For $i \in 2$ let $f_i : H \to H_i$ be an isomorphism. The graph G obtained from $H_0 \cup H_1$ by adding all the edges of the form $\{f_0(v), f_1(w)\}$ with $v, w \in V(H)$ but $\{v, w\} \notin E(H)$ is n-regular and of size $2n$.

Every $\sigma \in \mathrm{Aut}(H)$ induces automorphisms σ_0 and σ_1 of H_0 and H_1, respectively. Now $\sigma_0 \cup \sigma_1$ is an automorphism of G. The map assigning to every $\sigma \in \mathrm{Aut}(H)$ the automorphism $\sigma_0 \cup \sigma_1$ of G is a homomorphism.

Lemma 3. *For every finite graph H of size n there is a vertex transitive graph G of size at most $2^{2n \log n}$ such that both $H \subseteq^P G$ and $H \subseteq_{\mathrm{bi}} G$ hold.*

Proof. Let H be a graph of size n. By the previous lemma, H is a bi-embedded subgraph of an n-regular graph H' of size $2n$. In the case of a regular graph H', the Herwig-Lascar proof [10] of Hrushovski's theorem proceeds as follows:

Let $X = E(H')$. Since H' is n-regular and of size $2n$, $|X| = n^2$. Let G be the graph on the set $V(G) = [X]^n$ of vertices where two distinct sets $A, B \in [X]^n$ are connected by an edge iff $A \cap B \neq \emptyset$. The size of G is bounded by $n^{2n} = 2^{2n \log n}$, which is as promised in the statement of the lemma.

The map

$$f : V(H') \to V(G); v \mapsto \{e \in X : e \text{ is adjacent to } v\}$$

is an embedding of H' into G as an induced subgraph.

Herwig and Lascar now show that f witnesses $H' \leq^P G$, implying that $f \restriction V(H)$ witnesses $H \leq^P G$. We want to show that f is a bi-embedding. Let $\sigma \in \mathrm{Aut}(H')$. Let σ^* be the permutation of X that maps every edge $\{v, w\}$ of H' to $\{\sigma(v), \sigma(w)\}$. The permutation σ^* induces an automorphism $h(\sigma)$ of G by the formula

$$h(\sigma)(\{e_1, \dots, e_m\}) = \{\sigma^*(e_1), \dots \sigma^*(e_m)\}.$$

It is easily checked that $h : \mathrm{Aut}(H') \to \mathrm{Aut}(G)$ is a homomorphism.

For all $\sigma \in \mathrm{Aut}(H')$ and all $v \in V(H')$ we have

$$(h(\sigma) \circ f)(v) = \sigma^*(f(v)) = f(\sigma(v)).$$

It follows that f is indeed a bi-embedding. Now by the transitivity of $\subseteq_{\mathrm{bi}}$, $f \restriction V(H)$ is a bi-embedding, too.

Theorem 3. *For every finite graph H with n vertices and $r \geq 1$ there exists a vertex transitive graph G with no more than $2^{2rn\log n}$ vertices such that*

$$G \rightarrowtail^{P}_{\mathrm{bi}} (H)^1_r.$$

Proof. Given a graph H with n vertices, find by Lemma 3 a vertex transitive graph Z such that $H \subseteq^P Z$, $H \subseteq_{\mathrm{bi}} Z$, and $|V(Z)| \leq 2^{2n\log n}$. The graph $G := Z^{\otimes r}$ is vertex transitive and $G \rightarrowtail_{\mathrm{bi}} (Z)^1_r$ by Lemma 1. As $H \subseteq^P Z$ and $H \subseteq_{\mathrm{bi}} Z$, $G \rightarrowtail^P_{\mathrm{bi}} (H)^1_r$ follows.

Hrushovski [13] showed that an exponential lower bound on $|G|$ in terms of $|H|$ is required for the relation $H \subseteq^P G$, even if one restricts from P to permutations of free subsets of H. Thus, clearly, an exponential lower bound is required in the partition relation just proved.

2.2. The random graph Γ

Lemma 4. *Every countable graph bi-embeds into the random graph Γ.*

Proof. Given a countable graph G, define by induction on n a sequence of countable graphs $G^n = (V^n, E^n)$, starting with $G^0 = G$ and satisfying that for all n, G^n is an induced subgraph of G^{n+1}. Let

$$V^{n+1} := V^n \cup \{v^{n+1}_X : X \subseteq V^n \text{ and } |X| \text{ is finite}\}$$

with pairwise distinct v^{n+1}_X and

$$E^{n+1} := E^n \cup \{(v, v^{n+1}_X) : v^{n+1}_X \in V^{n+1} \setminus V^n \text{ and } v \in X\}.$$

Clearly, $G^\infty := (V^\infty, E^\infty) := (\bigcup_{n\in\mathbb{N}} V^n, \bigcup_{n\in\mathbb{N}} E^n)$ is countable, and $G = G^0$ is an induced subgraph of G^∞.

Let us see that G^∞ is isomorphic to Γ. For any two disjoint finite sets $X, Y \subseteq V^\infty$ there is some n so that $X \cup Y \subseteq V^n$ and now the vertex $v^{n+1}_X \in V^{n+1}$ is connected by edges to all points in X and to no point in Y. This property, together with the countability of G^∞, implies that $G^\infty \cong \Gamma$.

Finally, let $\sigma \in \mathrm{Aut}(G)$ be given. Let $\sigma^0 := \sigma$ and let $\sigma^{n+1} \in \mathrm{Sym}(V^{n+1})$ be the unique permutation of V^{n+1} which extends σ^n and satisfies $\sigma^{n+1}(v_X) = v_{\sigma^n[X]}$. Now $\sigma^\infty = \bigcup_n \sigma^n$ is an automorphism of G^∞ which extends σ.

A straight-forward induction on n shows that the maps

$$h^n : \mathrm{Aut}(G) \to \mathrm{Aut}(G^n); \sigma \mapsto \sigma^n$$

are group homomorphisms. It follows that also

$$h^\infty : \mathrm{Aut}(G) \to \mathrm{Aut}(G^\infty); \sigma \mapsto \sigma^\infty$$

is a homomorphism of groups. Hence $G \subseteq_{\mathrm{bi}} \Gamma$.

This provides an easy direct proof of the fact, due to Truss [20], that the infinite symmetric group $\mathrm{Sym}(\mathbb{N})$, which is the automorphism group of a countably infinite complete graph, embeds into $\mathrm{Aut}(\Gamma)$. Note that every automorphism group of a countable structure in a countable vocabulary can be realized as an automorphism group of a countable graph (see [11, Theorem 5.5.1] and [19, Lemma 4.2.2]).

Remark 1. The embedding of $G = G^0$ as a $\subseteq^A$-subgraph of G^∞ obtained in the proof of Lemma 4 satisfies the additional property that every vertex $v \in V^\infty \setminus V^0$ has only finitely many neighbors in V^0. Since we only add countably many vertices to G^0 when passing to G^∞, in order to extend every automorphim of G^0 to G^∞ it is necessary that every vertex in $V^\infty \setminus V^0$ has a neighborhood in G^0 whose orbit under $\mathrm{Aut}(G^0)$ is countable. This is clearly satisfied by the finite neighborhoods of our construction.

Another way to embed G symmetrically into Γ is to realize not only all finite sets but all definable subsets of V^n (definable in G^n with parameters from V^n) as neighborhoods of vertices in $V^{n+1} \setminus V^n$. This works because the collection of definable subsets of a countable graph is countable and closed under automorphisms.

In this case, there are vertices in $V^\infty \setminus V^0$ whose set of neighbors in V^0 is infinite.

Remark 2. If G is a countable graph and $H \subseteq^A G$, then, as indicated above, every neighborhood of a vertex $v \in V(G) \setminus V(H)$ in H has a countable orbit under $\mathrm{Aut}(H)$, namely the collection of neighborhoods in $V(H)$ of the images of v under extensions of automorphisms of H to G.

If H is a countably infinite complete graph or a countably infinite graph without any edges, then the only sets with a countable $\mathrm{Aut}(H)$-orbit are the finite sets and their complements. These sets are also exactly the definable sets.

More generally, the *Kueker-Reyes Theorem* [16] states that whenever $\mathcal{M}$ is a countable, homogeneous structure and A is a subset of the underlying set M of $\mathcal{M}$ that has an orbit of size $< 2^{\aleph_0}$ under the action of the automorphism group of $\mathcal{M}$, then there is a finite sequence $(a_1, \dots, a_n)$ of elements of M such that every automorphism of $\mathcal{M}$ that fixes all the a_i, $i \in \{1, \dots, n\}$, leaves the set A invariant. By the homogeneity of $\mathcal{M}$, every finite partial isomorphism of $\mathcal{M}$ extends to an automorphism of $\mathcal{M}$. It follows that if $a, b \in M$ have the same type over $(a_1, \dots, a_n)$, then either both a and b are elements of A or both are not. In other words, A is the union of a collection of subsets of M that are definable in $\mathcal{M}$ with parameters among $a_1, \dots, a_n$.

Applying this to the random graph Γ, we see that for every set $A \subseteq V(\Gamma)$ with a countable $\mathrm{Aut}(\Gamma)$-orbit, there are vertices $v_1, \dots, v_n \in V(\Gamma)$ such that A is a union of subsets of $V(\Gamma)$ definable with parameters among $v_1, \dots, v_n$. But by *elimination of quantifiers*, every set subset of $V(\Gamma)$ that is definable in Γ with parameters among $v_1, \dots, v_n$ is a Boolean combination of neighborhoods of the v_i and singletons of the form $\{v_i\}$. In particular, there are only finitely many such sets. It follows that A itself is a Boolean combination of neighborhoods of the v_i and singletons of the form $\{v_i\}$ and therefore definable.

So, for some countable graphs, for instance homogeneous graphs with quantifier elimination, the definable sets are the only sets of vertices that have a countable orbit under the action of the automorphism group of the graph. It follows that in general, in our construction of G^∞ the collection of definable subsets of $G = G^0$ is the largest collection of sets that can serve as neighborhoods in G^0 of vertices in the extension G^∞. This observation also places significant restrictions on how the random graph or, for example, the countably infinite complete graph can sit inside a countable graph as a $\subseteq^A$-subgraph.

Theorem 4. *For every natural* $r \geq 1$*,*

$$\Gamma \rightarrowtail^P_{\mathrm{bi}} (\Gamma)^1_r.$$

Thus for every vertex coloring of Γ *by* r *colors, there is a single color which contains induced symmetrized copies of all countable graphs.*

Proof. Let $r \geq 1$ be given. $\Gamma^{\otimes r}$ is countable, and as Γ is vertex transitive, $\Gamma^{\otimes r} \rightarrowtail_{\mathrm{bi}} (\Gamma)^1_r$ by Lemma 1. Fix $G \subseteq_{\mathrm{bi}} \Gamma$ with G isomorphic to $\Gamma^{\otimes r}$ by Lemma 4. For every vertex coloring of Γ by r colors, there is an induced monochromatic copy of Γ, $G' \subseteq_{\mathrm{bi}} G \subseteq_{\mathrm{bi}} \Gamma$. For every countable graph H it holds that $H \leq_{\mathrm{bi}} G'$ by Lemma 4. So by transitivity of $\subseteq_{\mathrm{bi}}$ we have $\Gamma \rightarrowtail_{\mathrm{bi}} (\Gamma)^1_r$. Since $H \subseteq^P \Gamma$ holds for every induced subgraph H of Γ, we are done.

3. Edge Colorings

It is well known [5] that $\Gamma \not\rightarrow (\Gamma)^2_2$. In particular, there is a countable graph H such that for all countable graphs G and all sets F of partial isomorphisms of H, $G \rightarrowtail^F (H)^2_2$ fails. Therefore, in the case of edge colorings, we are only interested in finite graphs.

Theorem 5. *There is a constant c such that for every finite graph H with n vertices there is a graph G with no more than $2^{2^{cn(\log n)^2}}$ vertices so that*

$$G \rightarrowtail^P (H)^2_2.$$

Proof. By [14] (see also [6]), there is a constant d such that given a graph H with n vertices, we can fix a graph G' with no more than $m := 2^{dn(\log n)^2}$ vertices such that $G' \rightarrowtail (H)^2_2$. By Lemma 3, fix a graph G with at most $2^{2m \log m}$ vertices such that $G' \subseteq^P G$. Thus, for some constant c,

$$\begin{aligned} |V(G)| \leq 2^{2m\log m} &= 2^{2^{dn(\log n)^2+1} dn(\log n)^2} \\ &= 2^{2^{dn(\log n)^2 + \log(dn(\log n)^2)+1}} = 2^{2^{dn(\log n)^2+\log d+\log n+2\log\log n+1}} \\ &\leq 2^{2^{cn(\log n)^2}} \end{aligned} \quad (6)$$

We check that $G \rightarrowtail^P (H)^2_2$. Given an edge-coloring of G by 2 colors, consider its restriction to the edges of G'. As $G' \rightarrowtail (H)^2_2$, there is a monochromatic $H' \subseteq G'$ which is isomorphic to H. As the relation $\subseteq^P$ is hereditary on the left hand side to induced subgraphs, we have $H' \subseteq^P G$.

We conclude this section with the following:

Theorem 6. *For every finite graph H and every finite number r of colors there is a finite graph G such that*

$$G \rightarrowtail^P (H)^d_r$$

holds for both $d = 1$ and $d = 2$.

Proof. By induction on r, we show that there is a finite graph G_1 such that $G_1 \rightarrowtail^P (H)^2_r$.

Namely, suppose that for some $r \geq 1$, we have a finite graph F such that $F \rightarrowtail^P (H)^2_r$. This is trivially satisfied if $r = 1$. By Theorem 5 there is a finite graph F' such that $F' \rightarrowtail^P (F)^2_2$. We claim that $F' \rightarrowtail^P (H)^2_{r+1}$. Given a coloring c of the edges of F' with $r + 1$ colors, define a coloring c' of the edges of F' with two colors by letting $c'(e) = 0$ if $c(e)$ is one of the first r colors of c and $c'(e) = 1$ if $c(e)$ is the last color of c. By the choice of F', there is copy of F that is a c'-monochromatic $\subseteq^P$-subgraph of F'. On this copy of F, c assumes at most r different values. Hence, by $F \rightarrowtail^P (H)^2_r$, this copy of F contains a c-monochromatic $\subseteq^P$-subgraph isomorphic to H. By the transitivity of $\subseteq^P$, this shows $F' \rightarrowtail^P (H)^2_{r+1}$.

Now fix r as in the theorem and choose a finite graph G_1 such that $G_1 \rightarrowtail^P (H)^2_r$. By Hrushovski's theorem there is a finite, *transitive* graph G_2 such that $G_1 \subseteq^P G_2$. Now let $G = G_2^{\otimes r}$.

By Lemma 1, $G \rightarrowtail^A (G_2)^1_r$. So, since $H \subseteq^P G_2 \subseteq^A G$, it holds that $G \rightarrowtail^P (H)^1_r$. The relation $G \rightarrowtail^P (H)^2_r$ holds because $G_2 \rightarrowtail^P (H)^2_r$ holds and $G_2 \leq^A G$.

3.1. Bipartite graphs

A *bipartite graph* is a triple $B = \langle L, R, E\rangle$ where $\langle L \cup R, E\rangle$ is a graph and $|e \cap L| = |e \cap R| = 1$ for all $e \in E$. L is the *left side* of B and R is the *right side* of B. We assume automorphisms and partial isomorphisms of bipartite graphs to preserve sides.

Hrushovski's theorem also holds for bipartite graphs. This follows either from adapting the original proof to bipartite graphs, or from Herwig's extension [9] of Hrushovski's theorem to relational structures. Thus, one can obtain the analog of Theorem 5 via a similar proof, using Nešetřil and Rödl's bipartite induced Ramsey theorem [17]. However, it is not necessary to use either Hrushovski's or Herwig's theorem at all in this case, as the monochromatic induced bipartite graph given by the proof in [17] is symmetrized. This is shown in the proof of Theorem 7.

With vertex colorings of bipartite graphs the situation is slightly different. One can color all vertices in L by red and all vertices in R by blue to avoid monochromatic bipartite subgraphs altogether. Theorem 8 below shows that monochromatic sides can be guaranteed on some symmetrized induced bipartite subgraph.

Definition 2. *The* n-th symmetric power *of a bipartite graph $B = \langle L, R, E\rangle$, introduced in [18] (see also [7] p. 119), denoted by $B^{(n)}$, is the bipartite graph $\langle L^n, R^n, E^{(n)}\rangle$ where $\{\bar{v}, \bar{u}\} \in E^{(n)}$ iff for all $i < n$ it holds that $\{v(i), u(i)\} \in E$.*

The mapping $\langle\langle v(i) : i < n\rangle, \langle u(i) : i < n\rangle\rangle \mapsto \langle\{v(i), u(i)\} : i < n\rangle$ is a natural 1-1 correspondence between $E^{(n)}$ and E^n, the set of all sequence of length n of edges from E.

For bipartite graphs, the notions *bi-embedding*, $\subseteq_{\mathrm{bi}}$, and $\rightarrowtail_{\mathrm{bi}}$ have the obvious meaning.

Theorem 7. *For every finite bipartite graph $B = \langle L, R, E\rangle$ and every finite number r of colors there exists a number n so that $B^{(n)} \rightarrowtail_{\mathrm{bi}} (B)^2_r$.*

Proof. First, let us assume that B has no isolated vertices. This can be achieved by adding a unique neighbor in L to every isolated vertex in R and vice versa. The original graph is bi-embedded in this extension without isolated vertices.

Let $n = HJ(|E|, r)$, the Hales-Jewett number of $|E|$ and r. A coloring of $E^{(n)}$ by r colors can be considered as a coloring of E^n via the natural correspondence above.

Let $W \in (E \cup \{X\})^n$ be a word such that $I = \{i : W(i) = X\} \neq \emptyset$ and so that the combinatorial line defined by W in E^n is monochromatic. The line defined by W is $\{W(e) : e \in E\}$ where $W(e)$ is obtained from W by substituting e for every occurrence of X in W.

Define an embedding ϕ of B into $B^{(n)}$ as follows. For $v \in L$ let

$$\phi(v) = \langle W(i) \cap L : i \in n \setminus I\rangle \cup \langle v : i \in I\rangle,$$

that is, the sequence of left vertices from $W(i)$ for all i such that $W(i) \neq X$ and constantly v for all $i \in I$. For $u \in R$, $\phi(u)$ is defined similarly.

As B has no isolated vertices, the map ϕ clearly embeds B into $B^{(n)}$ as an induced subgraph. Furthermore, this subgraph is monochromatic, as the combinatorial line defined by W is. We argue that this induced monochromatic copy of B is also symmetrized. Indeed, let $\sigma \in \mathrm{Aut}(\phi(B))$ be given, and let $\bar{\sigma} : L^n \cup R^n \to L^n \cup R^n$ be defined as follows: for $\bar{v} = \langle v(i) : i < n \rangle$,

$$\bar{\sigma}(\bar{v}) = \langle v(i) : i \in n \setminus I \rangle \cup \langle \sigma(v(i)) : i \in I \rangle,$$

and similarly for $\bar{u} \in R^n$. This is an automorphism of $B^{(n)}$: if $\{\bar{v}, \bar{u}\} \in E^{(n)}$ then $\{\bar{v}(i), \bar{u}(i)\} \in E$ for all $i < n$. For $i \in n \setminus i$ it holds that $\bar{\sigma}(\bar{v})(i) = \bar{v}(i)$ and $\bar{\sigma}(\bar{u})(i) = \bar{u}(i)$, while for $i \in I$ it holds that $\{\bar{\sigma}(\bar{u})(i), \bar{\sigma}(\bar{v})(i)\} \in E$ because $\sigma \in \mathrm{Aut}(\phi(H))$, so $\{\bar{\sigma}(\bar{u}), \bar{\sigma}(\bar{v})\} \in E^{(n)}$. Similarly, non-edges are preserved. Clearly, $\bar{\sigma}$ extends σ. Also, the uniformity of the definition of $\overline{\sigma}$ from σ shows that the map $\sigma \mapsto \overline{\sigma}$ is a homomorphism between the automorphism groups of B and $B^{(n)}$.

Theorem 8. *For every finite bipartite graph B and $r > 0$ there exists n such that for every coloring of vertices of $B^{(n)}$ by r colors there is a bi-embedded bipartite subgraph of $B^{(n)}$ that is isomorphic to B and whose left side and right side are both monochromatic (possibly of different color).*

Proof. Assume, as in the previous proof, that $B = \langle L, R, E \rangle$ has no isolated vertices and let $n = HJ(|E|, r^2)$. Given a vertex coloring c of $B^{(n)}$ by r colors, define an edge coloring of B by r^2 colors by assigning the color $\langle c(e \cap L^n), c(e \cap R^n) \rangle$ to an edge $e \in E^{(n)}$. By the previous theorem, there is a bi-embdedded bipartite subgraph B' of $B^{(n)}$ that is isomorphic to B and all whose edges have the same color. But this implies that the set of vertices of B' on the left side and the set of vertices of B' on the right side are both monochromatic.

4. Discussion

Hrushovski considers in [13] the case $F = \{f\}$, that is, when one is required to implement by $\mathrm{Aut}(G)$ a single arbitrary partial isomorphism of H, and proves:

Theorem 9. *There are constants $c < c'$ so that if one defines $g(n)$ to be the least m such that for every graph H with n vertices and every isomorphism f between subgraphs of H there exists a graph G with at most m vertices so that $H \subseteq^{\{f\}} G$, then*

$$c(n \log n)^{1/2} \leq \log g(n) \leq c'(n \log n)^{1/2}.$$

It is of interest to know if an upper bound in the order of magnitude of Hrushovski's function g can be found for the number of vertices of G in $G \rightarrowtail^{\{f\}} (H)^1_2$, for arbitrary f.

Considering the fact that we have a polynomial upper bound for the size of G in $G \rightarrowtail^{P_1} (H)^1_2$ but that, by Theorem 9, even for a single unrestricted partial isomorphism f we have a super-polynomial lower bound for the size of G in $G \rightarrowtail^{\{f\}} (H)^1_2$, it would be interesting to know if there is a number k such that the upper bound for $G \rightarrowtail^{P_k} (H)^1_2$ is polynomial but the lower bound for $G \rightarrowtail^{P_{k+1}} (H)^1_2$ is not polynomial, where, of course, $P_k = P_k(H)$ is the set of all isomorphisms between two subgraphs of H, each of cardinality k.

The upper bound of $2^{cn(\log n)^2}$ for the size of G in $G \rightarrowtail (H)^2$, where n is the size of H, was obtained in [14] using a randomly constructed graph. Randomly constructed graphs tend to be rigid. Fox and Sudakov [6] recently re-established this upper bound using the explicit Paley graph, which has quite a few automorphisms. Can one find a better upper bound than the one obtained in Theorem 5? Can pseudo-random bipartite graphs be used to give a good upper bound for the bipartite graph relation, instead of the upper bound coming from the Hales-Jewett theorem?

A question that is not so much concerned with upper bounds is whether one can prove a version of our Theorem 5 on edge colorings that yields a monochromatic bi-embedded subgraph.

Finally, it is of interest to find lower bounds for the vertex-coloring theorems. Is the n^r bound tight for vertex transitive graphs in Theorem 2?

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References

1. N. Alon, H. Lefmann and V. Rödl. *On an anti-Ramsey type result*, Colloq. Math. Soc. János Bolyai 60; Sets, Graphs and Numbers, Budapest (Hungary), 1991, 9-22.
2. László Babai, *An anti-Ramsey theorem.* Graphs Combin. 1 (1985), no. 1, 23–28.
3. László Babai and Vera T. Sós. *Sidon sets in groups and induced subgraphs of Cayley graphs.* European J. Combin. 6 (1985), no. 2, 101–114.
4. Z. Chadzidakis and E. Hrushovski. *Model theory of differential fields.* Tran. AMS. 351(8) (1999) 2997–3071.
5. P. Erdős, A. Hajnal and L. Pósa. *Strong embeddings of graphs into colored graphs.* Infinite and finite sets (Colloq. Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. 1, pp. 585–595.
6. Jacob Fox and Benny Sudakov. *Induced Ramsey-type theorems*, preprint.
7. Ronald L. Graham, Bruce L. Rothschild and Joel H. Spencer. **Ramsey Theory.** Jon Wiley & sons, 1990.
8. J. de Groot. *Groups represented by homeomorphism groups.* Mathematische Annalen 138 (1959), 80–102.
9. Bernard Herwig. *Extending partial isomorphisms on finite structures.* Combinatorica 15 (1995), 365–371.
10. B. Herwig and D. Lascar. *Extending partial automorphisms and the profinite topology on free groups.* Tran. AMS, 5 (2000) pp. 1985–2021.
11. W. Hodges. **Model theory**. Encyclopedia of Mathematics and its Applications, 42. Cambridge University Press, Cambridge, 1993. xiv+772 pp.
12. W. Hodges, I. Hodkinson, D. Lascar and S. Shelah. *The small index property for ω-stable, ω-categorical structures and the random graph.* J London Math Soc 48 (1993) pp. 204–218.

13. Ehud Hrushovski. *Extending partial isomorphisms of graphs.* Combinatorica 12 (4) (1992) 411–416.

14. Y. Kohayakawa, H. Prömel and V. Rödl. *Induced Ramsey numbers.* Combinatorica 18 (1998) pp. 373–404.

15. M. Kojman and S. Shelah. *Homogeneous families and their automorphism groups.* Journal of the London Mathematical Society (2) 52 (1995) 303–317.

16. D. Kueker, *Back and forth arguments in infinitary logic.* In memoriam Carol Karp, Springer Lecture Notes in Mathematics, vol. 429, edited by D.W. Kueker, Springer-Verlag, Berlin, 1969, pp. 17–71.

17. J. Nešetřil and V. Rödl. *Two proofs of the Ramsey property of the class of finite hypergraphs.* European J. Combin. 3 (1982), no. 4, 347–352.

18. J. Nešetřil and V. Rödl. *The Ramsey Property for Graphs with Forbidden Complete Subgraphs.* J. Comb. Th. (B) 20 (1976), 243–249.

19. S. Thomas, **The Automorphism Tower Problem**. Unpublished manuscript (September 2010). 231 pages.

20. J. Truss, *The group of the countable universal graph.* Mathematical Proceedings of the Cambridge Philosophical Society 98 (1985), no. 2, 213–245.

21. V.V. Uspenskij. *On the group of isometries of the Urysohn universal metric space.* Comment. Math. Univ. Carolin. 31 (1990), no. 1, 181–182.

22. V.V. Uspenskij. *On subgroups of minimal topological groups.* Topology Appl. 155 (2008), no. 14, 1580–1606.