SINGULAR CARDINALS: FROM HAUSDORFF'S GAPS TO SHELAH'S PCF THEORY

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1 PREFACE

The mathematical subject of singular cardinals is young and many of the mathematicians who made important contributions to it are still active. This makes writing a history of singular cardinals a somewhat riskier mission than writing the history of, say, Babylonian arithmetic.

Yet exactly the discussions with some of the people who created the 20th century history of singular cardinals made the writing of this article fascinating. I am indebted to Moti Gitik, Ronald Jensen, István Juhász, Menachem Magidor and Saharon Shelah for the time and effort they spent on helping me understand the development of the subject and for many illuminations they provided. A lot of what I thought about the history of singular cardinals had to change as a result of these discussions.

Special thanks are due to István Juhász, for his patient reading for me from the Russian text of Alexandrov and Urysohn's *Memoirs*, to Salma Kuhlmann, who directed me to the definition of singular cardinals in Hausdorff's writing, and to Stefan Geschke, who helped me with the German texts I needed to read and sometimes translate.

I am also indebted to the Hausdorff project in Bonn, for publishing a beautiful annotated volume of Hausdorff's monumental *Grundzüge der Mengenlehre* and for Springer Verlag, for rushing to me a free copy of this book; many important details about the early history of the subject were drawn from this volume.

The wonderful library and archive of the Institute Mittag-Leffler are a treasure for anyone interested in mathematics at the turn of the 20th century; a particularly pleasant duty for me is to thank the institute for hosting me during my visit in September of 2009, which allowed me to verify various details in the early research literature, as well as providing me the company of many set theorists and model theorists who are interested in the subject.

I an grateful to all colleagues who read early versions of this manuscript and who made many valuable comments. Aki Kanamori, who spent too many hours of his time reading and correcting my writing deserves my deepest gratitude. Needless to say, all mistakes which remained in this text are solely mine.

Every mathematician has come across Hausdorff's name, typically in more than one area of mathematics. That was the case with me as well, but it was not until the writing of this chapter that I truly grasped the influence of Hausdorff's work on modern set theory and, reading his letters, had the chance to appreciate from afar his delicate sense of humour and modesty. This article is devoted to his memory.

2 INTRODUCTION

Singular cardinals appeared on the mathematical world stage two years before they were defined. At the third international congress of mathematicians in Heidelberg, on August 10, 1904, Julius König allegedly refuted Georg Cantor's two major conjectures simultaneously, by proving that the continuum could not be well-ordered in any of the order-types in Cantor's list of alephs.

König's proof went as follows: Suppose that the continuum is some $\aleph_\beta$. Look at the sequence $\aleph_{\beta+1}, \aleph_{\beta+2}, \ldots, \aleph_{\beta+n}, \ldots$ for all natural numbers n. Its limit $\aleph_{\beta+\omega}$ is equal to the sum $\sum_n \aleph_{\beta+n}$.

König next presented his newly proved special case of what is now known as "König's inequality", which states that

$$\prod \aleph_{\beta+n+1} > \sum \aleph_{\beta+n}, \tag{1}$$

to derive the inequality

$$(\aleph_{\beta+\omega})^{\aleph_0} > \aleph_{\beta+\omega}. \tag{2}$$

So far, König is absolutely correct. The general theorem now known as "König's inequality" was obtained by Zermelo and presented in Göttingen in the same year, but published only four years later [Zermelo, 1908]. Next, König quotes a theorem from Felix Bernstein's Göttingen dissertation: $(\aleph_\alpha)^{\aleph_0} = \aleph_\alpha \cdot 2^{\aleph_0}$ for every ordinal α. Substituting $\beta + \omega$ for α and using $2^{\aleph_0} = \aleph_\beta$ one gets

$$(\aleph_{\beta+\omega})^{\aleph_0} = \aleph_{\beta+\omega} \tag{3}$$

Now (2) and (3) contradict each other. Ergo: $2^{\aleph_0}$ is not an aleph!

This had a dramatic effect:

> Though mathematics rarely makes a ripple in the daily press today, the local papers then were full of reports describing König's sensational discovery.

This is from Cantor's biography by Dauben [1990, p.249], where Cantor's consternation with the situation is described in detail (p.248):

> As Schoenflies once put it, belief that the power of the continuum was equal to $\aleph_1$ was a basic dogma with Cantor. Not only did König's proof challenge this dogma, but it further implied that the continuum

> could not be well-ordered. This rendered doubtful another article of Cantorian faith—that *every* set could be well-ordered.

Cantor was immediately suspicious of König's proof, particularly of Bernstein's lemma. Making a pun on König's last name, "Cantor quipped that whatever had been done to produce the alleged proof, he suspected the king less than the king's ministers." (Dauben [1990, p.249])

Indeed, Bernstein's lemma was false for exactly the type of cardinals required in König's use of it. Felix Hausdorff, after coming back from Wengen, Switzerland, where he and Hilbert had spent a short vacation after the Heidelberg congress, poetically described König's and Bernstein's combined flaw in a letter to Hilbert in the following month (cf. [Hausdorff, 2002, p.11]):

> After the continuum problem plagued me in Wengen almost like an obsession, my first look here was naturally directed to Bernstein's dissertation. The bug lay exactly in the expected place, on page 50 [...] Bernstein's argument employs a recursion from $\aleph_\mu$ which does not hold for alephs that have no immediate predecessor, that is, exactly those alephs which König required.[1]

That is, Bernstein's flaw lies in the fact that exactly the limit cardinal for which König uses Bernstein's equality—$\aleph_{\beta+\omega}$—escapes the recursion Bernstein uses to prove the equality, as it has no immediate predecessor.

As [Hausdorff, 2002] mentions, it was for a while erroneously thought that Zermelo was the one who detected the error in König's false proof. The debate continues whether it was actually Hausdorff (cf. [Hausdorff, 2002, p.10]) or someone else. Recently Moore [2009, p.825] suggested that a letter by Otto Blumenthal to Émile Borel dated December 1, 1904, may indicate that "König himself was the first to realize that his proof was not valid, followed (independently of each other) by Cantor, Bernstein and Zermelo."

Singular cardinals could not have received a louder introduction than the one they received in the 1904 Heidelberg episode: at an international congress, with all other parallel sessions canceled by the organizers to allow everyone to attend König's sensational lecture, in the presence of Cantor and Hilbert, and in direct relation to the two most important problems in set theory of the time.

Furthermore, disregarding the circumstances and considering only the mathematical concepts, *infinite products of distinct regular cardinals*, which are the most important objects in the modern elementary theory of singular cardinals, were introduced in König's pioneering result. König showed that the product $\prod_n \aleph_{\beta+n}$ is embeddable in the power set of $\aleph_{\beta+\omega}$ and diagonalizable against the sum of

[1] Nachdem das Continuumproblem mich in Wengen beinahe wie eine Monomanie geplagt hatte, galt hier mein erster Blick natürlich der Bernsteinschen Dissertation. Der Wurm sitzt genau an der vermutheten Stelle, S. 50 [...] Bernsteins Betrachtung giebt eine Recursion von $\aleph_{\mu+1}$ auf $\aleph_\mu$, versagt aber für solche $\aleph_\mu$, die keinen Vorgänger haben, also gerade für die Alephs, für die Herr J. König sie nothwendig braucht.

cardinals $\sum_n \aleph_{\beta+n} = \aleph_{\beta+\omega}$. He takes Cantor's diagonal argument for the inequality $2^\kappa > \kappa$ an important step further and applies it in a setting with a singular cardinal. Using the fact that $\aleph_{\beta+\omega}$ is a countable sum, he can use diagonalization to show that the total number of countable subsets of $\aleph_{\beta+\omega}$ exceeds $\aleph_{\beta+\omega}$. Zermelo [1908] extended this result to general products and observed that it implies Cantor's theorem that the power set of any set had larger cardinality than the set itself. Thus, the König-Zermelo inequality held all the information that Cantorian diagonalization could give at the time about cardinal exponentiation, and surprisingly enough continued to do so for many more decades.

The importance of this interplay between infinite products and infinite sums, which König discovered and Zermelo phrased in full generality, goes well beyond cardinal arithmetical considerations—it is fundamental for every mathematical structure of singular cardinality. Even a superficial examination of König's argument shows than it actually proves more that it states — e.g., that for every collection of $\aleph_{\beta+\omega}$ small subsets of $\aleph_{\beta+\omega}$ there exists a countable subset with finite intersection with each set in the family. A central tool for the study of singular cardinals has been introduced by König already in 1904 in Heidelberg, but was not put to appropriate use at that time.

König's Heidelberg episode clearly brought out the issue that would accompany the development of singular cardinals theory for many years to come. Singular cardinals were regarded as a curiosity, an obstacle, a threat to Cantor's conjectures. They are the points in the list of alephs at which Bernstein's inductive argument and many other arguments that would be attempted in the years to come break down, exactly because of the fact that some of the small subsets of the cardinal are unbounded in it — a property described as "singularity" by König [1905], and later adapted by Hausdorff as the name for these cardinals.

It is instructive to compare König's diagonal argument to the Baire category theorem, proved just three years earlier: both are proved via variations on Cantor's diagonal argument for $2^{\aleph_0} > \aleph_0$, with the Baire argument using first category sets instead of singletons and the König argument using subsets of cardinality smaller than the singular cardinal instead of singletons. Both theorems had short, simple proofs that were accessible to everyone and both were immediately accepted as true. One can argue that both theorems had similar aesthetic appeal. Yet, the Baire theorem rapidly became a part of the general mathematical language—it is probably one of the most frequently used theorems in mathematics and has numerous applications and generalizations—and many other uses of Cantorian diagonalization in analysis followed the Baire theorem example. But hardly any research at all followed the König-Zermelo inequality for many decades. The next development in singular cardinal arithmetic in the direction that the inequality marked came only with Silver's theorem in 1974 (see below).

Why did not König himself, Cantor, or, more realistically, Zermelo, who published the general formulation of König's inequality, or Hausdorff, pursue König's discovery further? König, at 1904 only one year before his retirement, is determined to disprove Cantor's well-ordering conjecture as much as Cantor is determined to prove it. He would spend the last eight years of his life trying to refute the Axiom of Choice, which Zermelo phrased and used to prove the Well-Ordering theorem shortly after the Heidelberg congress.

Cantor, who had previously been on good terms with König and appreciated his work, was agitated and probably angry that König put him in such public discomfort in Heidelberg, not having told him beforehand about the contents of his talk. Neither of them pursues König's discovery further.

Zermelo may have been more interested in the Axiom of Choice and in axiomatization issues in general than in exploring singular cardinal arithmetic further.

Hausdorff, who was an ideal candidate for studying the products that appeared in König's proof, with which he was well acquainted, and who would soon launch an intensive study of the infinite product ω^ω—concludes his letter to Hilbert as follows (cf. [Hausdorff, 2002, p.11]):

> I had written to this effect to Herr König while still on the road and explained this as far as I could without using Bernstein's work, but so far received no reply. Now I am yet even more convinced that König's proof is wrong and I believe König's theorem to be the peak of impossibility. On the other hand, I am sure you too do not believe that Herr Cantor discovered during these last few weeks what he has been looking for in vain for the last 30 years. Thus, your problem No. 1 seems to remain after the Heidelberg congress exactly where you left it after the Paris congress.
>
> But perhaps, as I am writing this, one of the combating parties is already in possession of the truth. I am very curious to see the printed proceedings of the congress.[2]

In König's paper, published in 1905 in *Mathematische Annalen*, he indeed retracted his claim and pointed to Bernstein's mistake (see below for further discussion of this paper).

Once the doubts that König's announcement cast had been lifted, effort to solve Cantor's two big open problems resumed. What is important to observe is that apart from its potential hazard to Cantor's problem, Hausdorff says nothing about König's inequality. This mathematically valuable statement seems to have been welded to the wrong proof in which it played a part.

Would König's important discovery have received wider attention had it been presented on its own, and not as part of the false refutation of the continuum

[2] Ich hatte in diesem Sinne, soweit Ich ohne Benutzung der Bernsteinschen Arbetin konnte, schon von unterwegs an Herrn König geschrieben, aber keine Antwort erhalten, bin also umso mehr geneigt, den König'schen Beweis für falsch und den König'schen Satz für den Gipfel des Unwahrscheinlichen zu Halten. Anderseits werden wohl auch Sie kaum den Eindruck gewonnen haben, dass Herr Cantor das, was er seit 30 Jahren vergeblich sucht, in den letzten Wochen gefunden haben sollte, und so scheint Ihr Problem Nr. 1 nach dem Heidelberger Congress genau dort to stehen, wo Sie es aur dem Pariser Congress verlassen haben.

Aber vielleicht ist, während ich dies schriebe, doch schon eine der streitenden Parteien im Besitze der Wahrheit. Ich bin sehr gespannt auf die gedruckten Verhandlungen des Congresses.

hypothesis? The circumstances so dictated that the mathematical contents of König's inequality was overshadowed by König's erroneous use of it. By the time König's congress paper appeared in 1905, all König claimed to have proved was that the continuum could not be well-ordered in an order-type of the form $\aleph_{\beta+\omega}$. However, his argument essentially proved that the continuum could not be well-ordered in a type of countable cofinality; as "cofinality" had not been yet defined, he could not state this. (Hausdorff later attributed the full statement to König.) But this claim is incomparably less dramatic that his original statement that the continuum could not be well-ordered at all.

The name "singular cardinal" and the circumstances of its emergence as a name for non-regular limit cardinals requires some explanation. This name appears for the first time in König's [1905] following his congress lecture, in a passage in which he explains the mistake in his lecture:

> It follows that the assumption that the continuum is equivalent to a well-ordered set would be certainly false if Bernstein's theorem was correct n general. Unfortunately its proof has a significant gap since for $\aleph_\omega$ and each of the "singular" well-ordered sets considered above, the assumption that each countable subset is included in a [proper] initial segment is not correct anymore.
>
> I mention this mainly in order to retract the conclusion that I drew from Bernstein's theorem in my talk at the congress under the assumption that the theorem was correct.

From a distance of over a hundred years the scorn in König's articulation is still audible: if not for these "singular" order-types, as König puts it, his proof would still be correct. Later in the paper König brings up the possibility that the continuum is not well-orderable and (as a non-well-ordered cardinal) larger than every well-orderable set. In this case, he writes, Bernstein's theorem follows.

Technically, König's original use of "singular" means "an uncountable cardinal with unbounded countable subsets", that is, singular of countable cofinality. The name "singular" was adopted by Hausdorff to describe cardinals whose cofinality type was smaller than themselves, once cofinality had been defined in his 1906 paper on order-types, and became the standard name for these cardinals ever since. This name was far from neutral. Hausdorff's division of cardinals to "regular" and "singular" suggested that regular cardinals were the objects that deserved serious attention, and that the singulars were less important. This view was sustained for a surprisingly long time.

Zermelo [1904] introduced the Axiom of Choice and proved that every set could be well-ordered, and Hausdorff launched an intensive project of classifying order-types, perhaps hoping he could solve Cantor's continuum problem in this way. Hausdorff's first result in set theory (obtained independently by Bernstein) was that the number of countable order-types is equal to the continuum. Moore [1989] expresses the opinion that Hausdorff was doing this so that he could prove that the number of countable well-order-types, which Cantor had already shown to be $\aleph_1$, was equal to the total number of countable order-types, and in this way solve Cantor's continuum problem, but Steprans [2010] feels that Hausdorff was "primarily interested in the classification of linear order-types, but . . . nevertheless recognized the significance of his work to the Continuum Problem."

Although Hausdorff did treat singular cardinals in his work, they were cast aside by him and the other researchers of the time in favor of the more burning issue of the continuum. The (consistent) possibilities that the continuum itself was a singular or a regular limit cardinal were not raised at that time at all. This may be partly due to the confusion between the $\aleph$ operation and the $\beth$ operation; the continuum is clearly not closed under the latter, therefore (if the two are the same, as GCH claims) it is not closed under the former. It took a stimulus from measure theory to bring the possibility of a limit continuum into the foreground (see Sect. 7.1).

Neither fashion nor prejudice could hide singular cardinals from the mathematical eye for long. The interrelation between singular cardinalities and basic properties of standard mathematical objects is so profound and varied that singular cardinals inevitably appeared in set-theoretic investigation in several mainstream branches of mathematics during the long period of time in which they were ignored by the organized set-theoretic research. Some of these investigations foundered for decades in the absence of the right set-theoretic tools. In Section 4 we shall look at some of these occurrences of singulars.

Set theory had to experience several remarkable metamathematical developments before set theorists developed an elementary theory of singular cardinals. It had to be realized first that there were no more rules for regular cardinal arithmetic than those given by the König-Zermelo inequality (and the trivial weak monotonicity). It was actually the singular cardinals that possessed a theory, and hence deserved a close study. A theory of singular cardinal arithmetic was finally developed without metamathematical means, via the algebraic study of small infinite products of different regular cardinals by Shelah in the 1980s and 1990s under the name of *pcf theory*. *A posteriori*, Shelah's theory shed an explanatory light on the instances in which singular cardinals appeared naturally in topology, algebra and elsewhere, and in many cases could be used to directly continue the early investigations of singulars.

A central ingredient in Shelah's pcf theory is the theory of exact upper bounds of increasing sequences of ordinal functions modulo some ideal on an infinite set. Given an ideal I of subsets over an infinite set A one defines a quasi-ordering on the class of functions from A to the ordinal numbers, On^A. For two such functions f and g, f is smaller modulo I than g if except for a set in I it holds for all $a \in A$ that $f(a) < g(a)$. This relation is denoted by $<_I$ and similarly $\leq_I$ and $=_I$ are defined. A typical example is the ideal of finite sets—in which case the relations $<_I$, $\leq_I$ and $=_I$ are customarily denoted by $<^*$, $\leq^*$ and $=^*$.

A central notion in pcf theory is that of a *scale*—a sequence of functions in an infinite product of regular cardinals which is linearly ordered and dominating for the relation $<_I$ for some ideal I. Shelah's results on scales are remarkably similar to what Hausdorff was looking for in his study of sequences of functions from natural numbers to natural numbers modulo the ideal of finite sets (cf. [Steprans, 2010]). Hausdorff knew that the existence of an ω_1-scale in $(\omega^\omega, <^*)$ followed from CH, but failed to prove the existence of such a scale without CH. He later managed to prove without CH (after having first proved with CH) the existence of an (ω_1, ω_1^*)-gap in $(\omega^\omega, <^*)$. A (κ, κ^*)-gap in a partially ordered set is a pair of sequences of length κ, the first increasing, the second decreasing, with each element in the first sequence being smaller than every element in the second, and with no member of the partially ordered set between the sequences.

The proof of the existence of an (ω_1, ω_1^*)-gap in $(\omega^\omega, <^*)$ was one of Hausdorff's most important results—partly so because Shelah's work in the 1990s showed that Hausdorff could not have proved more than that without using additional axioms. The irony is exquisite. Hausdorff, who discovered everything one could discover without metamathematics about gaps in $(\omega^\omega, <^*)$, is unaware that this is all that can be proved about gaps. Shelah, who knows everything there is to know about the independence phenomena surrounding the continuum proves that there are no ZFC gaps in the product $\prod_n \omega_n$ except the gaps Hausdorff had discovered in 1909, and proves the existence of an $(\aleph_{\omega+1})$-scale in a product of the form $\prod_{n\in B} \omega_n$ by totally elementary means. As Shelah writes in the introduction to his *Cardinal Arithmetic* [1994a], "Cantor should have no problems understanding and (so I feel) appreciating the theorems and even most proofs in this book."

Did Hausdorff miss the opportunity to discover pcf theory? Could he have extended his study of scales and gaps in ω^ω to scales and gaps in the (much better behaved) product $\prod_n \omega_n$? Perhaps so; there is no *mathematical* reason for not discovering at least some of the elementary theory of singular cardinals in Hausdorff's lifetime. We shall see in Section 4 that early investigations of structures related to singular cardinals came very close to discovering elements from pcf theory.

One should recall that the general notion of an ideal was not yet available to Hausdorff, that the Łoś ultraproduct theorem appears much later,[3] that a good part of the ZFC combinatorics Shelah employs was motivated by the combinatorial principles which Ronald Jensen discovered with fine structure theory in Gödel's constructible universe in the 1970s (like $\diamondsuit$, $\square$ and others), and that the most fundamental concept of "covering" was not arrived at in Hausdorff's lifetime. But it is still not hard to imagine Hausdorff discovering some of the basic facts about products of distinct regular cardinals. The sophistication and difficulty of what Hausdorff achieved in his 1909 construction of an (ω_1, ω_1^*)-gap certainly indicates that he would have discovered quite a bit of what there was to discover about infinite products of regular cardinals which arise in the study of singular cardinals,

[3]It is, however, remarkable that Hausdorff [1909] discovered a substantial part of the structure of a reduced product modulo the ideal of finite sets in his construction of an "arithmetic pantachie". See [Plotkin, 2005, p.264].

had he studied the appropriate products.[4]

I will risk stating that it was not mathematical ability that Hausdorff lacked; rather, it was the independence results which blew away the chaff from the wheat in cardinal arithmetic in the 1960s. History shows that singular cardinal arithmetic was pursued only after Cohen's, Solovay's and Easton's results made it absolutely clear that there was no elementary theory of regular cardinal arithmetic.[5]

What is more important, from the historical point of view, is that the impressive research about the structure of the product ω^ω which Hausdorff, Rothberger and later researchers have conducted, whose discoveries were later delineated by the huge body of complementary independence results that followed Cohen's discovery of forcing, created an intuition about the structure of ω^ω whose analog for the structure of the product $\prod_n \omega_n$ were totally misleading. The structural properties of each of these two infinite products are very different from those of the other and the independence phenomenon has a totally different effect in each (as will be described in Section 6). Thus, the influence of the understanding of $(\omega^\omega, <^*)$ on the understanding of $(\prod_n \omega_n, <^*)$ was more by contrast than by analogy.

Sociologically, rather than mathematically, before the method of forcing was discovered and used by Easton to show that apart from weak monotonicity and the König-Zermelo inequality there were no other rules governing regular cardinal exponentiation, attention was not diverted to singular cardinal exponentiation.

After the discovery of forcing and the development of large cardinals theory, the understanding of singular cardinal arithmetic progressed simultaneously in three tracks—consistency results were proved with large cardinal forcing, ZFC results were proved by a variety of methods, first non-elementary, then elementary, and consistency strength was achieved via the inner-model program.

Chronologically, the metamathematical investigations of singular cardinals preceded the development of their elementary theory. Still, in this history the elementary theory is described first. The reader should not forget that Prikry and Silver proved the consistency of the failure of the Singular Cardinal Hypothesis and Magidor proved the consistency of its failure at $\aleph_\omega$ *before* Silver's theorem was discovered (see Section 5.1 below)—this fact will be emphasized below.

Although the modern elementary theory of singular cardinals is accessible, in principle, to Hausdorff or to Cantor, the part played in its development by a variety of sophisticated metamathematical means is significant.

In retrospect, Cantor, König and Zermelo discovered by 1904 all the rules that could be discovered for *regular* cardinal exponentiation. The fact that no more new

[4]I recently heard from Ronald Jensen the following words on this matter: "After Silver's result was obtained I found an elementary proof of it, which I wrote down and circulated. As I was writing the proof I thought, 'Hausdorff could have proved this theorem!' "

[5]This dynamics of set-theoretic discoveries in the modern era is evident also in the working mathematician's private sphere, rather than that of the community at large. I heard from Shelah in the 1980s the view that it is good mathematical practice to try and prove by forcing the *negation* of your intended theorem before trying to prove it. He claimed that many of his ZFC proofs came after attempts to force the consistency of the negation had failed, leaving behind valuable clues about how the proof should go.

rules were discovered between 1904 and the 1970s was not because set theorists were less skilled in these decades, but because there are no other rules to discover. Had Hausdorff known that the König-Zermelo inequality was all that one could prove about regular cardinals, he and others who followed in his footsteps might have turned their attention to investigating singular cardinals.

Among the duties that kept König busy in 1905 as a member of the Hungarian Academy of Science was his appointment to serve on the committee of the newly established Bolyai prize. In that year, to mark the centennial anniversary of János Bolyai, and also partly stirred by the new Swedish Nobel Prize and the new Russian Lubachevsky prize, the Hungarian Academy of Science established the Bolyai prize, to be awarded to an important treatise or book published in the ten years preceding the award.

Two candidates were considered in 1905 for the Bolyai prize: Henri Poincaré and David Hilbert. Poincaré won the 1905 prize, and Hilbert was re-nominated and won the next prize in 1910.[6] The prize regulations required that the prize committees each consist of two Hungarian academy members and two foreign academy members. The two foreign members in 1905 were the Frenchman Jean-Gaston Darboux and the German Felix Klein, and in 1910 they were Poincaré, the 1905 prize winner (who became a foreign academy member after his win) and the Swede Gösta Mittag-Leffler. In both the 1905 and 1910 Bolyai prize committees served the same two Hungarian academy members: Gusztáv Rados and Gyula (Julius) König.

How is all this related to singular cardinals? Well, it is quite surprising that the third winner of the Bolyai prize, in the year 2000, was Saharon Shelah, for his book *Cardinal Arithmetic* [1994a]. The Bolyai prize had been discontinued for almost a century, and was resumed in 2000. The first time it was awarded in the new era was for a monograph on singular cardinals.

The history of the Bolyai prize demarcates the story of singular cardinals in the 20th century. All the story unfolded below happens between the year in which the founder of the theory of infinite products of different regular cardinals served on the Bolyai prize committee and the year 2000, in which Shelah won the prize for his monograph on the study of singular cardinals via the algebraic theory of such products of sets of regular cardinals, called pcf theory.

The two *mathematical* events which demarcate the development of singular cardinals theory in the 20th century would have to be Hausdorff's famous construction of an (ω_1, ω_1^*)-gap inside $(\mathbb{R}^\omega, <^*)$[7] and Shelah's construction of an $\aleph_{\omega+1}$-scale in

[6] The mathematical politics of that time is quite fascinating. It is perhaps worthwhile to mention in the context of this article that Poincaré opposed modern set theory, and declared that "Cantorism was dead", as Hausdorff states in a letter to Hilbert in 1907 (cf. [Plotkin, 2005]). Hilbert, in contrast to Poincaré, was the most prominent defender of set theory from the attacks on which were made, in Hausdorff's words, "by such medieval means!" ([Plotkin, 2005]).

[7] I choose Hausdorff's gap theorem and not König's inequality, because although inequality introduced the products used in the study of singular cardinals, no development followed it for many decades. Hausdorff proved his gap theorem for sequences of real numbers—that is, he worked in the structure $(\mathbb{R}^\omega, <^*)$. It had soon become clear that one could work with $(\omega^\omega, <^*)$

$(\prod_{n\in B}\omega_n,<^*)$ for some infinite set $B\subseteq\omega$. Hausdorff had previously hoped to construct an ω_1-scale and had also previously [1908] constructed such a gap with the aid of CH, and regarded his elimination of CH from the proof as a major achievement by which "the first closer relationship between the continuum and the second number class [i.e. ω_1] would be established". We know in hindsight that this was all the relationship Hausdorff could hope to find between the continuum and ω_1. Shelah, on the other hand, had also first proved the existence of an $\aleph_{\omega+1}$-scale with the aid of an additional assumption (that $2^{\aleph_0}\le\aleph_{\omega+1}$) but continued to discover that this was in fact a ZFC theorem. At that point, the structural analogy between $(\omega^\omega,<^*)$ and $(\prod_n\omega_n,<^*)$ broke down; the independence over ZFC of the existence of an ω_1-scale in the former versus the existence in ZFC of an $\aleph_{\omega+1}$-scale in a sub-product of the latter, made the difference between the theory of ω^ω and the theory of $\prod_n\omega_n$ manifest.

More than 80 years after their discovery, singular cardinals did receive their proper status. By the time Shelah published his book on singular cardinal arithmetic, it was sufficiently clear to everyone that there is no theory of regular cardinal arithmetic, that there was no need for Shelah to title his book "Singular Cardinal Arithmetic"; "Cardinal Arithmetic" was quite enough.[8] The absence of the word "singular" in Shelah's title symbolizes thus the singular cardinals' triumph as a central set-theoretic subject.

3 THE BEGINNING: HAUSDORFF'S WORK

Hausdorff, whose interest in set theory had begun shortly before the 1904 meeting, proceeded after 1904 to advance the theory of Cantor's alephs, and grounded them in the broader setting of linearly ordered sets. It is hard to exaggerate the importance of his 1906–1909 work on order-types. To this day, this work is fundamental.[9]

For our purposes, though, the most important part of this work is Hausdorff's [1906] definition, for the first time, of the *cofinality type* of a linearly ordered set, and the division of cardinals into *regular* cardinals—those whose cofinality types are equal to themselves—and *singular* cardinals, whose cofinality types are smaller than themselves. Here too Hausdorff is being politely aware of König's earlier imprecise use of the word "type" in saying that the continuum could not be well-ordered in any of the types $\aleph_{\beta+\omega}$. The right statement is, of course, that the continuum could not be well-ordered in any type of countable cofinality, which now after Hausdorff's definition is expressible precisely.

or even $(2^\omega,\le^*)$. The latter structure is indeed the one Hausdorff uses in publishing his second construction of a gap in [1936] (see also the English translation in [Plotkin, 2005]). A discussion of these issues in the context of set theory of the real numbers is found in [Steprans, 2010].

[8] Shelah reports that he had considered also the names "Cofinality Arithmetic" or "The Skeleton of Set Theory" for a while.

[9] The whole corpus of Hausdorff's work on ordered sets is fortunately available in English translation in Plotkin's lovely book [2005].

Hessenberg [1906] also defined cofinality independently of Hausdorff in the same year, in a text intended for general readership. Hessenberg uses a different terminology than Hausdorff. He talks about the "kernel" Kern(M) of a linearly ordered set M (a cofinal subset of M).

Hausdorff proved that all successor cardinals are regular, or, equivalently, that all singular cardinals are limit. The modern eye may overlook the fact that Hausdorff did not hesitate to use Zermelo's newly formulated Axiom of Choice, which is today generally accepted as a true axiom, but which at that time was still in dispute. Hausdorff had to use the Axiom of Choice, as without the axiom it may consistently hold that *all* uncountable cardinals are singular, as proved by Moti Gitik [1980] from the assumption of the consistency of existence of a proper class of large cardinals.

Hausdorff was aware of the possibility of a *regular limit* cardinal. He writes in his *Gründzuge der Mengenlehre* about regular limit cardinals ([Hausdorff, 2002, p.131]):

> Wenn es also reguläre Anfangzahlen mit Limesindex gibt (und es is bisher nicht gelungen, in dieser Annahme einen Widerspruch zu entdecken), so is die kleinste unter ihnen von einer so exorbitanten Größe, daß sie für die üblichen Zwecke der Mengenlehre kaum jemals in Betracht kommen wird.

Hausdorff describes the regular limit cardinals "whose existence so far did not lead to any contradiction" as being "of such an exorbitant magnitude that they can hardly be taken into account for the usual purposes of set theory." It is completely clear that Hausdorff does not consider the possibility that the continuum itself may be a regular limit cardinal. He also does not make a distinction between a strong limit regular—a "strongly inaccessible", that is, a limit which is closed under exponentiation—and a regular limit, a limit cardinal whose cofinality is equal to itself.

It is very reasonable to speculate that Hausdorff did not consider the possibility of a singular continuum either. As König proved, the possibility that the continuum was singular of countable cofinality did not exist, but there still remained the (consistent) possibility that the continuum was singular of uncountable cofinality.

This speculation is consistent with Hausdorff's formulation of the Generalized Continuum Hypothesis (GCH) in his [1908], which was the fourth in a sequence of works on order-types. Hausdorff calls GCH *Cantor's Aleph Hypothesis* and formulates it as the statement that $\kappa^{<\kappa} = \kappa$ for every *regular* κ. He warns his readers not to forget that he only requires it for regular cardinals; we know that he himself was aware of the possibility of a regular limit cardinal. He does mention in a footnote that for a singular cardinal μ the equality $\mu = \mu^{<\mu}$ does not hold, and generously credits König's 1905 paper for this fact, although, strictly speaking, König stated this only for singular cardinals of type $\aleph_{\beta+\omega}$, that is, the ω-th successor of some cardinal, whereas the full generality of the statement for all singular cardinals follows from Zermelo's formulation.

It is not clear whether Hausdorff regarded GCH as a "dogma" in the sense that Schoenflies ascribed to Cantor's view of CH (see quotation above). It is quite certain that at the time Hausdorff defines singular cardinal he does not consider the possibility that CH could be false, and in particular does not consider the possibility of a singular continuum. Hausdorff's results in the 1909 paper on order-types illustrated the usefulness of GCH, whose consistency was established by Kurt Gödel in 1937, and it is possible that Hausdorff also regarded GCH as true in the same sense.[10]

Among the striking pioneering results Hausdorff [1908] proved is the result, assuming GCH and AC, that in every infinite cardinal a *universal linearly ordered set* exists.[11] The argument Hausdorff uses is a saturation argument. Model theorists would most likely not declare Hausdorff as the first model theorist, but Hausdorff in fact counts types and realizes them to obtain saturated order-types, many years before the formal birth of model theory. Saturation arguments were formulated and used by Vaught and Morley in the 1950s and are central in model theory, but they appear for the first time in Hausdorff's work on linearly ordered sets.

For limit cardinals Hausdorff introduces the first construction of a "special model", in model-theoretic terminology. In a previous paper he proved the uniqueness of saturated linearly ordered sets and their universality without deciding their existence, and in [1908] establishes the existence of such orderings in all infinite cardinalities from GCH. For this construction, it is important that the limit cardinal at hand is a *strong limit*, that is, closed under the $\beth$ function, which indeed follows from his assumption of GCH. In this way Hausdorff generalized Cantor's treatment of countable order-types and initiated the uniform treatment of the class of all infinite cardinals.

It is important to acknowledge Hausdorff's contribution here to establishing the study of the class of cardinal numbers as standard mathematical object, similar to the way all natural numbers are studied in number theory. In his work on order-types, Hausdorff for the first time proves a structure theorem, one could even say, a model-theoretic theorem, concerning *all* infinite cardinals, *and* for the first time phrases and applies GCH. Similar proofs are now standard in modern algebra, topology, Boolean algebra and so on.

Hausdorff declared the regular well-order-types and their inverses as being "fundamental building blocks", or "atoms" in the structural theory of linearly ordered sets (as, say, is demonstrated by his work on scattered order-types). This is certainly true, but may be misleading about the function cardinals serve in measuring properties of structures (e.g. the weight of a topological space, the cofinality of a *partially* ordered set, etc.). It is unclear to what extent this work of Hausdorff's influenced the general perception of singular cardinals, as most of his achieve-

[10] The use of GCH for combinatorial arguments was made routine by Erdős, who elegantly used this assumption numerous times in his papers. Andras Hajnal coined the humorous acronym "ZFE", "Zermelo-Fraenkel-Erdős", for ZFC + GCH.

[11] A universal linearly ordered set of cardinality $\aleph_\alpha$ is a linearly ordered set of cardinality $\aleph_\alpha$ with the property that every linearly ordered set of cardinality $\aleph_\alpha$ order-embeds into it.

ments here were not included in his famous and very influential book *Grundzüge der Mengelehre* (see below).

Another aspect of Hausdorff's use of CH is quite modern and merits attention. The Continuum Hypothesis is the equation $2^{\aleph_0} = \aleph_1$, which asserts that the cardinality of the real line ($2^{\aleph_0}$) is equal to that of the first uncountable ordinal. This equation has since been used often for deriving consequences about the structure of the real line and its many related structures. The Polish school, mainly Sierpiński, excelled in this subject. It was much later that uses of CH (and more generally, of GCH) became popular for proving (combinatorial, model-theoretic, algebraic) theorems about the ordinals themselves in branches of modern mathematics in which the notion of an infinite cardinal played a central role, like infinite Abelian group, infinite combinatorics and model theory. Hausdorff's construction of a universal linearly ordered set of cardinality $\aleph_1$ is an early and rare example in which CH is used to prove a property of $\aleph_1$, rather than of $\mathbb{R}$.

Finally, another remarkable feature of Hausdorff's work with CH is that he obtained the first, and what would remain forever a rare instance of a "verifiable consequence" of CH. Let us recall Gödel's [1947] objection to CH as not having enough "verifiable consequences", that is, consequences which turn out to be ZFC theorems. Among the myriad consequences of CH listed in Sierpinski's book [1934], almost none proved to be a ZFC theorem.[12] Hausdorff first constructed an (ω_1, ω_1^*)-gap with CH, and then was able to prove the existence of such a gap in ZFC. It is now known that he could not hope to construct any other gap in ZFC! The relation between ω_1 and $2^{\aleph_0}$ stands, of course, in strong contrast to the relation between $(\aleph_\omega)^{\aleph_0}$ and $\aleph_{\omega+1}$, as in ZFC there exists an $\aleph_{\omega+1}$-scale inside $(\prod_n \omega_n)$ (see below for further discussion).

In 1914 Hausdorff published his very influential *Grundzüge der Mengenlehre*, which was dedicated to "the creator of set theory, Herr Georg Cantor". In this surprisingly concise volume Hausdorff presents the basics of modern set theory, Borel sets, metric space theory, measure theory and many other subjects that have become indispensable for modern mathematics. But no extensive treatment of singular cardinals is given. His theorem about universal order-types in all cardinals from GCH, for one example, stayed outside of this book (see [Plotkin, 2005] for further discussion of this issue). It could be that Hausdorff did not consider singular cardinals as sufficiently relevant to the central problem of set theory—Cantor's Continuum Hypothesis. Although Hausdorff himself defined the singular cardinals rigorously and made important contributions to their theory, his influential volume on set theory maintained the early approach to singular cardinals as a marginal subject.

[12] I know of exactly one: number 46, communicated to me by E. Grzegorek (cf. his [1985]).

4 EARLY OCCURRENCES OF SINGULAR CARDINALS IN MATHEMATICS

Cardinals are more than just abstract objects which one studies for their own sake. They are used to measure the size, and other properties, of mathematical structures. One expects that the arithmetical properties of the cardinality or other cardinal characteristics of a mathematical structure provide information about the structure itself. In the case of finite structures, this is hardly surprising. Recall, for instance, that a finite group whose cardinality divisible by a prime p has p-subgroups or that if the cardinality of a finite group is a power of a prime then the group must have a non-trivial center—that is, the cardinality of a finite group provides information about the group's algebraic structure.

Long before the elementary arithmetic of singular cardinals was discovered, certain arithmetical properties of singular cardinals surfaced in various well-known classes of mathematical structures—topological spaces, Abelian groups and Boolean algebras are some of the examples—as holding significant structural information. All these occurrences of singular cardinals are understood better in retrospect from the point of view of the elementary theory of singular cardinals, but it is interesting to assess them in the context in which they emerged.

We shall review a limited number of such occurrences in chronological order, leaving out many others, perhaps just as interesting and important. It appears that a singular cardinal characteristic of a structure was often more informative than a regular one. In some of the examples we examine in this section one may feel that singular cardinals were treated with a certain resentment, even when they occurred quite naturally. An exception to that general impression is found in the work of the Hungarian school in infinite combinatorics—Erdős, Rado, Hajnal and others.

4.1 Singular cardinals in topology: the work of Alexandrov and Urysohn

Two of the many young mathematicians who read Hausdorff's newly published *Grundzüge* carefully were Pavel Alexandrov and Pavel Urysohn from the new Moscow School of mathematics.[13] Their interest in the notion of *topological compactness*, which was formulated by Fréchet and presented in Hausdorff's book, led them to reformulate this notion by quantifying over the class of infinite cardinals in their *Memoirs* [1929].

A point of complete accumulation of a subset $A \subseteq X$ is a point $x \in X$ each of whose open neighborhoods $u \ni x$ satisfies $|A| = |A \cap u|$. Alexandrov and Urysohn

[13] About the "Lusitania" group in Moscow, lead by Egorov and Luzin, and the development of descriptive set theory, see [Graham and Kantor, 2009]. As part of the Lusitania group's excessive interest in names, each member in the club was named by some aleph. "Recruits were called $\aleph_0$. ... Alexandrov and Urysohn soon achieved the high rank of $\aleph_5$. Luzin himself was given the name $\aleph_{17}$. Egorov was $\aleph_\omega$." [Graham and Kantor, 2009, p.116]

observed that for every topological space X the following three properties are equivalent, and called a space which satisfied them "bicompact":

A Every infinite subset has a point of complete accumulation.

B Every well-ordered decreasing intersection of nonempty closed sets has a nonempty intersection.

C Every open cover has a finite subcover.

Condition C is nowadays called "compactness". This term was used then for what now is called "countable compactness" (the property that every infinite set has an accumulation point). Condition B is of course equivalent to the statement that every open cover that is linearly ordered by inclusion has a finite subcover.

A reasonable strategy for proving the equivalence was to try to prove it *separately* for each cardinal λ being the relevant parameter. However, this equivalence for a single infinite cardinal λ holds only if λ is a *regular* cardinal. To prove the equivalence between A, B, and C as stated, that is, *for all cardinals together*, they considered for each of the conditions A, B and C its relativization to the class of regular cardinals A^r, B^r and C^r and actually prove the equivalence of all six conditions.

Using their equivalence and the well-ordering of the cardinals, Alexandrov and Urysohn obtain the following elegant equivalent definition of topological compactness: a space X is compact if and only if for every *regular* cardinal λ, X satisfies the λ-complete accumulation point property. This result is very modern in flavor. In particular, the Alexandrov-Urysohn characterization of topological compactness in terms of complete accumulation points allowed them to ignore sets of points of singular cardinality: a space X is compact iff every infinite subset of X of regular cardinality has a complete accumulation point iff every infinite subset—of regular or of singular cardinality—has a complete accumulation point. This was actually convenient, as the standard operations one does with countable *sequences* in a topological space generalize to sequences of regular length, but not to sequences of singular length (e.g., that a sequence can be thinned out to either a constant or a 1-1 sequence of the same order-type). Thus, not having to deal with sequences of singular length was an advantage.

Thus, this set-theoretic characterization of compactness is a very interesting early example of "elimination of singulars" from the natural mathematical discourse. Alexandrov and Urysohn skip over the singular cardinals elegantly. An open cover of singular cardinality μ can be made, by unionizing, an increasing cover of regular cardinality $\mathrm{cf}(\mu)$. The complete accumulation point property for $\mathrm{cf}(\mu)$ implies that the unionized cover has a strictly smaller subcover, and hence also the original cover has a strictly smaller subcover. The equation $\mu = \sum_{\alpha<\mathrm{cf}(\mu)} \mu_\alpha$ is found useful for the calculus of accumulation points.

As Juhász and Szentmiklóssy point out in their [2009], the property of having complete accumulation points satisfies a sort of additivity: if $\mu = \sum_{i<\kappa} \lambda_i$ where κ is regular and $\langle \lambda_i : i < \kappa \rangle$ is an increasing sequence of regular cardinals, then whenever a space X satisfies the complete accumulation point condition for κ and for all λ_i then it satisfies it also for μ. There is no need to go through the language of open covers. Thus, if the accumulation point condition holds for all *regular* cardinals, it holds for all cardinals.

For a topological space X let $\text{CAP}(X)$ be the class of all cardinals λ for which X satisfies the λ-complete accumulation point property. What Alexandrov and Urysohn discovered was that $\text{CAP}(X)$ is "closed under singular sums", that is, if κ and each κ_α for $\alpha < \kappa$ belong to $\text{CAP}(X)$ then also $\sum_{\alpha<\kappa} \kappa_\alpha$ belongs to $\text{CAP}(X)$.

It is quite ironic that in their elimination of singular cardinals from the discussion of accumulation points Alexandrov and Urysohn indirectly prove an important topological property of singular cardinals that rests on their arithmetical structure. One can compare their discovery to Euler's discovery that if every prime number is a sum of four squares, then also every composite positive integer is. In this analogy the regular cardinals correspond to primes, and the singular cardinals to composites. However, Euler did not think (I hope) that composite numbers were of secondary importance! In fact, the desired theorem, eventually proved by Lagrange, was that *every* positive integer was a sum of four squares. It seems that this point of view is missing in the *Memoirs* about the class of cardinal numbers.

This example, discovered by Alexandrov and Urysohn, is the first in a line of many examples of structural properties indexed by a singular cardinal which are inherited from smaller regular cardinals. Cardinal exponentiation, the freeness of an Abelian group and several other topological properties demonstrate that the fact a singular cardinal is built from smaller ones is significant to many mathematical structures of such cardinality.

Next in their discussion of compactness Alexandrov and Urysohn consider a localization of their complete accumulation point condition. They consider *finally compact* spaces, that is, spaces in which the complete accumulation point condition holds for all regular cardinals in an end-segment of the cardinals, and pay special attention to the case in which all uncountable regular cardinals are considered. It is right here that their elimination of singular cardinals fights back. They are unable to prove the full correspondence they had before, and have to settle for a weaker implication. To the modern eye, the reason is clear: eliminating the regular cardinal $\aleph_0$ influences the topological behavior of subsets of cardinality μ for all μ with countable cofinality, not only $\aleph_0$.

However, if one held the view that only regular cardinals mattered—a view that the characterization of compactness supported—then it made sense to conjecture that if the CAP condition held for all regular uncountable cardinals then every open cover could be reduced to a countable subcover, that is, that this property of a space X was equivalent to the *Lindelöf property*.[14]

If, indeed, $\aleph_0$ is removed from the class of *all* infinite cardinals, that is, X is required to satisfy the λ-complete accumulation point property for *every* $\lambda > \aleph_0$, then indeed X is Lindelöf (as the implication from accumulation points to smaller

[14] A topological space X satisfies the Lindelöf property if each of its open covers has a countable subcover.

subcovers does not require regularity). However, this property is too strong to *characterize* Lindelöfness, as there are Lindelöf spaces that do not satisfy the complete accumulation point for singular cardinals.[15] But as they required this only for *regular* $\lambda > \aleph_0$, Alexandrov and Urysohn were only able to prove that a space X which satisfies this requirement is *linearly Lindelöf*, that, is, each of its linearly ordered covers has a countable subcover.

The question of whether linearly Lindelöf and Lindelöf are the same remained unanswered for a long time. Alexandrov and Urysohn did not come up with an example of a space X which satisfied the complete accumulation point condition for every regular $\lambda > \aleph_0$ but failed to have the Lindelöf property. If such an example existed, then in it the iterated process of shrinking covers, which Alexandrov and Urysohn described, not to reach a subcover of cardinality $\aleph_0$, it would have to stop at a *singular* of countable cofinality. Thus, the topological weight[16] of such a space would have to be at least $\aleph_\omega$.

Such a non-Lindelöf space was indeed discovered in 1959 by the Russian mathematician A. Miščenko [1962]. His space was a subspace of the product $\prod_n (\omega_n + 1)$ with the product topology. In other words, Miščenko used a singular cardinal to construct his example, in fact, a product of distinct regular cardinals. An even easier example is provided in unpublished work of Gruenhage and Buzyakova from the 1990s: Let X be the space of all functions in $2^{\aleph_\omega}$ which satisfy $|f^{-1}(1)| < \aleph_\omega$, with the topology induced from standard product topology on $2^{\aleph_\omega}$. This space is linearly Lindelöf but not Lindelöf.

For our purposes, this historic development illustrates simultaneously the naturalness and fundamentality of singular cardinals on the one hand and the resistance to acknowledge them on the other hand. Linearly Lindelöf spaces are investigated in general topology and their properties are compared with those of Lindelöf spaces. The main point is that a topologist who is interested in the Lindelöf property will inevitably be driven to consider spaces of singular weight—a fact that at first glance may not seem very natural.

Quite recently, Juhász and Szentmiklóssy [2009] proved the following result: if a space X satisfies the complete accumulation point condition for all regular *uncountable* cardinals and in addition for $\aleph_\omega$, the smallest singular cardinal, then X satisfies the condition for *all* $\lambda > \aleph_0$ and is hence Lindelöf. The only tool they use which Alexandrov and Urysohn did not have is Shelah's pcf scales theorem. This is a good example of how very early results about singular cardinals in topology were continued both naturally and by elementary means after a remarkably long pause.

4.2 More topology in products: Rudin's space

Miščenko's space motivated the famous construction of a Dowker space in ZFC by Mary Ellen Rudin [1971]. The problem of Dowker spaces began with a homotopy

[15] Arhangelskii [2008] believes that this was known to Alexandrov and Urysohn.
[16] The *weight* of a topological space X is the smallest cardinality of a topological base for X.

extension theorem due to Karol Borsuk [1936] which uses the following assumption about a normal space[17] X: the product $X \times [0,1]$ is also normal. At the time, Borsuk did not know whether this indeed was an additional assumption. Perhaps all normal spaces X satisfied that their product with the closed unit interval is also normal.

In 1951 the British topologist Clifford Dowker [1951] gave an internal characterization of the normality of $X \times [0,1]$. For a normal X, $X \times [0,1]$ is normal if and only if X is countably paracompact.[18] The name "Dowker space" became a standard name for normal yet not countably paracompact spaces. The existence of such spaces was still unknown at the time Dowker obtained this characterization.

The first Dowker space in ZFC was constructed by Rudin in 1969 (cf. [1971]) and for over 20 years was the only known ZFC example. Rudin was motivated by Miščenko's construction of the subspace of $\prod_n(\omega_n + 1)$. Miščenko raised in his paper the question of whether his space was Dowker or not. Very soon it was observed by many that Miščenko's space was not normal, hence not a Dowker space.

Rudin made the following modifications to Miščenko's construction: she used only functions whose values are everywhere of uncountable cofinality, and she replaced the product topology by the box product topology.[19] Now she was able to prove normality, making essential use of the fact that her space was a *P-space*, that is, satisfied that any countable intersection of open sets was open. She also proved that her space was not countably paracompact by exposing a combinatorial property of unbounded subsets of her partially ordered space.

The cardinality of Rudin's Dowker space is $(\aleph_\omega)^{\aleph_0}$. For many years, Rudin herself promoted the "small Dowker space problem", that is, of finding a ZFC example of a Dowker space with "small" cardinal characteristics, like cardinality, weight and local character. She called her own space, of cardinality and weight $(\aleph_\omega)^{\aleph_0}$ and local character $\aleph_\omega$, "huge" and "ugly". When a ZFC Dowker space of cardinality $2^{\aleph_0}$ and weight $2^{2^{\aleph_0}}$ was constructed by Zoltan Balogh [1996] in a paper titled "A small Dowker space in ZFC" it was accepted by Rudin and many others in the set-theoretic topology community as a solution of the small Dowker space problem. Was the continuum provably smaller that $(\aleph_\omega)^{\aleph_0}$? Even after it was known that the continuum was not bounded in ZFC, and could be, for example, a regular limit cardinal, still the opinion prevailed that $2^{\aleph_0}$ was "small", while $\aleph_\omega$ was "large".

What is absolutely remarkable about Rudin's construction is that Rudin's clear resistance to singular cardinals did not stop her from implicitly discovering several important ingredients of the systematic theory of small products of regular

[17] A topological space X is *normal* if any two closed and disjoint subsets of X can be separated by two open sets.

[18] A normal space X is *countably paracompact* if for every sequence $\{D_n\}_n$ of closed subsets of X which satisfies $\bigcap_n D_n = \emptyset$ there are open sets $U_n \supseteq D_n$ such that $\bigcap_n U_n = \emptyset$.

[19] The *box product topology* on a product $\prod_{i \in I} X_i$ is the topology generated by all sets of the form $\prod_{i \in I} u_i$ where $u_i \subseteq X_i$ is open.

cardinals. Looking carefully through her proof of the failure of countable paracompactness, one can trace elements of pcf theory appearing. In fact, neither normality nor the absence of countable paracompactness required the use of *all* functions in the product. The Rudin Dowker space contains a closed Dowker subspace of cardinality $\aleph_{\omega+1}$ [Kojman and Shelah, 1998], smaller in cardinality than all but the first ω members of the class of possible values of the continuum.

The interested reader may read more about singular cardinals in topology and in Boolean algebra in Juhász's book [1980] and in Monk's book [1996], both dealing with cardinal characteristics. Singular cardinals often play a different role in this setting than regular cardinals—we shall mention as a final example just one result, due to Erdős and Tarski [1943], that a singular cellularity[20] of a topological space or of a complete Boolean algebra is always attained.

4.3 The Czech school's investigations of the algebra $\mathcal{P}_{<\mu}(\mu)$

A cardinal number is also referred to in the set-theoretic literature since Hausdorff's time as an "initial ordinal", that is, an ordinal which is not equinumerous with any smaller ordinal. A cardinal is, then, an ordinal which possesses "new" properties, not possessed by smaller ordinals.

It makes sense to discover the *genuinely new* information stored in *the power set* $\mathcal{P}(\kappa)$ of a cardinal κ by modding-out all smaller subsets of κ and inspecting the the quotient Boolean algebra. Let $\mathcal{P}(\kappa)/{<}\kappa$ denote the *completion* of the quotient algebra of $\mathcal{P}(\kappa)$ over the ideal of all subsets of κ of cardinality smaller than κ. For the case $\kappa = \aleph_0$ this algebra corresponds to $\beta\omega \setminus \omega$, where $\beta\omega$ is the *Čech-Stone* compactification of ω. These Boolean algebras were addressed by the Czech school of set theory, and several interesting achievements were obtained, some of which are highly relevant to singular cardinals.

Among the cardinal characteristics studied in this setting one finds the *distributivity number* of $\mathcal{P}(\kappa)/{<}\kappa$, which, in forcing terminology, is the smallest cardinality of a new sequence of ordinals which is added by extending the universe of sets with the Boolean algebra as a forcing notion. In 1972 Bohuslav Balcar and Petr Vopěnka began the study of distributivity in quotient algebras $\mathcal{P}(\kappa)/{<}\kappa$ for infinite cardinals κ. Their first interesting discovery about singular cardinals was quit surprising then. While for regular cardinals κ the distributivity number of $\mathcal{P}(\kappa)/{<}\kappa$ was undecidable over ZFC, Balcar and Vopěnka [1972] proved that for *singular* cardinals μ the distributivity number of $\mathcal{P}(\mu)/{<}\mu$ is completely determined in ZFC by the cofinality of μ as follows: for μ of countable cofinality the distributivity number is ω_1 and for μ of uncountable cofinality it is ω. Additional parameters of distributivity in $\mathcal{P}(\mu)/{<}\mu$ for a singular μ were computed in a series of papers [Balcar and Franěk, 1987, Balcar and Simon, 1989, Balcar and Simon, 1988], usually under additional set-theoretic assumptions.

[20] The *cellularity* of a topological space is the supremum over all cardinalities of families of pairwise disjoint nonempty open sets of the space.

The Balcar-Vopěnka discovery was made already at a time when the intuition about cardinal characteristics of the continuum emerged that almost everything about them was independent of the axioms. Thus, it was remarkable that the naturally derived structures $\mathcal{P}(\mu)/{<}\mu$ are much more stable for singular cardinals than for regular cardinals.

In their [1987], Balcar and Franěk were able to show that under certain additional axioms, the quotient algebra $\mathcal{P}(\mu)/{<}\mu$ was actually isomorphic to a collapse algebra $\mathrm{Col}(\omega_1, \mu^{\aleph_0})$:[21] in the case that μ had countable cofinality the algebra $\mathcal{P}(\mu)/{<}\mu$ was isomorphic to $\mathrm{Col}(\omega_1, \mu^{\aleph_0})$ and in the other case to $\mathrm{Col}(\omega, \mu^{+})$.

The question may have arisen naturally whether these two algebras could be isomorphic just in ZFC, if not for an earlier independence result due to Baumgartner [1973]. By Baumgartner's result on almost disjoint families a negative *consistent* answer was given to the isomorphism question: it was consistent that these two algebras had different cellularities, or, in more detail, that $\mathcal{P}(\mu)/{<}\mu$ has strictly bigger cellularity than that of $\mathrm{Col}(\omega_1, \mu^{\aleph_0})$ for a singular μ of countable cofinality. The higher cellularity, one should point out, resulted from subsets of μ of regular cardinality larger than ω.

Balcar and Simon [1995] then conjectured that it would turn out in ZFC that $\mathrm{Col}(\omega_1, \mu^{\aleph_0})$ was isomorphic to a *complete subalgebra* of $\mathcal{P}(\mu)/{<}\mu$. This was proved by them with various additional axioms, and was finally settled just in ZFC by Kojman and Shelah [2001] with pcf theory.

As was the case with the topological properties of Rudin's space, after the discovery of pcf theory it turned out that some of the Boolean-algebraic properties of $\mathcal{P}(\mu)/{<}\mu$ for a singular μ are not that related to the usual power-set function $\kappa \mapsto 2^{\kappa}$ but rather closely related to the "binomial" function on singular cardinals (see Section 5 below).

4.4 The Erdős-Rado work in the partition calculus and the Erdős-Hechler work on MAD families over a singular

Paul Erdős was one of the first mathematicians to discover the charm of singular cardinals. The combinatorial arguments he employed in his considerations of singular cardinals were often very elegant. There are two other characteristics of Erdős' approach to infinite combinatorics. The first is his insistence *not* to exclude singular cardinals from the general formulation of a theorem. The other is his readiness to assume GCH for counting arguments.

Among the early results in partition calculus one finds the Erdős-Dushnik-Miller [1941] partition relation, valid for *every* infinite cardinal κ:

$$\kappa \to (\kappa, \omega)^2.$$

[21] The collapse algebra $\mathrm{Col}(\omega_1, \kappa)$ is the unique complete Boolean algebra in which the partially ordered set of all countable partial functions from ω_1 to κ is densely embedded. This poset is the standard forcing notion for collapsing the cardinal κ to ω_1.

This symbol means that for every partition of unordered pairs from κ to two parts, if there is no infinite subset of κ all of whose pairs lie in the first part, then there exists a subset of κ of cardinality κ all of whose pairs lie in the other part.

Dushnik and Miller [1941] were interested in this relation for the purpose of showing that any two enumerations of a set in the order-type of its cardinality agree on a large set (which follows from this relation trivially), and were able to prove it for *regular* cardinals. Erdős provided the elegant argument for the *singular* case.

Erdős and Rado begin their monumental work *A Partition calculus in set theory* [1956] by quoting previous results. The very first is Ramsey's theorem. Then comes the relation by Dushnik and Miller above. The third is a negative[22] partition relation:

$$\aleph_{\omega_0} \not\to (\aleph_1, \aleph_{\omega_0})^2.$$

All along the way singular cardinals are treated in this work as equal to their fellow regular cardinals and many a time the combinatorial argument required for a singular case is actually prettier than in the regular case.

It is interesting to point out that a "polarized partition relation" which was proved for singular cardinals of countable cofinality in [Erdős *et al.*, 1965] and was left open for singular cardinals of uncountable cofinality (even with GCH) was finally proved by Shelah [1998] to follow from the assumption that the singular cardinal is a strong limit but its exponent is larger that its successor.[23]

Another beautiful example from the Hungarian school of infinite combinatorics is Hajnal's free set theorem. Erdős' review of Hajnal's paper tells the story completely:

> Let S be a set of power m. To every $X \in S$ there corresponds a subset $f(X)$ of S. We assume that there exists a cardinal number $n < m$ so that for every X, $f(X)$ has power less than m. A subset S' of S is called free if for every every X, $Y \in S'$, $X \notin f(Y)$ and $Y \notin f(X)$.
>
> Ruziewicz conjectured that there always exists a free subset of power m. The first question of this type is due to P. Turán and the first positive results are due to G. Grünwald and D. Lázár. If m is regular the conjecture was first proved by Sierpiński and if m is cofinal to ω it was proved by S. Piccard. Assuming the generalised continuum hypothesis the reviewer proved the conjecture in full generality. The author now proves the conjecture without any hypothesis in a surprisingly simple and ingenious way.

The theorem was first proved for regular cardinals, then for singular cardinals with cofinality ω, then for all cardinals ("in full generality") with GCH (by Erdős himself) and finally in ZFC for all cardinals. Obviously, it was the singular cardinals

[22] The negation line on the arrow means that the partition relation expressed by the arrow notation does not hold.

[23] That is, from the *negation* of the Singular Cardinal Hypothesis, for which see the next section.

case that required more ingenuity than the regular cardinal case, and, also obviously, it was not called "the free set theorem" before *all* cardinals were covered in ZFC.

Erdős and Steven Hechler have studied maximal almost disjoint families over a singular cardinal in their [1975]. For a singular cardinal μ denote by $\mathrm{MAD}(\mu)$ the collection of cardinalities of *maximal almost disjoint* families over μ, that is, cardinalities of families $\mathcal{F} \subseteq \mathcal{P}(\mu)$ satisfying that each member of the family has cardinality μ, the intersection of any two members in the family has cardinality smaller than μ and every set $A \subseteq \mu$ of cardinality μ has intersection of size μ with some member of the family. For obvious reasons, families whose cardinality is smaller than $\mathrm{cf}(\mu)$ are not considered.

Erdős and Hechler began their investigations with an observation that resembles very much the additivity of the complete accumulation point spectrum described in section 4.1 above, and with pretty much the "same" proof: they proved that $\mathrm{MAD}(\mu)$ is closed under singular suprema. That is, if $\{\lambda_i : i < \theta\} \subseteq \mathrm{MAD}(\mu)$ and $\lambda := \sum_{i<\theta} \lambda_i > \theta$, then $\lambda \in \mathrm{MAD}(\mu)$. Using this, they proved that for a singular μ it could be that $\mu \in \mathrm{MAD}(\mu)$—a relation which clearly cannot hold for a regular cardinal! The simplest case to state is that if $2^{\aleph_0} < \aleph_\omega$ then every cardinal in the interval $[2^{\aleph_0}, \aleph_\omega]$ belongs to $\mathrm{MAD}(\aleph_\omega)$.

Then Erdős and Hechler raised the question of whether their assumption was necessary, and in fact conjectured that $2^{\aleph_0} > \aleph_\omega$ together with Martin's Axiom would imply that $\aleph_\omega \notin \mathrm{MAD}(\aleph_\omega)$. Their conjecture was answered positively in [Kojman *et al.*, 2004], in which new ways to control $\mathrm{MAD}(\mu)$ for a singular μ were introduced using pcf theory. The Erdős-Hechler additivity method works plainly only up to μ itself; with pcf theory one can get almost-increasing *smooth* unions up to some cardinal larger than μ, and prove that the whole interval of cardinals from $\min \mathrm{MAD}(\mu)$ to that cardinal is contained in $\mathrm{MAD}(\mu)$.

4.5 Singular cardinal compactness and Whitehead's problem

The algebraic compactness that Shelah discovered and which shall be described now is a phenomenon which is unique to singular cardinals. Let us give a simple example: if an Abelian group G is of singular cardinality μ and G is *almost free*, that is, each of the subgroups of G of cardinality smaller than $|G|$ is free, then G itself is free. This implication does not hold, in general, for Abelian groups of regular cardinality. In the constructible universe, for example, this phenomenon of algebraic compactness does not hold for any regular cardinal other than the weakly compact cardinals, hence consistently does not hold for any regular cardinal at all. The implication "almost free" $\Longrightarrow$ "free" holds for all cardinals—regular or singular—above a compact cardinal.

The discovery of singular cardinal compactness was related to Shelah's solution of *Whitehead's problem* in infinite Abelian groups. Whitehead's problem in infinite Abelian group theory is the following: is every Whitehead group a free Abelian group? An Abelian group G is Whitehead if $\mathrm{Ext}(G, \mathbb{Z}) = 0$, that is, the functor

Ext on G is trivial. Every free Abelian group is Whitehead, and Whitehead's problem, which stood open for many years, was whether the converse implication is also true.

Shelah constructed a non-free Whitehead group from MA($\aleph_1$), and thus showed that the implication "Whitehead" $\Longrightarrow$ "free" could not be proved in ZFC (see the discussion in [Steprans, 2010] of the role of this development for the set theory of the continuum). Could this implication be *refuted* in ZFC? Since Shelah's counter-example utilized the additional axiom MA($\aleph_1$), it did not produce a refutation in ZFC.

Shelah proved that from Gödel's axiom of constructibility it followed that every Whitehead group is free, hence that Whitehead's implication cannot be refuted in ZFC. The proof was by induction on cardinality, utilizing the fact that the Whitehead property is hereditary. Given a Whitehead group of cardinality λ, each of its subgroups of smaller cardinality is Whitehead as a subgroup of a Whitehead group, and by the induction hypothesis is therefore free. For a regular cardinal λ, the constructibility axiom is used to assure that also G itself is free. But in the case λ is singular, Shelah discovered a very interesting fact: *every* Abelian group of singular cardinality all of whose subgroups of smaller cardinality are free, is itself free—just in ZFC.

A previous theorem by Paul Hill stated that a group of cardinality μ which is almost free is free whenever μ is singular of cofinality ω_0 or ω_1. It had also been known that every countable subgroup of $\mathbb{Z}^\omega$ is free, but that there are subgroups of $\mathbb{Z}^\omega$ of cardinality $\aleph_1$ which are not free (cf. [Eklof and Mekler, 2002]).

In fact Shelah has done much more. He phrased a simple set of axioms for an abstract notion of freeness, axioms which freeness in Abelian groups satisfies, and proved his *Singular Cardinals Compactness* theorem [Shelah, 1975]: Every structure of singular cardinality which is almost free is free. Many other natural problems fell under the scope of Shelah's set of axioms for freeness.

A reader who is interested in almost-freenes may want to look at the book [Eklof and Mekler, 2002] on this subject.

5 THE ARITHMETIC OF SINGULAR CARDINALS

Cantor and Hessenberg proved that for every infinite cardinal κ the equality $2^\kappa = \kappa^\kappa$, which implies trivially that whenever $\lambda \leq \kappa$ the exponentiation λ^κ is equal to 2^κ. Thus, exponentiations in which the base is not larger than the exponent, reduced to exponentiations with base 2.

The König-Zermelo inequality showed that for a singular μ, $\mu^{\mathrm{cf}(\mu)} > \mu$. Thus, there are nontrivial exponentiations in which the base is larger than the exponent. Zermelo proved in general that whenever $\lambda = \sum_{i<\theta} \lambda_i$ with the λ_i are increasing with i, the product $\prod_i \lambda_i$ has larger cardinality than λ.

Before continuing with the arithmetic of singular cardinals, it is important to note that the "usual" power set function $\kappa \mapsto 2^\kappa$ is not always sufficiently informative for understanding singular cardinals. Suppose that $\lambda = \sum_{i<\theta} \lambda_i$ where θ is regular and the λ_i are increasing. By the usual distributive law, known to Cantor, it follows that

$$2^{\sum_{i<\theta} \lambda_i} = \prod_{i<\theta} 2^{\lambda_i}. \tag{4}$$

In other words, the power set of a singular cardinal is itself representable as a product of the sort considered by König. However, the cardinalities of such products have lives of their own. While in the case of a product of *uncountably* many different cardinals there is some influence on the product from the exponentiations of its factors, in countable products this is more delicate. In particular, the first fixed point of the aleph function μ may be a strong limit and have an arbitrary large 2^μ.

The equation (4) above constituted an observation by Lev Bukovský [1965] and independently by Hechler (see [Jech, 2003]) that if in this case the function $\kappa \mapsto 2^\kappa$ is eventually constant below a singular, then its value persists at the singular itself.

Bukovský in fact proved that all exponentiations are reducible to the *gimmel function* given by $\kappa \mapsto \kappa^{\mathrm{cf}(\kappa)}$. For regular cardinals the gimmel function is exactly the power-set function, and for singular cardinals it is again related to König-type products. If one knows what are the values of 2^κ for regular κ and what are the values of $\mu^{\mathrm{cf}(\mu)}$ for singular μ, then one knows what is κ^λ for every κ and λ.

This indicates that for singular cardinals μ it is not the function $\mu \mapsto 2^\mu$ which matters, but rather $\mu \mapsto \mu^{\mathrm{cf}(\mu)}$. What can be said about this function? Well, the cardinal $\mu^{\mathrm{cf}(\mu)}$ is clearly the total number of functions from $\mathrm{cf}(\mu)$ to μ, which is equal to the total number of $\mathrm{cf}(\mu)$-subsets of μ, that is $|[\mu]^{\mathrm{cf}(\mu)}|$.

Let us now consider a different function, which Cantor neglected to introduce: the binomial $\binom{\mu}{\mathrm{cf}(\mu)}$ which we define as the *cofinality* of the partially ordered set $([\mu]^{\mathrm{cf}(\mu)}, \subseteq)$. This is the smallest cardinality of a collection of $\mathrm{cf}(\mu)$-subsets of μ which *covers* all of $[\mu]^{\mathrm{cf}(\mu)}$. It is justified to call this function "the binomial" because if you apply it to natural numbers $k \leq n$ it gives exactly the usual binomial (because two different k-subsets of n do not cover each other).

It is obvious that $\mu^{\mathrm{cf}(\mu)} = \max\{2^{\mathrm{cf}(\mu)}, \binom{\mu}{\mathrm{cf}(\mu)}\}$. Thus, one can use the binomial for singular cardinals rather than the gimmel function for determining all exponentiations.

König's original argument needs to be only slightly modified to prove, for a singular μ, that

$$\binom{\mu}{\mathrm{cf}(\mu)} > \mu;$$

in fact it gives more, namely that the almost disjointness number of $[\mu]^{\mathrm{cf}(\mu)}$ is larger than μ.

The Singular Cardinals Hypothesis (SCH) is, basically, the statement that exponentiation of singular cardinals assumes the smallest possible value. It was first phrased in a special way as a conditional:

$$\bigwedge_{\kappa<\mu} 2^\kappa < \mu \Longrightarrow 2^\mu = \mu^+.$$

We find it in this form in Azriel Levy's review of [1970]:

> The corresponding question concerning the singular $\aleph_\alpha$'s is still open, and seems to be one of the most difficult open problems of set theory in the post-Cohen era. It is, e.g., unknown whether for all $n(n < \omega \to 2^{\aleph_n} = \aleph_{n+1})$ implies $2^{\aleph_\omega} = \aleph_{\omega+1}$ or not.

This formulation says nothing when μ is not a strong limit.

Then SCH was phrased in a more general way, which makes sense also in when μ is not a strong limit (see, e.g. [Jech, 2003, p.61]):

$$2^{\mathrm{cf}(\mu)} < \mu \Longrightarrow \mu^{\mathrm{cf}(\mu)} = \mu^+.$$

It can be put equivalently as $\mu^{\mathrm{cf}(\mu)} = \max\{\mu^+, 2^{\mathrm{cf}(\mu)}\}$, since for any cardinal $\kappa < \mu$ it holds that if $2^\kappa > \mu$ then $\mu^\kappa = 2^\kappa$. This formulation allows that $2^{(\mathrm{cf}(\mu))^+} > \mu$ while remaining meaningful. When $2^{\mathrm{cf}(\mu)} > \mu$, this formulation says nothing.

However, there is no need to make any conditions about $2^{\mathrm{cf}(\mu)}$ either: the Binomial Hypothesis (BH) is simply

$$\binom{\mu}{\mathrm{cf}(\mu)} = \mu^+.$$

This statement is meaningful also in the case $2^{\mathrm{cf}(\mu)}$ is larger than μ.

This separation of cardinal arithmetic to the exponent function on regular cardinals and to the binomial on singular cardinals enables the separate study of each of the functions independently of the other. Thus, for example, one may ask whether the Binomial Hypothesis for $\aleph_\omega$ holds or not regardless of what the continuum is. (One may observe that adding any number of Cohen reals, for example, has absolutely no effect on $\binom{\aleph_\omega}{\omega}$.)

The gimmel function was not dealt with before Bukovský [1965]. The function $\kappa \mapsto 2^\kappa$ for regular κ was completely understood with Easton's results, that followed Cohen and Solovay. Apart from the König-Zermelo inequality and weak monotonicity there are no rules governing $\kappa \mapsto 2^\kappa$ *on the regular cardinals.*

This created a drastically wrong intuition about the Singular Cardinals Hypothesis. Mathematicians in the 1970s expected that Easton's results would be extended to the singular cardinals. The truth turned out to be totally different. The following table, incorporating present knowledge, illustrates the vastly different behavior of $2^{\aleph_0}$ versus $\binom{\aleph_\omega}{\omega}$.

Properties \ Term	$2^{\aleph_0}$	$\binom{\aleph_\omega}{\omega}$
Possible values of the term	The class of all cardinals κ satisfying $\mathrm{cf}(\kappa) \neq \omega$	Only Regular Cardinals in the interval $[\aleph_{\omega+1}, \aleph_{\omega_4})$. Only regular cardinals below $\aleph_{\omega_1}$ are known to actually occur in models of ZFC.
Structural relations with least possible value	Hardly any structural relation between canonical structures on $2^{\aleph_0}$ and ω_1	Many structural relations between canonical structures on $(\aleph_\omega)^{\aleph_0}$ and $\aleph_{\omega_1}$
Consistency Strength of values other than the least possible one	All possible values are equiconsistent with ZFC	Any value other than $\aleph_{\omega+1}$ requires the consistency of a measurable κ of order at least κ^{++}
Implications of the negations of CH/BH	No information on ω_1 is available in ZFC from $\omega_1 < 2^{\aleph_0}$	$\aleph_{\omega+1} < \binom{\aleph_\omega}{\omega}$ has many implications for $\aleph_{\omega_1}$, some of which are strengthening of consequences of BH

The behavior of the power-set function at $\aleph_0$ is not unique to this cardinal—it is shared by every regular cardinal. There is no bound for 2^κ for a regular κ, and worse, by Easton's result. The binomial of singular cardinals behaves differently. On singular cardinals which are not fixed points of the aleph function there are bounds. At the least fixed point there is no bound. However, above the first cardinal which is compact for Abelian groups (that is, above it, every almost free Abelian group is free) BH holds [Shelah, 1994a]. One can say that some chaos is displayed in the behavior of the binomial, but only for a "small" part of the universe, and at a high consistency-strength cost.

It had already been amply confirmed from Sierpiński's days that CH, that is, the equation $2^{\aleph_0} = \aleph_1$, has many consequences for the structure of the continuum. Less evident is that there is information stored in this equation for $\aleph_1$. However, the negation of CH has very little information about what happens at that cardinality. If Martin's Axiom holds, that all cardinals which are smaller than the continuum "look like" $\aleph_0$. Nothing of the structure of the continuum is inherited by its subsets of cardinality $\aleph_1$. For example, if CH holds then by Hausdorff's work there is a universal linearly ordered set of cardinality $\aleph_1$, a fact which does not necessarily hold without CH. Another example is that by CH there are many pairwise incomparable separable, $\aleph_1$-dense order-types; but if the continuum is larger than $\aleph_1$ there may be only a single isomorphism type of such a set, by [Baumgartner, 1973]. Finally, the existence of an ω_1-scale in $(\omega^\omega, <^*)$, which follows from CH, is not a ZFC theorem, nor is the existence of scale of any length. All this background makes Hausdorff's fantastic discovery of a ZFC (ω_1, ω_1^*)-gap shine in solitude.

In strong contrast to this situation, much of what is true for $\aleph_{\omega+1}$ when BH holds remains true when BH fails. For example, the existence of an $\aleph_{\omega+1}$-scale in $(\prod_n \aleph_n, <^*)$, which is a triviality from BH, is actually a ZFC theorem, that is, holds whether BH holds or not. Furthermore, if BH fails, not only does an $\aleph_{\omega+1}$-scale exist, but actually one exists with additional, combinatorial properties which follow from $V = L$ but are not true for all scales in general—a scale which is called "very good", that is, in which a closed unbounded set of initial segments of cofinality ω_1 are equivalent to strictly increasing sequences (see [Shelah, 1994a, Chap. II]).

An even more intriguing example is the polarized partition relation for a singular with uncountable cofinality [Shelah, 1998], which at the moment is known to follow *only* from the assumption that the singular is strong limit which violates SCH. As Shelah described it:

> This is a good example of a major thesis from [Shelah, 1994a]:
>
> **Thesis 1.2** Whereas CH and GCH are good (helpful, strategic) assumptions having many consequences and, say, ¬CH is not, the negation of GCH at singular cardinals (i.e. for μ strong limit singular $2^\mu > \mu^+$ or, really, the strong hypothesis: $\mathrm{cf}(\mu) < \mu \Longrightarrow \mathrm{pp}(\mu) > \mu^+$) is good (helpful, strategic) assumption.

The fact that the negation of BH is also useful allows one to prove theorems by dividing the proof to cases: one argument if BH holds and another if it fails. (Such, for example, was Shelah's first proof of the existence of an $\aleph_{\omega+1}$-scale.)

In summary, most of the intuition that was gathered about CH and the negation of CH in many decades simply shatters when approaching BH. BH is hard to violate, its negation has consequences, and there are deep, unbreakable connections between $\binom{\aleph_\omega}{\omega}$, $(\aleph_\omega)^{\aleph_0}$, and $\aleph_{\omega+1}$ with many consequences. Several mathematical structures constructed on $(\aleph_\omega)^{\aleph_0}$ reflect down to (absolute) objects of cardinality $\aleph_{\omega+1}$ because the machinery which dictates their properties is binomial rather than exponential. For example, a closed Dowker space of cardinality $\aleph_{\omega+1}$ was constructed inside Rudin's original space, whose cardinality is $(\aleph_\omega)^{\aleph_0}$ [Kojman and Shelah, 1998], with a pcf scale and without forming unrestricted infinite products, which appear in Rudin's construction.

5.1 Annus Mirabilis: the development of singular cardinal arithmetic in 1974

The beginning of modern singular cardinal arithmetic can certainly be dated to Silver's 1974 theorem about the exponentiation of singulars of uncountable cofinality. It is not only the theorem itself which marks the beginning of the modern era, but the realization that followed its publication, that there actually was a theory of singular cardinal arithmetic to be discovered.

By 1974 Solovay had already proved that SCH holds above a compact cardinal, and Magidor had proved that SCH may fail at $\aleph_\omega$, starting from a supercompact cardinal. But Magidor himself, like everyone else, thought that large cardinals were not necessary for such consistency results and that with time it would be discovered that also singular cardinal arithmetic did not really exist. Jensen was investigating his morasses with the intention of putting them to use to proving the consistency of the negation of SCH without appealing to the consistency of large cardinals.

Magidor arrived in Berkeley in 1974 shortly after Jensen left Berkeley for Bonn. Magidor, who was interested in ultrapowers of the universe of sets, was trying to establish that *non-regular ultrafilters* on ω_1 could not exist, by examining ultraproducts of the universe taken with such an ultrafilter. Rather than disproving the existence of such ultrafilters, his efforts led to a surprising discovery, that if GCH held below $\aleph_{\omega_1}$ it would continue to hold at $\aleph_{\omega_1}$ itself. By modifying Magidor's proof, Silver proved in ZFC that GCH does not fail for the first time at a singular of uncountable cofinality.

Silver's own account of the discovery of his famous theorem is as follows:

> The immediate stimulus for this result was some work of Kanamori and Magidor[1] concerning nonregular ultrafilters on ω_1. The other principal influences were the result of Scott concerning GCH at measurable cardinals, Some work of Keisler on ultrapowers . . . , the two-cardinal theory developed by several model-theorists, some work of Prikry and Silver on indecomposable ultrafilters . . . as well as Cohen's and work on nonstandard models of set theory . . . [24]

---

[24] The footnote below the line is Silver's and so is the formulation of the properties of the ultrafilter.

> [1]The result of Magidor states, in particular: If there is a regular, nonuniform ultrafilter over ω_1 and $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all $\alpha < \omega_1$ then $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$.

We shall not survey all of the developments that Silver lists in this passage; rather, the reader is invited to appreciate the enormous quantity and sophistication of techniques that went into stimulating what very soon turned out to be an elementary theorem.

Magidor had presented in a seminar the following proof: Suppose that there is a non-regular ultrafilter on ω_1. By a theorem of Kanamori [1976] who built on earlier work by Ketonen [1976], also a weakly normal such ultrafilter exists. Hence, in the ultraproduct of the universe of sets (which is not necessarily well founded) the constant functions with values ω_α for $\alpha < \omega_1$ are cofinal below the identity function.

Taking a non-wellfounded ultraproduct of the universe of sets, Magidor showed that if GCH was true below $\aleph_{\omega_1}$, then it could not fail at $\aleph_{\omega_1}$. Magidor was hoping to obtain a contradiction with these considerations.

Silver, who was present at the seminar, observed that one did not need to assume that there was such an ultrafilter; rather, the situation which Magidor obtained from this assumption could be forced, without any assumptions, by collapsing 2^{ω_1} to ω, and rather than a real ultrapower of the universe, a *generic* ultrapower of the ground universe sufficed.

The result was stunning:

THEOREM 1 (Silver). *If* GCH *holds below* $\aleph_{\omega_1}$ *then it holds at* $\aleph_{\omega_1}$.

This result should be viewed in the light of the prevailing view in those days, that Easton's result would soon be generalized also to singular cardinals. Baumgartner and Prikry [1977], and independently Jensen, discovered elementary proofs of Silver's theory shortly after its discovery. Baumgartner and Prikry soon published a self-contained, purely combinatorial proof of Silver's theorem and Jensen circulated mimeographed copies of essentially the same elementary proof.

More mimeographed notes were being circulated from Bonn by Jensen, who was studying the consequences of Silver's result. After several preliminary partial results, Jensen realized that what he was doing in showing that L computes cofinality correctly amounted to showing that any subset from V was covered by a subset in L of the same cardinality plus $\aleph_1$. Jensen's covering lemma was discovered:

THEOREM 2 (Jensen's covering lemma). *If* $0^\#$ *does not exist, then every set of ordinals* A *is covered by a set of ordinals* $A' \in L$ *such that* $|A'| \leq |A| + \aleph_1$.

$0^\#$ is a set of integers whose existence is implied by large cardinals and implies that L is very small compared to V (cf. Sect. 7.2). The lemma provides a surprisingly weak condition for the existence of $0^\#$.

The consequences of this lemma were dramatic. We quote from the introduction to the paper by Devlin and Jensen [1975]:

> Jensen's effort to produce a positive solution [getting SCH to fail by forcing] over L led to total failure. Silver's work then led him to consider the problem from a new perspective. He discovered the statement "$0^{\#}$ does not exist" ... implies a negative solution to all cases of the singular cardinal problem. But then there cannot be a positive forcing solution over L, since every generic extension of L by a set of conditions satisfies $\neg 0^{\#}$. ... Our main theorem says, in effect, that if $\neg 0^{\#}$ then the "essential structure" of cardinalities and cofinalities in L is retained in V.

Jensen continued his work and together with Dodd published in [1981] their work on the core model, from which it followed that for Magidor's result at least the assumption of a measurable cardinal was necessary.

Magidor and Solovay obtained a *bound* on the power of a strong limit singular cardinal of uncountable cofinality from the assumption of a Ramsey cardinal, but their proof was soon subsumed by the proof of Galvin and Hajnal [1975] of a bound in ZFC. Galvin and Hajnal proved, e.g., that if $\aleph_{\omega_1}$ is a strong limit cardinal then $2^{\aleph_{\omega_1}} < \aleph_{(2^{\aleph_1})^+}$. Jech and Prikry [1976] were able to extend such bounds to more cases, like fixed points of the aleph function, using the additional assumption of a precipitous ideal on ω_1, and Magidor [1977a] obtained the bound $(\aleph_{\omega_1})^{\aleph_1} < \aleph_{\omega_2}$ from Chang's conjecture.

Both Silver's theorem and Solovay's theorem would turn out to be special cases of more general pcf theorems. This is the time to describe this theory, to which we devote a separate section.

6 SHELAH'S PCF THEORY

Before describing the historical development of Shelah's pcf theory it is perhaps useful to describe the theory itself. The name "pcf" stands for "possible cofinalities". The basic discovery in Shelah's pcf theory is that products of small sets of regular cardinals have a basis for their set of *possible cofinalities.*

The notion of cofinality was introduced by Hausdorff for linearly ordered sets, as the smallest well-order-type of a *cofinal* subset of the linearly ordered set. With linearly ordered sets, "cofinal" and "unbounded" are the same. Not so with partially ordered sets, where these two parameters may differ.

In a partially ordered set with no finite unbounded set, the least cardinality of an unbounded subset is always a regular cardinal, which is called the *bounding number* of the poset. The cofinality, or *dominating number*, of a poset is the least cardinality of a cofinal, or *dominating*, set, and this time may be of singular cardinality. The bounding and dominating numbers are denoted by $\mathfrak{b}(P, \leq)$ and $\mathfrak{d}(P, \leq)$ respectively.

When in a poset it happens that the bounding number and the potentially larger dominating number are the same number, it is said that the poset has *true cofinality* that number, which is denoted by $\mathrm{tcf}(P, \leq)$.

An infinite product of cardinals $\prod_{i\in A}\lambda_i$ consists of all functions $f : A \to \mathrm{On}$ which satisfy that $f(i) \in \lambda_i$ for all $i \in A$. There is a natural partial ordering on this product, namely, $f < g \iff f(i) < g(i)$ for all $i \in A$. As mentioned earlier, this partial ordering can be "modded out" using an ideal I on A to get $f <_I g$ iff $\{i \in I\colon g(i) \le f(i)\} \in I$. In the well-studied case $(\omega^\omega, <^*)$ where $<^*$ is $<_I$ for the ideal I of finite subsets of ω, the dominating number is uncountable, but undecidable in ZFC. It may be singular. The bounding number is also uncountable, not greater than the dominating number and undecidable in ZFC. Whether $\mathfrak{b}(\omega^\omega, <^*) = \mathfrak{d}(\omega^\omega, <^*)$ or not is also not decidable in ZFC. Lastly, it may be the case that for every ultrafilter U over ω, when modding out by (the dual of) U, hence making $(\omega^\omega, <_U)$ a linearly ordered set in which bounding and domination agree, this number is always strictly larger than $\mathfrak{b}(\omega^\omega, <^*)$.

This discussion may be technical, but it is important to mention these facts as the results which pcf theory obtained go against the intuition that may have been created by earlier results about ω^ω.

In the product $(\prod_n \omega_n, <^*)$, for example, the bounding number is equal to $\aleph_{\omega+1}$ in ZFC (that is, absolutely), the dominating number is *always* regular and for *every* ideal I over ω the bounding number $\mathfrak{b}(\prod_n \omega_n, <_I)$ is equal to the cofinality modulo some ultrafilter disjoint to I, actually to the true cofinality of some non-maximal ideal extending I. The exact details are less important than the general theme that in the product $\prod_n \omega_n$ there exists decidable structure, while in ω^ω there is none.

We quote now the "basis theorem" for all possible cofinalities of structures of the form $(\prod_n \omega_n, <_I)$ for some ideal I:

THEOREM 3 (Shelah's pcf theorem). *Let A be a set of regular cardinals which satisfies $|A| < \min A$. Let* $\mathrm{pcf}(A)$ *be the set of all bounding numbers of all reduced products $(\prod A, <_I)$ modulo some proper ideal $I \subseteq \mathcal{P}(A)$. Then for every $\lambda \in$* $\mathrm{pcf}(A)$ *there is a set $B_\lambda \subseteq A$, called the* pcf generator for λ, *such that, letting $J_{<\lambda}$ be the ideal generated by $\{B_\theta : \theta \in \mathrm{pcf}(A)$ and $\theta < \lambda\}$:*

- $\lambda = \mathfrak{b}(\prod A, <_{J_{<\lambda}}) = \mathrm{tcf}(\prod B_\lambda, <_{J_{<\lambda}})$.
- *For every proper ideal $I \subseteq \mathcal{P}(A)$ the bounding number $\mathfrak{b}(\prod A, <_I)$ is the least λ for which $B_\lambda \notin I$ and $\mathfrak{d}(\prod A, <_I)$ is the last λ for which $B_\lambda \notin I$.*
- $\mathrm{tcf}(\prod A, <_I) = \lambda$ *iff* $\lambda \in \mathrm{pcf}(A)$ *is unique such that* $B_\lambda \notin I$.

Implicit in the theorem is that every bounding number is also a true cofinality. With this theorem it is easy to determine what is the bounding number modulo an ideal I and whether or not it is also a true cofinality: the bounding number of $(\prod A, <_I)$ is simply the least λ for which $B_\lambda \notin I$ and true cofinality exists if and only if for every $\theta \neq \lambda$ in $\mathrm{pcf}(A)$ it holds that $B_\theta \in I$.

Shelah's pcf theorem is an algebraic basis theorem for the space of all linearly ordered homeomorphic images of the partially ordered product $(\prod A, <)$. It is *not* something one expects to find if guided by the intuition accumulated in the area of independence results surrounding ω^ω, or the continuum in general.

The modern way to prove this theorem uses the density of exact sequences, that is, for every $<_I$-increasing sequence $\langle f_\alpha : \alpha < \lambda\rangle$ for regular λ there is a $<_I$-increasing sequence $\langle f'_\alpha : \alpha < \lambda\rangle$ such that $f_\alpha \le f'_\alpha$ for all α and $\langle f'_\alpha : \alpha < \lambda\rangle$ has an exact upper bound $g : A \to \mathrm{On}$. The existence of the latter sequence is proved using some ZFC combinatorics, typically, a stationary set in an intrinsically important ideal $I[\lambda]$, with which an exact sequence is constructed so that two of the three possibilities in Shelah's Trichotomy Theorem are ruled out, leaving only the exact upper bound possibility.

The relation of pcf to arithmetic is as follows: for a singular μ let $\mathrm{pp}(\mu)$ be the supremum of all $\mathrm{tcf}(\prod A, <_I)$ for $A \subseteq \mu$ an unbounded set of regular cardinals of cardinality $\mathrm{cf}(\mu)$ and I an ideal over A which contains all bounded subsets of A. For many singular cardinals it is known that $\mathrm{pp}(\mu)$ is the binomial $\binom{\mu}{\mathrm{cf}(\mu)}$. If fact, the consistency of the opposite is not known.

In particular, if some end-segment A of all regular cardinals below a singular μ has order-type smaller than μ and hence without loss of generality satisfies $|A| < \min A$, then $\max \mathrm{pcf}(A) = \mathrm{cf}([\mu]^{\mathrm{cf}\,\mu}, \subseteq)$. In other words, the binomial function at μ is equal to $\max \mathrm{pcf}(A)$. For such μ,

$$\mu^{\mathrm{cf}(\mu)} = \max \mathrm{pcf}(A) + 2^{\mathrm{cf}(\mu)}.$$

This part of the theory is obtained with the "convexity" or "no hole" theorem, which states that if the set A is an interval of regular cardinals then also $\mathrm{pcf}(A)$ is an interval of regular cardinals.

Another important pcf theorem is that for regular uncountable products, e.g. $\prod_{\alpha<\omega_1} \aleph_{\alpha+1}$, true cofinality exists and is equal to the least possible value, that is, in the case quoted, $\mathrm{tcf}(\prod_{\alpha<\omega_1} \aleph_{\alpha+1}, <_{\mathrm{NS}}) = \aleph_{\omega_1+1}$.[25] This is the underlying basis for, e.g., Silver's theorem.

The pcf theorem is a ZFC theorem, but did not start out as a ZFC theorem. It was not discovered all at once. We describe now the gradual emergence of its components. Let us identify now the exact place where pcf theory was born: Shelah's [1978] construction of a Jonsson algebra on $\aleph_{\omega+1}$ from the assumption $2^{\aleph_0} \le \aleph_{\omega+1}$.

A Jonsson algebra is an algebra of functions—an infinite set with countably many (finitary) operations—that has no proper subalgebra of the same cardinality as the whole algebra. There is a Jonsson algebra on ω and if there is a Jonsson algebra on a cardinal κ there is one on κ^+. Thus a straightforward induction gives that there are Jonsson algebras on $\aleph_n$ for all $n \in \omega$. What about $\aleph_\omega$? This is still an open problem! The next cardinal in line is, then, $\aleph_{\omega+1}$. Can one combine the Jonsson algebras on each $\aleph_n$ to create one on $\aleph_{\omega+1}$? From modern pcf theory the answer is obviously positive: if $\langle f_\alpha : \alpha < \aleph_{\omega+1}\rangle \subseteq \prod_{n\in B_{\aleph_{\omega+1}}} \omega_n$ is $<^*$-increasing and $<^*$-cofinal in the product then every subset of this scale of the same cardinality

[25] NS denotes the ideal of non-stationary subsets of ω_1.

as the whole scale achieves unbounded values in unboundedly many ω_n, and so a Jonsson algebra is easily created from such a scale.

Does such a scale exist? This is exactly the third in a list of three questions with which the paper [Shelah, 1978] begins, the first being whether a Jonsson algebra on $\aleph_{\omega+1}$ exists in ZFC.

Shelah defines a precursor of the pcf operation, $Psc_D(\overline{\lambda})$ where $\overline{\lambda} = \langle \lambda_i : i < \delta \rangle$ is a sequence of regular cardinals and D is a filter containing all co-bounded sets of δ. Now Psc,"possible scales", is the set of all cardinals λ obtainable as follows: for some subsequence of $\langle \lambda_i : i < \delta \rangle$, replace λ_i in the product by 1. This corresponds to changing the filter D to its restriction $D \restriction B$ to a subset $B \subseteq \delta$. Now the cardinal which is obtained is simply the bounding number of the product $(\prod_{i<\delta} \lambda_i, <_{D \restriction B})$. Since a single filter is considered, there are at most 2^δ localizations $D \restriction B$, hence the set of possible scales has cardinality at most 2^δ.

What Shelah does next is to prove the first version of the "no-hole" theorem, using the assumption that the continuum is small:

THEOREM 4. *Suppose* $\overline{\lambda} = \langle \lambda_i : i < \kappa \rangle$ *is increasing,* κ *regular, and* $\lambda_* = \sum_{i<\kappa} \lambda_i$. *If* $\lambda \in Psc_D \overline{\lambda}$, $\lambda_* < \mu < \lambda$, μ *regular,* D $\aleph_1$*-complete or* $2^\kappa < \mu$, *then* $\mu \in Psc_D \langle \lambda'_i : i < \kappa \rangle$ *for some* $\lambda'_i \leq \lambda_i$, $\langle \lambda'_i : i < \kappa \rangle$ *is not D-trivial.*

Thus he gets a maximal filter D on ω with $\mathrm{cf}(\prod_n \omega_n, <_D) = \aleph_{\omega+1}$, and uses this to get a Jonsson algebra.

We first find here the main themes of pcf theory, but should also appreciate, in retrospect, the way Shelah managed to circumvent the difficulty which may arise by choosing inadvertently a maximal filter which creates cofinality larger than $\aleph_{\omega+1}$. One cannot just prove that for an arbitrary maximal filter the cofinality shall be as desired.

The consideration of $D \restriction B$ already predicts the concept of a pcf generator. Also the concept of a least upper bound is present and least upper bounds are obtained by using the cardinal arithmetic assumptions.

It is quite fascinating to look at this early work and trace the exact point in history at which pcf theory emerged. It is also important to observe that the motivation for this development was a concrete combinatorial problem; not a systematic study of cardinal arithmetic.

In 1980 Shelah was working on stronger versions of Jensen's covering lemma. One of the discoveries made in this context led to the next important step in the development of pcf theory. Shelah discovered that the intersection of a sufficiently closed elementary submodel M of size $\aleph_1$ (for example) with the ordinals below $\aleph_\omega$, say, is *determined* by the model's *characteristic function* $\chi_M(n) := \sup M \cap \omega_n$. By carefully choosing particular pcf scales and employing a sophisticated argument, Shelah obtained control over characteristic functions of sufficiently closed submodels. Now Shelah could use pcf theory to bound $(\aleph_\omega)^{\aleph_0}$.

Thus we find the full scheme of the pcf proof for upper bounds for singular cardinal exponentiation appearing in Shelah's 1980 result:

$$\forall n (2^{\aleph_n} < \aleph_\omega) \Longrightarrow 2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}.$$

pcf theory of a set A was developed here with ultrafilters and with the additional assumption that $2^{|A|} < \min A$. That was a reasonable assumption in the context of the application, but Shelah was aware then of the importance of separating pcf theory from the exponentiation of regular cardinals.

This proof appeared in Shelah's book *Proper Forcing* [1982] right after the sections dealing with the strong covering lemma. The ideal $J_{<\lambda}(A)$ was defined as the collection of all $B \subseteq A$ such that for every ultrafilter U containing B the cofinality of the product mod U is smaller than λ. Also the notions of exact upper bound and true cofinality of an ideal appear.

Shelah obtained his result in 1980 while visiting Ohio State University, where Hugh Woodin had just presented a proof using large cardinals that $(\aleph_\omega)^{\aleph_0}$ could be arbitrarily large with $\aleph_\omega$ being strong limit. After leaving Ohio State University, Woodin discovered a gap in his proof, and was trying to fill it for a while until Shelah's proof of the bound was communicated to him.

Shelah presented his proof of the bound at the logic meeting in Patras, Greece, 1980. Woodin was not present in that meeting, but his earlier announcement created the suspicion (or the hope, for some) that large cardinals would soon be proved inconsistent. However, all large cardinals involved in this incident are still alive and well.

The remaining developments in pcf theory we shall refer to here appear in Shelah's book *Cardinal Arithmetic* [1994a]. In [1990], which became Chapter I of [1994a], Shelah replaces the assumption $2^{|A|} < \min A$ by $|A| < \min A$. The existence of a single generator for $J_{<\lambda^+}$ over $J_{<\lambda}$ for $\lambda \in \mathrm{pcf}(A)$ is not yet proved. This is an important step forward in separating the exponent function from the binomial function.

A major advance is made in Chapter II of [1994a]; Shelah isolated the Trichotomy Theorem for sequences of ordinal functions which are increasing modulo some ideal. This discovery is presented as "Claim 1.2" on page 41 of the book, a presentation which does little justice to its formidable contents. Shelah's trichotomy says, roughly, that for *any* ideal I over an infinite set A and *any* $<_I$-increasing sequence of ordinal functions $\langle f_\alpha : \alpha < \lambda \rangle$ on A of regular length $\lambda > |A|^+$, either the sequence is equivalent modulo some larger ideal to a sequence inside the product of small sets (that is, in case A is countable, the sequence is inside a copy of ω^ω); or there is explicit evidence that the saturation of I exceeds λ; or there is an exact upper bound to the sequence modulo I. If the ideal I is maximal or if $\lambda > 2^{|A|}$, neither of the latter two alternatives can hold, and thus the sequence has an exact upper bound.

This is a remarkable discovery: the class of ordinal functions on a set A satisfies Dedekind completeness for all increasing sequences of length larger than $2^{|A|}$ when quasi-ordered modulo *any* ideal over A. This means that Hausdorff could not have discovered any gap other than the one he discovered in On^ω. Even if Hausdorff was allowed to choose any ideal he wanted over ω and search for gaps of any cardinal length λ in the class of *all* cardinal functions on ω, he would find no other gap in ZFC than the one he discovered: If CH holds, then every sequence of ordinal functions on ω of length $\geq \omega_2$ which is increasing modulo some ideal I over ω has an exact upper bound modulo the same ideal I. Put more bluntly: Hausdorff had discovered the single exceptional case in the whole multitude of $<_I$-increasing sequences of ordinal functions on ω for any ideal I!

Using his trichotomy, Shelah proceeds to prove one of the most useful pcf theorems, the existence of a μ^+-scale for every singular μ, i.e. a sequence which is increasing and cofinal of length μ^+ inside a product $(\prod_{i<\mathrm{cf}(\mu)} \kappa_i, <_{bd})$ for an increasing sequence $\langle \kappa_i : i < \mathrm{cf}(\mu) \rangle$ of regular cardinals with supremum μ. This had many applications, e.g. the existence of a Jonsson algebra on $\aleph_{\omega+1}$ in ZFC, as mentioned earlier.

The development here resembles the steps that Hausdorff had taken some eighty years earlier. Hausdorff first sought an ω_1-scale—which he found, with the aid of CH, but could not find in ZFC—and then moved to construct his gap. Shelah first got an $\aleph_{\omega+1}$-scale from $2^{\aleph_0} \leq \aleph_{\omega+1}$, and then proceeded to discover that such a scale existed in ZFC. One cannot help but wonder what would have happened if Hausdorff was driven to study the product $(\prod_n \omega_n, <^*)$.

To apply the strength of the Trichotomy Theorem inside the product $(\prod A, <_I)$ two ingredients were needed: club guessing and having stationary sets in an intrinsically important ideal $I[\lambda]$. Club guessing was discovered by Shelah in 1988 in his model-theoretic work—to prove that a non-superstable theory has many pairwise non-embeddable models at a singular cardinality. It is a prediction principle which is a formal weakening of Jensen's diamond principle, weaker in the sense that it predicts only closed unbounded subsets (rather than all subsets) of a cardinal with the prediction requirement weakened from equality to inclusion. Formally, if $\kappa < \lambda$ are regular cardinals, a club guessing sequence on $S^\lambda_\kappa := \{a < \lambda \colon \mathrm{cf}(\alpha) = \kappa\}$ is a sequence $\overline{C} = \langle c_\delta : \delta \in S^\lambda_\kappa \rangle$ such that for each $\delta \in S^\lambda_\kappa$, c_δ is a closed unbounded subset of δ of order-type κ and such that for every closed unbounded $E \subseteq \lambda$ there is a $\delta \in S^\lambda_\kappa$ such that $c_d \subseteq E$. The main gain in weakening Jensen's diamond principle in this way is, as Shelah proved, that if $\kappa^+ < \lambda$ and both κ and λ are regular, then a club-guessing sequence on S^λ_κ exists *in* ZFC.

Club guessing appears in [Shelah, 1994a], but the existence of a stationary set $S \subseteq S^\kappa_\lambda$ in $I[\lambda]$ for $\kappa^+ < \lambda = \mathrm{cf}(\lambda)$ does not appear in the book—although Shelah knew this for a long time before the book was printed. These two facts are related: the existence of a stationary set in $I[\lambda]$ is proved using club guessing. With these two facts one proves the density of exact sequences in products $(\prod A, <_I)$ and can prove with them the existence of pcf generators.

In Chapter VIII of [Shelah, 1994a], Shelah proves the existence of a single pcf generator without using the later approach. The mechanism of the proof of a stationary set in $I[\lambda]$ appears in disguise in the proof of the existence of a generator. The other property of pcf that is proved in the chapter is the *localization* theorem, which states that if $\lambda \in \mathrm{pcf}(B)$ where $B \subseteq \mathrm{pcf}(A)$ then there is a subset $C \subseteq B$ of cardinality at most $|A|$ such that $\lambda \in \mathrm{pcf}(C)$. In the very first section of the chapter Shelah proves a pcf generalization of the Galvin-Hajnal theorem for his pp function.

By the Fall of 1989 all ingredients were ready to the next great achievement of pcf theory—the $\aleph_{\omega_4}$ bound. Shelah was visiting Mathematical Sciences Research Institute in Berkeley, where, according to testimony in his own book, he was eager to contradict Leo Harrington's opinion that cardinal arithmetic was "dead" ([Shelah, 1994a, p.359]):

> I was interested in this particularly since 80, but the last impetus was my failure in Fall 89 to convince Harrington to retract his 86 statement: "cardinal arithmetic, this was a great problem, [but is now] essentially finished."

The set $\mathrm{pcf}(\{\aleph_n : n \in \omega\})$ is an interval of regular cardinals on which the pcf operation defines a closure operation, hence a topology. This topology is countably-tight,[26] by the localization theorem. Furthermore, this topology is related to the order topology by the representation theorem of μ^+. Now a purely combinatorial argument proves that such a structure has cardinality smaller than ω_4.

This proves that $\max \mathrm{pcf}(\{\aleph_n : n \in \omega\}) < \aleph_{\omega_4}$. By the equality of max pcf with the binomial it follows that

$$\mathrm{cf}([\aleph_\omega]^\omega, \subseteq) < \aleph_{\omega_4}.$$

Thus, unlike the exponentiation $2^{\aleph_0}$ which has no bound in the list of alephs, the binomial of $\aleph_\omega$ is bounded by $\aleph_{\omega_4}$.

This gives immediately the following bounds:

$$2^{\aleph_0} < \aleph_\omega \Longrightarrow (\aleph_\omega)^{\aleph_0} < \aleph_{\omega_4};$$

$$\forall n (2^{\aleph_n} < \aleph_\omega) \Longrightarrow 2^{\aleph_\omega} < \aleph_{\omega_4}.$$

The significant improvement in bound $\aleph_{\omega_4}$ over Shelah's 1980 bound $\aleph_{(2^{\aleph_0})^+}$ is that the subscript ω_4 does not contain any exponentiation. This is perhaps Shelah's best known application of pcf theory.

As Shelah writes about the problem of what the actual bound is:

> So really our problem breaks to two:
>
> (a) Is it consistent to have such [a pcf] structure on specific $\gamma(*)$?
>
> (b) If there is such a structure can we then force a suitable cardinal arithmetic such that it is the structure corresponding to it?
>
> What is accomplished here ... is that in (a), for $\aleph_\omega$, necessarily $\gamma(*) < \aleph_4$. ... But (b) is widely open, and it does not seem unreasonable at all, to me at least.

[26] Countable tightness in a topological space X is the property that for every subset $A \subseteq X$ and a point x in the closure of A there is a countable $B \subseteq A$ with x in the closure of B.

The same proof works to bound the binomial of every singular which is not a fixed point of the aleph function. For the first fixed point the theory breaks down, as there is no interval of regular cardinals of the right form unbounded in this singular cardinal. As Shelah puts it:

> ... before 1980 we could have thought that for example for strong limit λ of cofinality $\aleph_0$ there are no restrictions on 2^λ ... we know now there are some but have less clear idea what to expect for fix points.

In the same chapter the relation between $\mathrm{pp}(\mu)$ and the binomial of μ is clarified for singular μ. Earlier, Shelah [1986] proved that if μ is strong limit singular of uncountable cofinality with no weakly inaccessible cardinal below it then 2^μ is smaller than the first weakly inaccessible cardinal.

In [1983] Shelah had already proved that for the first fixed point of order ω_1 of the aleph function of cofinality $\aleph_1$[27] this was the best upper bound. Also, for every limit δ of uncountable cofinality, for club many $\alpha < \delta$ of countable cofinality $\mathrm{pp}(\beth_\alpha)$ is equal to the binomial of $\beth_\alpha$. Finally, if $\mathrm{cf}(\delta) = \aleph_0$ and the binomial of $\beth_\delta$ is strictly larger than $\mathrm{pp}(\beth_\delta)$, then $\mathrm{pp}(\beth_\delta)$ must be very large: there is some fixed point μ of the aleph function of uncountable cofinality between $\beth_\delta$ and its pp.

A remarkable result of Moti Gitik [2005] showed that the least fixed point of the aleph function, which is the least cardinal for which pcf theory does not give a bound on its binomial, indeed has no bound on its binomial. This result required the full machinery of large cardinal forcing developed by Magidor and Gitik in over 20 years. This result of Gitik closed a circle that began in 1980 when Woodin thought he could prove that there was no bound on the power set of a strong limit $\aleph_\omega$. In exactly the same year Shelah discovered the $\aleph_{(2^{\aleph_0})^+}$ bound. Gitik proved for the first fixed point of the aleph function what Woodin tried to prove for $\aleph_\omega$.

In the very first section of [1994a] Shelah proves that the least μ for which $\mathrm{pp}(\mu) > \mu^+$ satisfies that $\mathrm{pp}(\mu)$ is equal to its binomial. This was the missing part for completing the work of Gitik and Mitchell on the consistency strength of the Singular Cardinal Hypothesis. Gitik and Mitchell proved that the negation of SCH is equiconsistent with a measurable κ of order κ^{++}.

Shelah continued to develop pcf theory in many directions. He [1997] investigated relaxations of the condition $|A| < \min A$ and considered the theory in models without AC. Research on pcf theory and on its application is conducted presently by set-theorists and by mathematicians from other fields.

It should be made clear that only a small part of pcf theory had been described here. Not even all of the material in *Cardinal Arithmetic* has been described, while the volume of results in pcf theory and its application has doubled since the appearance of the book. Shelah produces formidable results faster than what the author of this paper can read, and even a well-known text like the book contains

[27] A cardinal $\aleph_\alpha$ is a fixed point if $\aleph_\alpha = \alpha$. It is a fixed point of order two if it is a fixed point and is equal to its own index in the increasing enumeration of fixed points, and so forth.

too many results to digest in a life time. A reader who wishes to learn more about pcf theory should consult the *analytical guide* at the end of the book and Shelah's archive: http://shelah.logic.at.

6.1 The revised GCH *above* $\beth_\omega$

A result of special status is Shelah's [2000b]. In this work Shelah provides a new perspective on cardinal arithmetic by proving in ZFC that a natural cardinal arithmetic function asymptotically assumes the smallest possible value.

We may recall the value of $2^{\aleph_0}$ is undecidable in a formidable way: the axiom system ZFC allows $2^{\aleph_0}$ to be any cardinal $\aleph_\alpha$ for which $\mathrm{cf}(\aleph_\alpha) \neq \omega$. Thus CH is very far from being "true in ZFC". The situation with GCH on regular cardinals is not much better, as by Easton's result there are no restrictions on regular cardinal exponentiation other than König's lemma and weak monotonicity.

With full exponentiation being separated into the binomial on singular cardinals and exponentiation on regular cardinals, it made sense to clarify what rules governed the binomial. As long as the singular at hand is not a fixed point of countable cofinality of the aleph function, there is a bound (whether the "4" in the bound is best possible or not is a separate issue) and the binomial of the first fixed point of countable cofinality is *unbounded*.

In 1990, the British model theorist Wilfrid Hodges communicated to Shelah a problem posed by another British mathematician, Kishor Kale: Do the cardinalities of countably generated topologies satisfy the continuum hypothesis? Suppose that τ is a topology, not necessarily Hausdorff, nor even T_0, which has a countable basis. Does it follow that the number of open sets with respect to τ is either countable or equal to the continuum?

Shelah's answer was positive and appeared in [1993], but that was not the end of this investigation. He examined what happens if one replaces $\aleph_0$ by $\beth_\omega$. In [1994b] Shelah proved the analogous theorem for $\beth_\omega$: if a topology τ has a base of cardinality at most μ and for some strong limit cardinal λ of cofinality ω and $\lambda \leq \mu < 2^\lambda$, then the size of τ is either $\leq \mu$ or $\geq 2^\lambda$. Shelah's first proof used cardinal arithmetic assumptions that were shortly later replaced by the revised GCH theorem.

In [2000b] Shelah formulated a different binomial function: $\lambda^{[\kappa]}$ is the least cardinality of a family of κ-subsets of λ with the property that every κ-subset of λ is contained in the union of fewer that κ members of the family. Clearly, for every singular cardinal μ it holds that $\mu \leq \mu^{[\mathrm{cf}(\mu)]} \leq \binom{\mu}{\mathrm{cf}(\mu)}$. However, a totally new phenomenon emerged with this revised form of the binomial. This new function seemed to be able to absorb almost all independence fluctuations, and assumes asymptotically the smallest possible values. Shelah proved in ZFC that for every $\lambda \geq \beth_\omega$, for every sufficiently large $\kappa < \beth_\omega$ it holds that $\lambda^{[\kappa]} = \lambda$. This result is Shelah's "revised GCH in ZFC".

This theorem has applications (some of which are described below) and will have more applications in the future, but its primary significance is in changing the attitude towards the axiomatic system ZFC in the context of infinite cardinal arithmetic. Placing Easton's result that there are no rules of singular cardinal exponentiation at one extreme, this theorem secures the opposite extreme position. The revised GCH in ZFC demonstrates that a natural cardinal arithmetic function assumes the smallest possible value asymptotically.

The asymptotic minimality of the revised binomial is not an isolated cardinal arithmetic fact, but actually has applications which change the standard intuition about using cardinals in mathematics. Let us quote an example.

A Boolean algebra B is μ-linked if $B \setminus \{0\}$ is the union of at most μ sets, in each of which any two elements are compatible (have non-zero intersection); it is ccc, i.e. satisfies the countable chain condition, if in every uncountable subset of the algebra there are two elements with non-zero intersection. Hajnal, Juhász and Szentmiklóssy in their [1997] proved the following interesting structure theorem for Boolean algebras: If $\mu = \mu^{\aleph_0}$ then every ccc Boolean algebra B whose cardinality satisfies $|B| \leq \mu^{+\omega} < 2^\mu$ is μ-linked.

What about larger cardinalities below 2^μ? Well, the same authors proved that this was independent: for $|B| = \mu^{+\omega+1}$ the condition could hold or could fail. This may indicate that nothing more can be proved on the relation between chain condition and μ-linkedness, as Shelah writes ([2000a, Sect. 8]):

> This gives the impression of essentially closing the issue, and so I would have certainly thought some years ago,...

Now Shelah proves: If $\mu = \mu^{<\beth_\omega}$ and B is a ccc Boolean algebra whose cardinality satisfies $|B| < 2^\mu$ then B is μ-linked. Here the result is gotten for *all* cardinals below 2^μ, rather than only for the first $\omega + 1$ of them. The proof uses the revised GCH above $\beth_\omega$.

What has changed in Shelah's perspective? It is the awareness that things do not always become more complicated along the sequence of cardinals. Once Hajnal, Juhász and Szentmiklóssy pushed their proof up to $\mu^{+\omega}$ and showed that the next case is consistently false, then why should anyone bother to look higher? What was shown is that there is a qualitative difference between the condition $\mu = \mu^{\aleph_0}$ and the condition $\mu = \mu^{\beth_\omega}$.

This is a new sort of intuition about infinite cardinals. One may say, more mature, or less naïve. One finds such a point of view among the mathematicians who look at the natural numbers. There it is not expected that simple intuitions will guide the researcher correctly. Many years of number theoretic investigations have accustomed number theorists to expect surprises. With cardinal arithmetic this is only beginning.

Kurt Gödel [1947] introduced "verifiable consequences" as a means by which new axioms should be evaluated. A new axiom could help mathematicians easily prove theorems which may later be proved without it. Such is, for example, the Axiom of Choice. In the same paper Gödel uses his own criterion to reject CH. Among all consequences of CH, he explains, none turned out to be "really" true. That of course projected on Gödel's own constructibility axiom $V = L$, which implies CH. Later mathematical discoveries, like Scott's theorem (see below) also indicated that L may be very far from the "real" universe of sets.

However, as one moves up the list of cardinal numbers, the axiom $V = L$ seems to have more and more "verifiable consequences", if not literally, then at least metaphorically. Asymptotic regularity along the sequence of cardinal numbers emerged in many combinatorial discoveries of Shelah in the sense that combinatorial principles which hold in L by Jensen's work had weaker forms that are true in ZFC—from some point onwards.

Shelah's revised GCH in ZFC is the culmination of the "eventual regularity" theme, or, the verifiability of $V = L$ theme, which emerged from previous pcf discoveries. The standard GCH is of course not a ZFC theorem, and even the binomial hypothesis is not a ZFC theorem, though its violation requires consistency strength, but the revised GCH is a ZFC theorem—above $\beth_\omega$. A. Kanamori writes in his review of [Shelah, 2000b]:

> Bewildered by the sheer volume of [Shelah's] work, one can lose the sense for those few among the many achievements that might have an even higher claim to significance. The current paper under review, number 460, is such a paper, and although it was written in the early 1990s, it has not appeared in print until now. ... This is a remarkable ZFC result about the small size of covering families. Whether or not one regards the result as an appropriate "solution" to Hilbert's first problem, it is certainly a triumph for the author's pcf theory, as developed in his book.

In spite of its evident significance, Shelah's [2000b] claim that this result was a solution to Hilbert's first problem did not stir any serious discussion. On the one hand, this is certainly not Hilbert's original problem. On the other hand, after Easton's result, which Hilbert of course was not acquainted with, the problem, viewed broadly as "to discover the rules of cardinal exponentiation", needed to be reformulated. If one adopts Hilbert's problem to "is cardinal arithmetic simple" or even "do the arithmetic functions assume their least possible values" then this is indeed a partial solution: a natural cardinal arithmetical function assumes *asymptotically* its least possible values, in every universe of sets which satisfies ZFC.

As an application of his theorem, Shelah proved the following: for every $\lambda \geq \beth_\omega$, $2^\lambda = \lambda^+$ if and only if Jensen's diamond holds on λ^+. Kanamori addresses in his review this new connection between GCH and Jensen's diamond:

> The two principles had long been known to be distinct at small cardinals, but Shelah's result says that for sufficiently large λ, they are equivalent! There are further applications to topology and to model theory. Once again, the power of the author's pcf theory is demonstrated with surprising and elegantly concise results provable in ZFC.

Shelah's discovery of the revised GCH in ZFC enabled many new proofs by induction along the cardinals. Some of these inductions have to begin above $\beth_\omega$, though. A new type of intuition has formed: there is eventual regularity in ZFC. While in the small cardinals there may be a radically different behavior in V than the behavior in L, asymptotically V behaves in many ways like L.

Most of Shelah's diamond consequences were superseded by an even more surprising theorem of his, from 2007: for every uncountable λ, $2^\lambda = \lambda^+$ if and only if diamond holds on λ^+. Thus, $\aleph_0$ is the *only* infinite cardinal on which and instance of GCH may hold *without* being accompanied by Jensen's diamond on its successor.

This is, philosophically, an interesting example in the development of the subject, as the "simplest case" was again, as in other cases, not discovered first: Shelah first proved the equivalence of instances of GCH with Jensen's diamond for fairly large cardinals, and only later, and with a simpler proof, proved it for all successors of uncountable cardinals.

7 FROM MEASURABLE CARDINALS TO LARGE CARDINAL FORCING

Measurable cardinals are never singular! But measurable cardinals theory nevertheless induced several major developments in singular cardinal theory.

7.1 *Ulam's discovery of measurable cardinals and a regular limit continuum*

The discussion of Ulam's [1930] discoveries about measurable cardinals belongs naturally to the opening lines of a history of large cardinals. This chapter of modern set theory is, however, relevant to the history of singular cardinals.

The first relevance is that by proving that the least real-valued measurable cardinal is either at most the continuum or else a strong limit cardinal, Ulam directed attention for the first time to the possibility that the continuum may be a limit cardinal. Ulam discovered that in yet another branch of mathematics—measure theory—the relation between a cardinal and its cofinality is useful. Thus, the least cardinal which carries a real-valued measure cannot be a successor, and cannot be singular. This makes the least real-valued measurable cardinal a regular limit—one of those "exorbitantly big" cardinal numbers which Hausdorff predicted would not be related to any meaningful mathematical consideration. Furthermore, if the real-valued measure is atomless, the continuum is at least as large as the least real-valued measurable.

7.2 *Scott's theorem*

Set theory developed as a modern discipline from Gödel's work on the constructible universe. The role of formal logic and the concept of the universe of set theory became central, and the line of research of set theory as a *set of axioms* became a central branch of investigation.

Łoś's discovery of ultrapowers and Ulam's work on measurable cardinals were combined by Scott [1961] to introduce the construction of the ultraproduct embedding of the universe of set theory into a transitive class. Two theorems were proved by Scott: first, that if GCH fails at a measurable cardinal κ then it fails in many cardinals below κ; second, that if a measurable cardinals exists, then $V \neq L$.

This was the first time in which the way L sits inside V was investigated, and in which it was shown that L could be different from V—if large cardinals exist.

After Scott's theorem further work was done by Gaifman and by Rowbottom, who showed that if large cardinals exist not only did L differ from V but actually L was very small—-e.g. contains only countably many real numbers.

Silver's Ph.D. thesis completed this line of investigation by showing that a small L was *very* small—that assuming a set of integers $0^{\#}$ exists as given by large cardinals, all cardinals from V, for instance, are indiscernible, inaccessible and so on in L. In particular, singular cardinals in V are regular from the point of view of L. This meant that singular cardinals could be much larger in an inner model than what they seemed in V. This possibility was manifested much more sharply after Prikry showed how a measurable could be turned into a singular of countable cofinality by forcing while preserving all cardinals as cardinals; yet already at that stage of history, the importance of singular cardinals to the study of the relation between V and L became evident.

The role singular cardinals played in this branch of set theory became much more central when Jensen derived the first consequences from the assumption that $0^{\#}$ did not exist after Silver's theorem about singular cardinals. Jensen's covering lemma was discovered under direct influence of Silver's theorem, and implied that if $0^{\#}$ did not exist, that all singular cardinals in V are also singular in L. Thus, although their elementary theory was not yet explored in depth, singular cardinals served as a central tool for detecting the distance between L and V.

7.3 Generic sequences: Prikry's discovery

After the invention of forcing, measurable cardinals were found related to singular cardinal in a surprisingly intimate way.

In 1970 Karel Prikry made a formidable discovery: that an unbounded ω-sequence to a measurable cardinal κ could be introduced by a forcing notion, called ever since its discovery "Prikry forcing", without collapsing the cardinal, without introducing bounded subsets of κ and without changing the universe above κ. In short, Prikry's forcing does virtually nothing else other than changing the cofinality of κ to ω. In other words, the singular of cofinality ω is a strong limit regular—in fact measurable—cardinal in an inner model which is hidden below the full universe V by a thin crust.

The unbounded sequences in a singular cardinal received a totally different meaning after Prikry's discovery. Prikry sequences are still the basis to all modern large cardinal forcing constructions for increasing the power of a strong limit singular cardinal. Prikry himself observed that his forcing could provide a counterexample to the SCH if it could be arranged to have a measurable cardinal κ which violates GCH. Indeed, Silver managed to start with an even larger large cardinal—a supercompact—and by forcing turn it into a measurable κ with $2^\kappa > \kappa^+$, actually, with 2^κ arbitrarily large. Now following Silver's forcing by Prikry's, this κ became a strong limit singular of cofinality ω which satisfied $2^\kappa > \kappa^+$. It is important to observe that the singular κ was very high on the list of alephs. In retrospect, this made a difference, as for strong limit singular cardinals below the least fixed point of the aleph function it is, as we now know, impossible to make their power set arbitrarily large. The feeling at the time was that the use of a large cardinal to obtain a violation of SCH would turn out to be unnecessary.

The next major development was Magidor's technique of changing the cofinality of a large cardinal. Magidor combined a system of Levy collapses with Prikry's method of adding an ω-sequence to a large cardinal and obtained in [1977b] a model in which $\aleph_\omega$ was strong limit and $2^{\aleph_\omega}$ was larger than $\aleph_{\omega+1}$, as large as $\aleph_{\omega+\omega+1}$. Thus, the first singular cardinal—$\aleph_\omega$—was shown to consistently violate SCH. The difference between the violation of SCH at a "large" singular by combining Prikry and Silver forcing, described above, to Magidor's result is not merely quantitative. In retrospect we know that the structure of the product of the ω_n's imposes strong restrictions on the power set of $\aleph_\omega$, while at larger singular cardinals, e.g. fixed points of the aleph function, such limitations are not present. Magidor's forcing construction had to overcome these difficulties, before pcf theory was discovered.

In a subsequent paper [1977c] Magidor started from a larger large cardinal, managed to obtain an even more stunning result: $\aleph_\omega$ is the first cardinal violating GCH. This means that for every $n \in \omega$ it holds that $2^{\aleph_n} = \aleph_{n+1}$ but $2^{\aleph_\omega} = \aleph_{\omega+2}$ in Magidor's model.

Still, in 1973, Magidor—like everyone else working in this field —was convinced that large cardinals were not truly needed to force a failure of SCH and that eventually Easton's result would be extended to all cardinals by some forcing over, say, L.

Magidor was able to obtain violations of SCH in an arbitrary long initial segment of the ordinals with the techniques he introduced. After Radin's forcing [1982] was introduced, with which a club is added to a large cardinal, Foreman and Woodin [1991] were able to improve on Magidor's techniques to obtain a model in which GCH fails at every cardinal. Cummings [1992] used Radin forcing to obtain a model in which GCH holds at regular cardinals and fails at singular cardinals.

In [1983] Shelah built on Magidor's work to increase the lower bound on the power of a strong limit $\aleph_\omega$. For an arbitrary $\alpha < \omega_1$ Shelah proved the consistency of $2^{\aleph_\omega} = \aleph_{\alpha+1}$ while $2^{\aleph_n} < \aleph_\omega$ for all n.

An important breakthrough in Prikry-type forcing was obtained by Woodin in the early 1980s (unfortunately, unpublished). Woodin was able to replace the supercompact cardinal by a $(\kappa, \kappa^{++}+1)$ extender in obtaining GCH up to $\aleph_\omega$ and a gap of 2 above. This was a major step nearing the exact consistency strength, which is a measurable κ of order κ^{++}.

Gitik and Magidor, who, as they write in the paper [1992], were "grateful for the opportunity to work together in Berkeley", introduced a new type of forcing to obtain GCH below $\aleph_\omega$ and $2^{\aleph_\omega} = \aleph_{\alpha+1}$ for an arbitrary countable ordinal α. They used a directed system of extenders related by projections so that a Prikry sequence of a higher one created the Prikry sequence of a lower one. The directedness secured the κ^{++}-chain condition for their forcing. They obtained the lower bound of Shelah, with GCH below $\aleph_\omega$, from a weaker large cardinal hypothesis.

In a series of papers Gitik investigated this type of forcing further. He managed to replace the Woodin $(\kappa, \kappa^{++}+1)$ extender by a (κ, κ^{++}) extender, and continued to tighten the lower bounds obtained by Mitchell on the consistency strength of the negation of SCH. By around 1990 Gitik and Mitchell (cf. [Gitik, 1989, Gitik, 1991]) completed the mission of computing the exact consistency strength of the failure of SCH at $\aleph_\omega$: a measurable κ of order κ^{++}.

With Gitik's proof that there was no bound on the binomial of the least fixed point of the aleph function, basically all cardinal arithmetic problems concerning $\kappa \mapsto 2^\kappa$ that were expressible before the discovery of pcf theory were solved.

8 THE 21ST CENTURY

Singular cardinals theory is very active in the present century. In 2004 a workshop titled "Cardinal arithmetic at work" was held in Jerusalem, where more than half of the results which were presented were about singular cardinals. In 2004 in Banff, a meeting was held which was solely devoted to "singular cardinals combinatorics".

Many important results have been obtained about singular cardinals in the last decade. Of these let us mention Matteo Viale's proof [2006] that the Proper Forcing Axiom implies SCH, which built on earlier work by Moore [2006]. Assaf Sharon's forcing construction of the failure of SCH with combinatorial conditions and finally Itai Neeman's proof [Neeman, 2009] of the consistency of the failure of SCH at a singular of countable cofinality μ together with the tree property at μ^+.

More work is being done by Shelah and others on pcf without AC. Applications of pcf theory to topology, measure theory and algebra continue to flow.

The mathematical subject of singular cardinals is most likely only in it beginning, and someone should certainly revise its history in the 22nd century.

9 SUMMARY AND CONCLUDING REMARKS

The understanding of singular cardinals developed in the 20th century in a chaotic way. The main feature of this development, as seen from the description above, is that existing techniques were not accompanied by suitable conjectures. In other words, not all that could be discovered with elementary methods was discovered; much of it was not even conjectured.

Singular cardinals appeared in many subjects, and their centrality to mathematics was well established. Within set theory itself, singular cardinals play a role in several contexts. First, they serve as gauges for the distance between L and V, and thus have metamathematical significance. Second, their arithmetic is where ZFC could restore some of its lost honor after it was discovered that it could prove next to nothing about *regular* cardinal arithmetic.

Finally, in tracing the major developments in the theory, one finds that concrete mathematical problems were several times the stimulus for important discoveries in set theory. This is true for set theory in general, of course, as Cantor discovered the uncountability of $\mathbb{R}$ in his investigations of uniqueness of trigonometric series, and is evident in the history of singular cardinals as well. Topological investigations related to homotopy led to the first constructions of spaces in the product $\prod_n \omega_n$ whose more advance properties touched pcf theory; the problem of a Jonsson algebra on $\aleph_{\omega+1}$ is where pcf theory is directly born; and Kishor Kale's innocent looking problem about the number of open sets in a countably generated topology let quite quickly to Shelah's discovery of the revised GCH in ZFC.

Many of the advanced techniques for treating singular cardinals are widespread. Large cardinal forcing, L-like combinatorics and pcf theory are being used daily by young set theorists, and perhaps soon by mathematicians from neighboring subjects. There is still much left to be explored.

Having traced the fascinating connection between Hausdorff's work and Shelah's work, it is fitting to conclude this article by the following mathematical problem which is related to the work of both mathematicians. This problem is a good benchmark for the existing techniques, and it illustrates that not all singular cardinals are alike.

Hausdorff proved that for every strong limit cardinal μ there is a universal linearly ordered set of cardinality μ. A partial converse was proved by Kojman and Shelah [1992] for an *initial segment* of the singular cardinals: For every singular μ below the least fixed point of second order, if μ is not strong limit, then a universal linearly ordered set of cardinality μ does not exist. (A fixed point is a cardinal $\mu = \aleph_\mu$ and a fixed point of the second order is a cardinal μ which is the μ-th fixed point.)

Does this converse hold for all cardinals? Does it fail at the least fixed point of second order? Does it fail higher?

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