1. Some notation and definitions

By a space we will mean a Hausdorff topological space. We will deal mainly with metrizable spaces. Recall that a space $X$ is arc-wise connected (or pathwise connected) if for every $a, b \in X$ there is a continuous map $f: [0, 1] \to X$ with $f(0) = a, f(1) = b$.

Let $B^{n+1}$ denote the $(n+1)$-dimensional closed unit ball in $\mathbb{R}^{n+1}$ and let $S^n$ denote its boundary, i.e. the $n$-dimensional sphere. Observe that arc-wise connectedness of $X$ means that every continuous map $f: S^0 \to X$ can be continuously extended to a map $g: B^1 \to X$. In the same way we can define higher dimensional analogues of arc-wise connectedness: a space $X$ is $k$-connected if every continuous map $f: S^k \to X$ can be extended to a continuous map $g: B^{k+1} \to X$. A space $X$ is $C^n$ if it is $k$-connected for each $k < n+1$ ($n$ is a natural number or $n = \infty$).

These notions can be defined locally: a space $X$ is locally $k$-connected if for each $p \in X$ and its neighborhood $U$ there is a smaller neighborhood $V$ of $p$ such that every continuous map $f: S^k \to V$ has a continuous extension $g: B^{k+1} \to U$. The local version of property $C^n$ is denoted by $LC^n$.

Let $X = [-1, 1] \cup \{(t, \sin \frac{\pi}{2} t) : t \in (0, 1)\}$. Then $X$ is compact and connected but not arc-wise connected. If we add to $X$ an arc joining points $(0, -1)$ and $(1, 0)$, then we obtain the Warsaw circle which is a compact arc-wise connected space which is not locally arc-wise connected.

2. Absolute Retracts and Absolute Extensors

Recall that a retraction is a continuous map $f: X \to Y$ such that $Y \subset X$ and $f|Y = \text{id}$. Then $Y$ is a retract of $X$.

Let $\mathcal{K}$ be a class of spaces and let $Y \in \mathcal{K}$. We say that $Y$ is an absolute retract for $\mathcal{K}$, if whenever $Y$ is a closed subset of $X$ and $X \in \mathcal{K}$ then $Y$ is a retract of $X$. We write $Y \in \text{AR}(\mathcal{K})$. If $\mathcal{K}$ is the class of all metrizable spaces then we write $Y \in \text{AR}$ and say that $Y$ is an absolute retract.

A space $Y$ is an absolute extensor for a class of spaces $\mathcal{K}$, if whenever $X \in \mathcal{K}$, $A$ is a closed subset of $X$ and $f: A \to Y$ is continuous then there exists a continuous map $g: X \to Y$ which extends $f$. We write $Y \in \text{AE}(\mathcal{K})$.

According to Tietze-Urysohn theorem, $[0, 1]$ and $\mathbb{R}$ are absolute extensors for the class of normal spaces. If, for instance, $\mathcal{K}$ is the class of Boolean spaces then an absolute extensor for $\mathcal{K}$ is called a Dugundji space. Completely metrizable spaces belong to the class of Dugundji spaces.

Let us note some basic properties of absolute extensors.

Proposition 1. Let $\mathcal{K}$ be a class of spaces. Then:

(a) If $Y \in \mathcal{K}$ and $Y \in \text{AE}(\mathcal{K})$ then $Y \in \text{AR}(\mathcal{K})$. 


(b) A retract of an absolute extensor for \( K \) is an absolute extensor for \( K \).
(c) A product of absolute extensors for \( K \) is an absolute extensor for \( K \).
(d) Assume that \( K \) contains closed \( k \)-dimensional balls for \( k \leq n + 1 \). Then each absolute extensor for \( K \) is \( C^n \).

**Proof.** (a) If \( Y \) is a closed subset of \( X \) and \( X \in K \) then the identity map \( \text{id}_Y : Y \to Y \) can be continuously extended to \( r : X \to Y \). Then \( r \) is a retraction and hence \( Y \) is a retract of \( X \).

(b) Let \( Z \) be a retract of \( Y \in \text{AE}(K) \), let \( r : Y \to Z \) be a retraction. Let \( f : A \to Z \) be continuous, where \( A \) is closed in \( X \) and \( X \in K \). There is a continuous map \( g : X \to Y \) which extends \( f \). Then \( h = r \circ g \) is also an extension of \( f \) and \( h[X] \subset Z \).

(c) Let \( Y = \prod_{\alpha<\kappa} Y_\alpha \) where \( Y_\alpha \in \text{AE}(K) \) for each \( \alpha < \kappa \). Let \( f : A \to Y \) be continuous, where \( A \) is closed in \( X \) and \( X \in K \). For each \( \alpha \) let \( \text{pr}_\alpha : Y \to Y_\alpha \) be the \( \alpha \)-th projection. For each \( \alpha < \kappa \) there is \( g_\alpha : X \to Y_\alpha \) which extends \( \text{pr}_\alpha \circ f \). Then the diagonal map \( g : X \to Y \) defined by \( g(x)(\alpha) = g_\alpha(x) \), is a continuous extension of \( f \).

(d) This follows from the definition of \( k \)-connectivity. \( \square \)

The last statement implies that if \( K \) contains \([0, 1]\) then every absolute extensor for \( K \) is connected. We will see that metric absolute retracts (i.e. absolute retracts in the class of metric spaces) have much more interesting properties.

The next lemma describes situations when absolute retracts are absolute extensors.

**Lemma 1.** Let \( K \) a class of spaces with the following property: for every \( Y \in K \) there is \( E \in K \) such that \( E \in \text{AE}(K) \) and \( Y \) is a closed subset of \( E \). Then for every \( Y \in K \), \( Y \in \text{AR}(K) \) if and only if \( Y \in \text{AE}(K) \).

**Proof.** The "if" part follows from Proposition 1(a). Fix \( Y \in \text{AR}(K) \) and a continuous map \( f : A \to Y \) from a closed subset of \( X \in K \). Assume that \( Y \) is a closed subset of \( E \in \text{AE}(K) \cap K \). There is a retraction \( r : E \to Y \), because \( Y \in \text{AR}(K) \). There is a continuous extension \( g : X \to E \) of \( f \), because \( E \in \text{AE}(K) \). Then \( r \circ g \) is a continuous extension of \( f \) and \( r \circ g : X \to Y \). \( \square \)

**Corollary 1.** Let \( C \) denote the class of all compact spaces and let \( Y \in C \). Then \( Y \in \text{AR}(C) \) iff \( Y \) is a retract of \([0, 1]^{\kappa} \) where \( \kappa = w(Y) \). Moreover \( Y \in \text{AR}(C) \iff Y \in \text{AE}(C) \).

**Proof.** Every compact space is normal (since we assumed that all spaces are Hausdorff), thus \([0, 1] \in \text{AE}(C) \) by Tietze-Urysohn theorem. By Proposition 1(c), a Tikhonov cube \([0, 1]^{\kappa} \) is an absolute extensor for \( C \) (in fact for the class of normal spaces). A compact space of weight \( \kappa \) can be embedded into \([0, 1]^{\kappa} \). Now the above statements follow from Proposition 1(a),(b) and Lemma 1. \( \square \)

We will be interested in absolute retracts in the class of metric spaces. We will show that the class of metric spaces has the property described in Lemma 1, namely a metric space is a closed subset of a normed linear space and a normed linear space is an AE. The first part is the theorem of Arens-Eells (known also as the theorem of Kuratowski-Wojdyslawski) and the second part follows from the theorem of Dugundji. For the study of the theory of absolute retracts and extensors we refer to Borsuk’s monograph [2] or van Mill’s book [11].
3. Michael’s Selection Theorem

Given a topological space $Y$, denote by $\text{CL}(Y)$ the collection of all nonempty closed subsets of $Y$. A multifunction is a map of the form $F: X \to \text{CL}(Y)$. Let $X$ be a topological space. A multifunction $F: X \to \text{CL}(Y)$ is lower semicontinuous (briefly lsc) if for every open set $U \subset Y$ the set $\{x \in X: F(x) \cap U \neq \emptyset\}$ is open in $X$. We write $U^-$ for the set $\{A \in \text{CL}(Y): A \cap U \neq \emptyset\}$. Then $F$ is lsc iff $F^{-1}[U^-]$ is open in $X$ whenever $U$ is open in $Y$. A selection for $F$ is a map $f: X \to Y$ such that $f(x) \in F(x)$ for every $x \in X$.

**Michael’s Selection Theorem** ([10]). Let $X$ be a paracompact Hausdorff space and let $Y$ be a Banach space. Then every lsc multifunction $F: X \to \text{CL}(Y)$ with convex values, admits a continuous selection.

**Corollary 2.** Every Banach space is an AE for the class of paracompact Hausdorff spaces.

**Proof.** Let $f: A \to Y$ be a continuous map defined on a closed subset of a paracompact space $X$ and let $Y$ be a Banach space. Define $F: X \to \text{CL}(Y)$ by setting

$$F(x) = \begin{cases} \{ f(x) \} & \text{if } x \in A, \\ Y & \text{otherwise.} \end{cases}$$

Observe that a selection for $F$ is an extension of $f$. Thus it suffices to check that $F$ is lsc. Fix open $U \subset Y$. There is open $W \subset X$ with $f^{-1}[U] = W \cap A$. Then we have $F^{-1}[U^-] = f^{-1}[U] \cup (X \setminus A) = (W \cap A) \cup (X \setminus A) = W \cup (X \setminus A)$. It follows that $f^{-1}[U^-]$ is open in $X$. □

The proof of Michael’s selection theorem is based on two lemmas. We assume that $X, Y$ are as above.

**Lemma 2.** Let $F: X \to \text{CL}(Y)$ be lsc with convex values and let $\varepsilon > 0$. Then there exists a continuous map $f: X \to Y$ such that $\text{dist}(f(x), F(x)) < \varepsilon$ for every $x \in X$.

**Proof.** Fix an open cover $\mathcal{U}$ of $Y$ consisting of sets of diameter $< \varepsilon$. Let $\mathcal{V} = \{ F^{-1}[U^-]: U \in \mathcal{U} \}$. As $F$ is lsc, $\mathcal{V}$ is an open cover of $X$. As $X$ is paracompact and Hausdorff, there exists a partition of unity $\mathcal{H}$ subordinated to $\mathcal{V}$. Specifically, $\mathcal{H}$ is a family of continuous functions from $X$ into $[0,1]$ such that $\{ h^{-1}(0,1) \}_{h \in \mathcal{H}}$ is a locally finite open cover of $X$; for each $h \in \mathcal{H}$ there is $U_h \in \mathcal{U}$ such that $h^{-1}(0,1) \subset F^{-1}[U^-]$ and $\sum_{h \in \mathcal{H}} h(x) = 1$ for every $x \in X$.

For each $h \in \mathcal{H}$ choose $y_h \in U_h$. Define $f: X \to Y$ by setting

$$f(x) = \sum_{h \in \mathcal{H}} h(x) y_h.$$  

Observe that $f$ is continuous. Fix $x \in X$ and set $\mathcal{H}_x = \{ h \in \mathcal{H}: h(x) > 0 \}$. Then $\mathcal{H}_x$ is finite and for each $h \in \mathcal{H}_x$ there exists $z_h \in F(x) \cap U_h$. We have $\| y_h - z_h \| < \varepsilon$ so for $z = \sum_{h \in \mathcal{H}_x} h(x) z_h$ we get $\| f(x) - z \| < \varepsilon$ and $z \in F(x)$. This shows that $\text{dist}(f(x), F(x)) < \varepsilon$. □

**Lemma 3.** Let $F: X \to \text{CL}(Y)$ be lsc and let $f: X \to Y$ be a continuous map such that $\text{dist}(f(x), F(x)) < \varepsilon$ for every $x \in X$. Then a multifunction $G: X \to \text{CL}(Y)$ defined by the formula

$$G(x) = \text{cl} \left[ B(f(x), \varepsilon) \cap F(x) \right],$$

is lower semicontinuous.
Proof. Fix \( x_0 \in X \) and open \( U \subset Y \) with \( x_0 \in G^{-1}[U] \). Choose \( p \in U \cap B(f(x_0), \varepsilon) \cap F(x_0) \). Let \( \delta > 0 \) be such that \( B(p, 2\delta) \subset U \cap B(f(x_0), \varepsilon) \). Fix a neighborhood \( W \) of \( x_0 \) such that for every \( x \in W \) we have \( B(p, \delta) \subset B(f(x), \varepsilon) \) and \( B(p, \delta) \cap F(x) \not= \emptyset \). Then \( W \subset G^{-1}[U] \). It follows that \( G^{-1}[U] \) is open. □

Using the above lemmas we can define sequences \( \{F_n\}_{n<\omega}, \{f_n\}_{n<\omega} \) such that \( F_n : X \rightarrow CL(Y) \) is an lsc multifunction with convex values, \( f_n : X \rightarrow Y \) is continuous and
\begin{align*}
&\text{(i) } \text{dist}(f_n(x), F_n(x)) < 2^{-n}, \\
&\text{(ii) } F_{n+1}(x) = \text{cl}[B(f_n(x), 2^{-n}) \cap F_n(x)],
\end{align*}
for every \( x \in X \). We start with a given multifunction \( F : X \rightarrow CL(Y) \). Conditions (i), (ii) imply that \( \{f_n\}_{n<\omega} \) is uniformly Cauchy. By the completeness of \( Y \), there is a continuous map \( f : X \rightarrow Y \) which is the limit of \( \{f_n\}_{n<\omega} \). We have \( \text{dist}(f_n(x), F(x)) \leq \text{dist}(f_n(x), F_n(x)) < 2^{-n} \) because \( F_n(x) \subset F(x) \). Thus \( f(x) \in F(x) \). Hence \( f \) is a selection for \( F \).

Remarks. (a) Observe that if \( Y \) is separable Banach space and \( X \) is normal and countably paracompact then Lemma 2 still holds, since we can consider countable open covers only. Lemma 3 is true for maps of arbitrary topological spaces. Thus we get another version of Michael’s theorem: every lsc multifunction from a countably paracompact normal space, with convex values in a separable Banach space, admits a continuous selection.

(b) Let \( K(Y) = \{ A \subset Y : A \text{ is compact, convex and nonempty } \} \cup \{ \emptyset \} \). Another Michael’s selection theorem says that if \( X \) is normal, \( Y \) is a separable Banach space then every lsc multifunction \( F : X \rightarrow K(Y) \) admits a continuous selection. However, the original proof in [10] contains a gap; a correct proof can be found in Nagata’s book [12]. This theorem implies that a separable Banach space is an AE for the class of normal spaces.

(c) If \( X \) is a strongly 0-dimensional paracompact space and \( Y \) is a complete metric space then we can prove a version of Lemma 2 for lsc multifunctions \( F : X \rightarrow CL(Y) \), because every open cover of \( X \) has a refinement consisting of clopen sets; then we do not need convex combinations in \( Y \). Thus, in this case, every lsc multifunction admits a continuous selection.

As a corollary, we obtain the result which says that every complete metric space is an AE for the class of Boolean spaces.

4. The Theorem of Arens-Eells

We prove the theorem of Arens-Eells, which is also known as the theorem of Kuratowski-Wojdyslawski, about embeddings of metric spaces into normed linear spaces.

Theorem 1. Every metric space can be embedded as a closed set in a normed linear space.

Proof. Let \( (X, d) \) be a metric space. We may assume that \( d \leq 1 \), otherwise we can take \( (X, d_1) \), where \( d_1(x, y) = \min\{d(x, y), 1\} \). Let \( L(X) \) be the set of all functions \( f : X \rightarrow [0, 1] \) which are nonexpansive, i.e. \( |f(x) - f(y)| \leq d(x, y) \) for all \( x, y \in X \). Let \( F(X) \) be the free real vector space generated by \( X \), i.e. \( F(X) \) is the set of all formal combinations \( \sum_{i<k} \alpha_i x_i \), where \( x_i \in X, \alpha_i \in \mathbb{R} \) for \( i < k \), equipped with the usual operations. If \( v \in F(X) \) and \( v = \sum_{i<k} \alpha_i x_i \), where \( x_0, \ldots, x_{k-1} \in X \) then we write \( \langle v, f \rangle \) for \( \sum_{i<k} \alpha_i f(x_i) \), whenever \( f : X \rightarrow \mathbb{R} \). Now define
\[ \|v\| = \sup_{f \in L(X)} |\langle v, f \rangle|. \]

Clearly \( (F(X), \| \cdot \|) \) is a normed space and each function of the form \( v \leftrightarrow \langle v, f \rangle \), where \( f \in L(X) \), is continuous. Observe that \( \|x\| = 1 \) for every \( x \in X \).
Now observe that \( \|x_0 - x_1\| = d(x_0, x_1) \) for every \( x_0, x_1 \in X \). Indeed, if \( f \in L(X) \) then \( |\langle x_0 - x_1, f \rangle| = |f(x_0) - f(x_1)| \leq d(x_0, x_1) \) and on the other hand \( \langle x_0 - x_1, h \rangle = d(x_0, x_1) \) for \( h = d(\cdot, x_1) \). It follows that \((X, d)\) is isometrically embedded into \((F(X), \|\cdot\||)\). It remains to show that \( X \) is closed in \( F(X) \).

Fix \( v \in F(X) \setminus X \). We may assume that \( \|v\| = 1 \), because \( X \) is contained in the unit sphere of \( F(X) \). Then \( v = \sum_{i<k} \alpha_ix_i \), where \( x_i \in X \) are pairwise distinct and \( \alpha_i \neq 0 \) for \( i < k \).

Suppose first that \( k = 1 \), so \( v = \alpha_0x_0 \). In this case \( |\alpha_0| = \|v\| = 1 \) and hence \( \alpha_0 = -1 \), because \( v \notin X \). Thus for \( f \in L(X) \) such that \( f(x_0) > 0 \) we have \( \langle v, f \rangle < 0 \) and setting \( V = \{w \in F(X) : \langle w, f \rangle < 0 \} \) we get a neighborhood of \( v \) disjoint from \( X \).

Suppose now that \( k > 1 \). Let \( U_0, U_1 \) be disjoint neighborhoods of \( x_0, x_1 \) respectively. Let \( f_i(x) = \text{dist}(x, X \setminus U_i) \). Then \( f_i \in L(X) \). Define

\[
V = \{w \in F(X) : \langle w, f_i \rangle > 0 \text{ for } i = 0, 1\}.
\]

Then \( V \) is open in \( F(X) \), \( v \in V \) and \( V \cap X = \emptyset \). This completes the proof. \( \square \)

5. Simplicial structures and Dugundji’s extension theorem

Let \( S \) be a finite set. A geometric simplex \( \sum(S) \) with vertices in \( S \) is the set of all formal convex combinations of the form \( \sum_{s \in S} \lambda_s s \), endowed with the usual topology. Now let \( \mathcal{A} \) be a collection of finite sets. Then \( \sum(\mathcal{A}) = \bigcup_{a \in \mathcal{A}} \sum(a) \) is an abstract polytope induced by \( \mathcal{A} \), \( \sum(\mathcal{A}) \) is endowed with the CW-topology: a subset \( U \subset \sum(\mathcal{A}) \) is open if \( U \cap \sum(a) \) is open in \( \sum(a) \) whenever \( a \in \mathcal{A} \). If \( \mathcal{A} = [Y]^{<\omega} \) for some \( Y \) then \( \sum(\mathcal{A}) \) is called a convex polytope. In this case we have \( \sum(\mathcal{A}) = \text{conv} Y \), where \( \text{conv} \) is the convex hull in the free real vector space generated by \( Y \). A CW-polytope is a topological space which is homeomorphic to \( \sum(\mathcal{A}) \) for some collection \( \mathcal{A} \) consisting of finite sets.

Now let \( Y \) be a topological space. A singular polytope in \( Y \) is a continuous map \( \varphi : \sum(\mathcal{A}) \rightarrow Y \), where \( \mathcal{A} \subset [Y]^{<\omega} \). If \( \mathcal{A} = \{a\} \) then \( \varphi \) is a singular simplex in \( Y \).

A simplicial structure in \( Y \) is a collection \( \mathcal{F} \) consisting of singular simplices in \( Y \), satisfying the condition

\[
(**) \quad \text{if } \sigma \in \mathcal{F} \text{ and } \tau \text{ is a subsimplex of } \sigma \text{ then } \tau \in \mathcal{F}.
\]

If \( \sigma \in \mathcal{F} \) then we denote by \( \text{vert } \sigma \) the set of vertices of \( \sigma \); we write \( \text{vert } \mathcal{F} \) for the collection \( \{\text{vert } \sigma : \sigma \in \mathcal{F}\} \). We say that a pair \((A, B)\) of subsets of \( Y \) is \( \mathcal{F}\)-convex if \([A]^{<\omega} \subset \text{vert } \mathcal{F} \) and \( \text{rng } \sigma \subset B \) whenever \( \sigma \in \mathcal{F} \) and \( \text{vert } \sigma \subset A \). Observe that in this case we have \( A \subset B \).

A simplicial structure \( \mathcal{F} \) is:

- **convex**, if vert \( \mathcal{F} = [Y]^{<\omega} \);
- **locally convex**, if for each \( p \in Y \) and its neighborhood \( V \) there is a neighborhood \( U \) of \( p \) such that \((U, V)\) is \( \mathcal{F}\)-convex;
- **regular**, if for each \( s \in \text{vert } \mathcal{F} \) and for each collection \( \{\tau_t : t \subset s\} \subset \mathcal{F} \) consisting of compatible simplices, there is \( \sigma \in \mathcal{F} \) which extends all \( \tau_t \)’s;
- **determined**, if for each \( s \in \text{vert } \mathcal{F} \) there is a unique \( \tau \in \mathcal{F} \) with \( s = \text{vert } \tau \).

These notions are due to Kulpa. He proved in [8] that every compact Hausdorff space with a convex and locally convex simplicial structure has the fixed-point property.

**Example 1.** Let \( E \) be a linear topological space. For each \( s \in [E]^{<\omega} \) let \( \sigma_s : \Sigma(s) \rightarrow E \) be the unique affine map which is the identity on \( s \). Let \( \mathcal{F} = \{\sigma_s : s \in [E]^{<\omega}\} \). Then \( \mathcal{F} \) is a convex determined simplicial structure in \( E \). Observe that \( \mathcal{F} \) is locally convex iff \( E \) is locally convex.
In the same way we can define a convex determined simplicial structure in every convex subset of $E$.

Let us note the following easy properties of simplicial structures.

**Proposition 2.** Let $F$ be a simplicial structure in a space $Y$. Then:

1. If $r : Y \to Z$ be a retraction then $F_Z = \{ r \circ \sigma : \sigma \in F \}$ is a simplicial structure in $Z$. Furthermore, if $F$ is regular (determined, locally convex, convex) then $F_Z$ has the same property.

2. If $U \subset Y$ is open then $F_U = \{ \sigma \in F : \text{rng} \sigma \subset U \}$ is a simplicial structure in $U$. If $F$ is regular (determined, locally convex) then $F_U$ has the same property.

Combining the above proposition, Example 1 and the theorem of Arens-Eells, we get the following.

**Corollary 3.** Every ANR [AR] has a determined locally convex [locally convex and convex] simplicial structure.

Our aim is to prove the converse. We prove a general version of Dugundji’s Extension Theorem for spaces with simplicial structures. This generalization is due to Kulpa [8], but a weaker result of this type was proved by Borges [1]. Also some similar but more complicated results can be found in some papers in mathematical analysis. Let us note that the classical version of Dugundji’s Extension Theorem concerns convex sets in locally convex linear topological spaces.

**Lemma 4.** If $F$ is a regular simplicial structure then there exists a determined simplicial structure $F_0 \subset F$ such that $\text{vert} F_0 = \text{vert} F$.

**Proof.** There exists a maximal (with respect to inclusion) determined structure $F_0 \subset F$. Suppose that $\text{vert} F_0 \neq \text{vert} F$ and let $s \in \text{vert} F \setminus \text{vert} F_0$ be a set of the smallest cardinality. Let $\tau_t \in F_0$, $t \subseteq s$ be such that $\text{vert} \tau_t = t$. By uniqueness, $\{ \tau_t : t \subseteq s \}$ are pairwise compatible so by the regularity of $F$ there exists a $\sigma \in F$ with $\text{vert} \sigma = s$ and which extends all of $\tau_t$’s. Now $F_0 \cup \{ \sigma \}$ is a determined structure and $F_0$ is not maximal, a contradiction. \( \square \)

Observe that every determined structure is regular. The above lemma says, in particular, that a regular locally convex (convex) simplicial structure contains a determined one with the same property.

**Dugundji’s Extension Theorem.** Let $Y$ be a topological space with a regular locally convex simplicial structure $F$. Let $f : A \to Y$ be a continuous map defined on a closed subset of a metric space $(X,d)$. Then there is a continuous extension $F : W \to Y$ of $f$ such that $W \supset A$ is open in $X$. If additionally $(f[A],Y)$ is $F$-convex then $W = X$.

**Proof.** (a) By Lemma 4 we can find a determined simplicial structure $F_0 \subset F$ such that $\text{vert} F_0 = \text{vert} F$. Then $F_0$ is locally convex and $(f[A],Y)$ is $F_0$-convex provided it is $F$-convex. Thus we may assume that $F$ is determined.

(b) Let $\mathcal{V} = \{ B(x, \frac{1}{2} \text{dist}(x,A)) : x \in X \setminus A \}$. Then $\mathcal{V}$ is an open cover of $X \setminus A$. Using paracompactness, we can find a locally finite partition of unity $\mathcal{H}$ subordinated to $\mathcal{V}$. For each $h \in \mathcal{H}$ pick $x_h \in X \setminus A$ and $a_h \in A$ such that $h^{-1}(0,1] \subset B(x_h, \frac{1}{2} \text{dist}(x_h,A))$ and $d(x_h,a_h) < 2 \text{dist}(x_h,A)$. 

6
(c) For each \( x \in X \setminus A \) denote by \( \mathcal{H}_x \) the set of all \( h \in \mathcal{H} \) such that \( h(x) > 0 \). Observe that \( \mathcal{H}_x \) is finite, because \( \mathcal{H} \) is locally finite. Let \( s_x = \{ f(a_h) : h \in \mathcal{H}_x \} \) and define \( M = \{ x \in X \setminus A : s_x \in \text{vert} F \} \).

Observe that \( M = X \setminus \) whenever \((f[A], Y)\) is \( F\)-convex. Now for each \( x \in M \) choose \( \sigma_x \in \mathcal{F} \) with \( s_x \subset \sigma_x \) and define \( F : A \cup M \to Y \) by setting \( F|A = f \) and

\[
F(x) = \sigma_x \left( \sum_{h \in \mathcal{H}_x} h(x) f(a_h) \right), \quad x \in M.
\]

(d) If \( x_0 \in M \) then there is a neighborhood \( U \) of \( x_0 \) such that \( U \subset X \setminus A \) and \( U \cap h^{-1}(0, 1] \neq \emptyset \) only for \( h \in \mathcal{S} \), where \( \mathcal{S} \subset \mathcal{H} \) is finite. Now \( F|\{U \cap M\} \) is a finite union of maps of the form

\[
\varphi_T(x) = \tau \left( \sum_{h \in T} h(x) f(a_h) \right),
\]

where \( \tau \in \mathcal{F} \) and \( T \subset \mathcal{S} \) is such that \( \{ f(a_h) : h \in T \} \subset \text{vert} \tau \). Each \( \varphi_T \) is continuous and has relatively closed domain. It follows that \( F \) is continuous at each point of \( M \).

(e) Fix \( a \in A \) and a neighborhood \( V \) of \( f(a) \). Let \( V_0 \) be such a neighborhood of \( f(a) \) that \((V_0, V)\) is \( F\)-convex. By the continuity of \( f \), we can find \( \varepsilon > 0 \) so that \( f[A \cap B(a, 6\varepsilon)] \subset V_0 \). Let \( U = B(a, \varepsilon) \). We show that \( U \subset A \cup M \) and \( F[U] \subset V \).

Fix \( x \in U \setminus A \) and fix \( h \in \mathcal{H}_x \). Then \( h(x) > 0 \) so \( x \in B(x_h, \frac{1}{2} \text{dist}(x_h, A)) \). Thus

\[
\varepsilon > d(a, x) \geq d(a, x_h) - d(x_h, x) \geq \text{dist}(x_h, A) - d(x_h, x)
\]

\[
\geq \text{dist}(x_h, A) - \frac{1}{2} \text{dist}(x_h, A) = \frac{1}{2} \text{dist}(x_h, A).
\]

Hence \( d(x, x_h) < \varepsilon \). Moreover

\[
d(x_h, a_h) \leq 2 \text{dist}(x_h, A) \leq 2d(x_h, a).
\]

Thus

\[
d(a, a_h) \leq d(a, x_h) + d(x_h, a_h) \leq 3d(a, x_h)
\]

\[
\leq 3(d(a, x) + d(x, x_h)) < 3(\varepsilon + \varepsilon) = 6\varepsilon.
\]

As \( h \in \mathcal{H}_x \) was fixed arbitrarily, it follows that \( s_x \subset V_0 \). Consequently \( x \in M \) and \( F(x) \in V \), because \((V_0, V)\) is \( F\)-convex. Thus we have showed that \( F[U \setminus A] \subset V \). Clearly \( F[U \cap A] \subset V \).

(f) By (d) and (e), \( F \) is continuous and each point of \( A \) has a neighborhood contained in \( A \cup M \). Thus \( A \subset \text{int}(A \cup M) \). This completes the proof. \( \square \)

**Corollary 4.** A space with a locally convex [locally convex and convex] regular simplicial structure is an ANE [AE] for metric spaces.

Now we are ready to prove an important result in the theory of metric absolute retracts.

**Theorem 2.** For a metrizable space \( Y \) the following conditions are equivalent:

(a) \( Y \in \text{AR} / Y \in \text{ANR} \).

(b) \( Y \) is a retract of [an open subset of] a normed linear space.

(c) \( Y \in \text{AE} / Y \in \text{ANE} \).

**Proof.** Implication (a) \( \implies \) (b) follows from the theorem of Arens-Eells. (b) \( \implies \) (c) follows from Dugundji’ Extension Theorem and from the fact that the AE [ANE] property is preserved under retractions. Clearly (c) \( \implies \) (a). \( \square \)
Lemma 5. If $Y$ is $C^\infty$ and $F$ is a regular simplicial structure in $Y$ then there exists a regular convex simplicial structure $F_1 \supset F$ such that $\{ \sigma \in F_1 : \text{vert } \sigma \in \text{vert } F \} \subset F$.

Proof. Let $F_1$ consists of all singular simplices $\sigma$ in $Y$ satisfying the condition: whenever $\tau$ is a subsimplex of $\sigma$ and $\text{vert } \tau \subset \text{vert } F$ then $\tau \subset F$. Clearly $F_1$ is a simplicial structure, $F \subset F_1$ and if $\sigma \in F_1$ and $\text{vert } \sigma \subset \text{vert } F$ then $\sigma \in F$.

If $s \in [Y]^{<\infty} \setminus \text{vert } F$ and $\{ \tau_t : t \subseteq s \}$ are pairwise compatible, $\tau_t \in F_1$, $\text{vert } \tau_t = t$, then - since $Y$ is $C^\infty$ - there exists a singular simplex $\sigma : \sum(s) \to Y$ such that $\tau_t \leq \sigma$ for $t \subseteq s$. Such a $\sigma$ belongs to $F_1$, hence $F_1$ is regular. Applying property $C^\infty$ once more, we see that every finite subset of $Y$ belongs to $\text{vert } F_1$.

Combining the above lemma and Dugundji’s Extension Theorem we obtain the following important result which allows us to recognize AR’s among ANR’s.

Corollary 5. Let $Y$ be a metrizable space. Then $Y \in \text{AR}$ if and only if $Y \in \text{ANR}$ and $Y \in C^\infty$.

6. A characterization of completely metrizable ANR’s

We recall some notation: Given any set $S$ we shall denote by $\Sigma(S)$ the union of all geometric simplices with vertices in $S$, endowed with the CW-topology. More precisely, $\Sigma(S)$ is the set of all formal convex combinations of the form $\sum_{s \in S} \lambda_s s$, where $\lambda_s = 0$ for all but finitely many $s \in S$. By a singular polytope in a topological space $Y$ we mean any continuous map $\varphi : P \to Y$ where $P$ is a subpolytope of $\Sigma(S)$, $S \subset Y$ and $\varphi(s) = s$ whenever $s \in S \cap P$. We denote by $\text{vert } \varphi$ the set of vertices of $\varphi$, i.e. $\text{vert } \varphi = S \cap P$. Let $U$ be a collection of subsets of $Y$. We write $A \prec U$ whenever $A$ is a set contained in some element of $U$. A singular polytope $\varphi$ in $Y$ is $U$-dense if $U \cap \text{vert } \varphi \neq \emptyset$ whenever $U \in U \setminus \{ \emptyset \}$. Let $U, V$ be two open covers of $Y$. We say that a singular polytope $\varphi$ is $(U, V)$-compatible if for every finite set $S \subset \text{vert } \varphi$ we have $\Sigma(S) \subset \text{dom } \varphi$ and $\varphi[\Sigma(S)] \prec V$ whenever $S \prec U$.

The results of this section are contained in my paper [7].

We say that a metric space $Y$ has property (B) provided there exists a sequence of open covers $\{ U_n \}_{n \in \omega}$ satisfying the following conditions:

(a) for each $n \in \omega$, $U_{n+1}$ is a star-refinement of $U_n$,
(b) $\sum_{n \in \omega} \text{mesh } U_n < +\infty$,
(c) for each $n \in \omega$ there is $m > n + 5$ and there exists a $U_{m+1}$-dense singular polytope in $Y$, each of which is simultaneously $(U_{m}, U_{n+5})$- and $(U_{n+4}, U_n)$-compatible.

If additionally, for some $n > 1$ there is a convex polytope satisfying (c) then we say that $Y$ has property (B'). By mesh $U$ we mean the supremum of diameters of members of $U$.

Theorem 3. Every complete metric space with property (B) is an absolute neighborhood retract. A complete metric space with property (B') is an absolute retract.

Proof. We start with two lemmas. We assume here that $Y$ is a metric space with property (B), $A$ is a closed subset of a metrizable space $X$ and $f : A \to Y$ is a fixed continuous map.

Lemma 6. Let $n > 0$ and suppose that $g : X \to Y$ is a continuous map such that $g|A$ is $U_{n+3}$-close to $f$. Then there exists a continuous map $g' : X \to Y$ which is $U_{n-1}$-close to $g$ and $U_{n+4}$-close to $f$ on $A$.
Proof. Let \( m > n + 5 \) be as in condition (c) of property (B). Set \( \mathcal{U} = \mathcal{U}_{m+1} \). For \( U \in \mathcal{U} \) define
\[
U^* = f^{-1}[U] \cup (g^{-1}[\text{star}(U,\mathcal{U}_{n+3})] \setminus A).
\]
Observe that \( f^{-1}[U] \subset U^* \) and \( \mathcal{U}^* = \{U^*\}_{U \in \mathcal{U}} \) is an open cover of \( X \). Let \( \{h_U\}_{U \in \mathcal{U}} \) be a locally finite partition of unity such that \( h_U^{-1}[(0,1)] \subset U^* \) for \( U \in \mathcal{U} \). By condition (c) of property (B), there exists a \( \mathcal{U} \)-dense polytope \( \varphi \) in \( Y \) which is simultaneously \( (\mathcal{U}_m,\mathcal{U}_{n+5}) \)- and \( (\mathcal{U}_{n+1},\mathcal{U}_n) \)-compatible. Choose \( y_U \in U \cap \text{vert } \varphi \) whenever \( U \in \mathcal{U} \) is nonempty.

Fix \( t \in X \) and consider \( \mathcal{U}_t = \{U \in \mathcal{U}: h_U(t) > 0\} \). Then \( g(t) \in \text{star}(U,\mathcal{U}_{n+3}) \) for \( U \in \mathcal{U}_t \). It follows that \( S_t = \{y_U: U \in \mathcal{U}_t\} \subset \text{star}(g(t),\mathcal{U}_{n+2}) \). Thus \( S_t \prec \mathcal{U}_{n+1} \) and hence \( \Sigma(S_t) \subset \text{dom } \varphi \).

Define a map \( g': X \to Y \) by setting
\[
g'(t) = \varphi\left(\sum_{U \in \mathcal{U}} h_U(t)y_U\right).
\]
Clearly \( g' \) is continuous and \( \mathcal{U}_{n-1} \)-close to \( g \) since \( \{g'(t)\} \cup S_t < \mathcal{U}_n \) and \( \{g(t)\} \cup S_t < \mathcal{U}_{n+1} \).

Suppose now that \( t \in A \). Then \( f(t) \in \bigcap \mathcal{U}_t \) and consequently \( S_t \subset \text{star}(f(t),\mathcal{U}) < \mathcal{U}_m \). Thus \( g'(t) \in \varphi[\Sigma(S_t)] < \mathcal{U}_{n+5} \) which means that \( g'|A \) is \( \mathcal{U}_{n+4} \)-close to \( f \).

Lemma 7. There exists an open set \( W \supset A \) and a continuous map \( g: W \to Y \) which is \( \mathcal{U}_4 \)-close to \( f \) on \( A \). If \( Y \) has property (B*) then we may assume that \( W = X \).

Proof. Applying condition (c) of property (B) (for \( n = 0 \)) we get \( m > 5 \) and a singular polytope \( \varphi \) which is \( \mathcal{U}_{m+1} \)-dense and \( (\mathcal{U}_m,\mathcal{U}_3) \)-compatible. Set \( \mathcal{U} = \mathcal{U}_{m+1} \).

By paracompactness, there is a locally finite open cover \( \{H_U\}_{U \in \mathcal{U}} \) of \( X \) such that \( A \cap \text{cl } H_U \subset f^{-1}[U] \) for every \( U \in \mathcal{U} \). Set
\[
V_U = H_U \setminus \{\text{cl } H_G: G \in \mathcal{U} \cap A \cap \text{cl } H_G = \emptyset\}.
\]
Observe that each \( V_U \) is open in \( X \), \( A \subset \bigcup_{U \in \mathcal{U}} V_U \) and \( V_{U_1} \cap V_{U_2} \neq \emptyset \) iff \( U_1 \cap U_2 \neq \emptyset \). The last property follows from the fact that if \( V_{U_1} \cap V_{U_2} \neq \emptyset \) then there is \( t \in A \cap \text{cl } H_{U_1} \cap H_{U_2} \) and consequently \( f(t) \in U_1 \cap U_2 \). Let \( W = \bigcup_{U \in \mathcal{U}} V_U \) and let \( \{h_U\}_{U \in \mathcal{U}} \) be a locally finite partition of unity in \( W \) such that \( h_U^{-1}[(0,1)] \subset V_U \) for every \( U \in \mathcal{U} \).

Now choose \( y_U \in U \cap \text{vert } \varphi \) whenever \( U \in \mathcal{U} \) is nonempty. Define
\[
(*) \quad g(t) = \varphi\left(\sum_{U \in \mathcal{U}} h_U(t)y_U\right), \quad t \in W.
\]
Observe that \( g \) is well-defined, since if \( \mathcal{U}_t = \{U \in \mathcal{U}: h_U(t) > 0\} \) then \( \{y_U: U \in \mathcal{U}_t\} \subset \text{star}(U_0,\mathcal{U}) \), where \( U_0 \in \mathcal{U} \) is arbitrary (because \( U_1 \cap U_2 \neq \emptyset \) whenever \( U_1, U_2 \in \mathcal{U}_t \)) and consequently \( \Sigma(\{y_U: U \in \mathcal{U}_t\}) \subset \text{dom } \varphi \). As in the proof of the previous Lemma, one can check that \( g|A \) is \( \mathcal{U}_4 \)-close to \( f \). Finally, if \( Y \) has property (B*) then we may assume that \( \varphi \) is a convex polytope, so formula (*) well defines a continuous map on the entire space \( X \). Thus, in this case we can set \( W = X \).

Theorem 3 follows immediately from Lemma 6 and Lemma 7. Indeed, using Lemma 7 we get a continuous map \( g_0: W \to Y \) which is \( \mathcal{U}_4 \)-close to \( f \), where \( W \subset A \) is open. If \( Y \) has property (B*) then \( W = X \). Now we can use inductively Lemma 6 to obtain a sequence of continuous maps \( g_n: W \to Y \) such that \( g_{n+1} \) is \( \mathcal{U}_{n-1} \)-close to \( g_n \) and \( \mathcal{U}_{n+4} \)-close to \( f \). By condition (b) of property (B), the sequence \( \{g_n\}_{n \in \omega} \) converges uniformly to a continuous map \( f': W \to Y \) and \( f' \) is an extension of \( f \) (here we have used the completeness of \( Y \)).

Now we show that every metric ANR/AR has property (B)/(B*).
**Proposition 3.** Let $Y$ be a metric ANR. Then there exists a singular polytope $\varphi$ in $Y$ with $\text{vert} \varphi = Y$ and there exists a sequence $\{U_n\}_{n \in \omega}$ of open covers of $Y$ such that for each $n \in \omega$, $\text{mesh}U_n \leq 2^{-n}$, $U_{n+1}$ is a star-refinement of $U_n$ and $\varphi$ is $(U_{n+1}, U_n)$-compatible. If additionally, $Y$ is an AR then $\varphi$ is a convex polytope.

**Proof.** By the theorem of Arens-Eells we can assume that $Y$ is a closed subset of a normed linear space $E$. Let $r: W \rightarrow Y$ be a retraction, where $W \supset Y$ is open in $E$. Define

$$P = \bigcup \{\Sigma(S): S \in [Y]^{<\omega} \& \text{conv}_E S \subset W\}.$$

Let $\psi: P \rightarrow E$ be the unique affine map with $\psi|Y = \text{id}_Y$ and let $\varphi = r \psi$. Let $U_0$ be any open cover of $Y$ with mesh $\leq 1$. Suppose that covers $U_0, \ldots, U_n$ are already defined. By the continuity of $r$, there exists an open cover $\mathcal{V}$ of $W$, consisting of convex sets and such that $\{r[V]: V \in \mathcal{V}\}$ is a refinement of $U_n$. Now let $U_{n+1}$ be a star-refinement of $U_n$ with mesh $\leq 2^{-(n+1)}$, which is also refinement of $\mathcal{V}$. Then $\varphi$ is $(U_{n+1}, U_n)$-compatible.

Finally, if $Y$ is an AR then $W = E$ and hence $P = \Sigma(Y)$. \qed

We describe an example of a separable metric space with property $(B^*)$, which is not an ANR.

**Example 2.** Consider the Hilbert cube $H = [0, 1]^{\omega}$ endowed with the product metric. There exists a sequence $\{A_n\}_{n \in \omega}$ of pairwise disjoint dense convex subsets of $H$. Indeed, if $\{B_n\}_{n \in \omega}$ is a decomposition of $\omega$ into infinite sets then we can set

$$A_n = \{x \in H: \exists i \in B_n \ (x(i) > 0 \& (\forall j > i \ x(j) = 0)\}.$$

Now, for each $n \in \omega$ choose finite $D_n \subset A_n$ which is $1/n$-dense in $H$ and define $Y = \bigcup_{n \in \omega} \text{conv}_H D_n$.

Let $\{U_n\}_{n \in \omega}$ be a sequence of finite open covers of $Y$ such that $\text{mesh}U_n \leq 2^{-n}$, $U_{n+1}$ is a star-refinement of $U_n$ and each element of $U_n$ is of the form $U \cap Y$, where $U \subset H$ is convex. Let $\varphi_k: \Sigma(D_k) \rightarrow \text{conv}_H D_k$ be the unique affine map which extends $\text{id}_{D_k}$. Then $\varphi_k$ is $(U_n, U_n)$-compatible for each $n \in \omega$; moreover $\varphi_k$ is $U_n$-dense for a sufficiently large $k$. It follows that $Y$ has property $(B^*)$.

On the other hand, $Y$ is not an ANR, because it is not locally arc-wise connected at any point. Indeed, every arc in $Y$ is contained in $\text{conv}_H D_n$ for some $n$, but these sets are nowhere dense in $Y$.

### 7. Application to simplicial structures

Recall that a collection $\mathcal{F}$ consisting of singular simplices in a space $Y$ is a **simplicial structure** in $Y$ provided $\sigma \in \mathcal{F}$ implies that every subsimplex of $\sigma$ is in $\mathcal{F}$. The pair $(Y, \mathcal{F})$ is then called a **simplicial space**. We write $\text{vert} \mathcal{F} = \{\text{vert} \sigma: \sigma \in \mathcal{F}\}$. A simplicial space $(Y, \mathcal{F})$ is **locally convex** if for each $p \in Y$ and its neighborhood $V$ there exists a smaller neighborhood $U$ of $p$ such that $[U]^{<\omega} \subset \text{vert} \mathcal{F}$ and for every $\sigma \in \mathcal{F}$, $\text{vert} \sigma \subset U$ implies $\text{im} \sigma := \sigma[\Sigma(\text{vert} \sigma)] \subset V$. A simplicial space $(Y, \mathcal{F})$ is **convex** if every finite subset of $Y$ is in $\text{vert} \mathcal{F}$. A theorem of Kulpa [8] says that every convex locally convex simplicial space has the fixed point property for continuous compact maps. We show that every compact metric space with such a property is an AR. This answers a question posed by Kulpa in [8].

**Theorem 4.** Every compact metric convex and locally convex simplicial space is an AR.
Proof. Fix an open cover \( \mathcal{U} \) of a compact metric space \( Y \) with a convex, locally convex simplicial structure \( \mathcal{F} \). Denote by \( R(\mathcal{U}) \) a fixed refinement \( \mathcal{V} \) of \( \mathcal{U} \) with the following property:

\[
(\forall V \in \mathcal{V})(\exists U \in \mathcal{U})(\forall \sigma \in \mathcal{F}) \text{ vert } \sigma \subset V \implies \text{ im } \sigma \subset U.
\]

Now define a sequence of open covers \( \mathcal{U}_n \) such that \( \mathcal{U}_{n+1} \) is a finite star-refinement of \( R(\mathcal{U}_n) \) consisting of sets of diameter \( \leq 2^{-n} \). Clearly, the sequence \( \{\mathcal{U}_n\}_{n \in \omega} \) satisfies conditions (a), (b) of property (B'). We check condition (c). Fix \( n \in \omega \) and let \( m = n + 6 \). There exists a \( \mathcal{U}_{m+1} \)-dense singular simplex \( \sigma \in \mathcal{F} \), since \( \mathcal{F} \) is convex. Observe that \( \sigma \) is \( (\mathcal{U}_{k+1}, \mathcal{U}_k) \)-compatible for each \( k \in \omega \). Indeed, if \( S \subset \text{ vert } \sigma \) and \( S \prec \mathcal{U}_{k+1} \) then \( S \prec R(\mathcal{U}_k) \) so \( \sigma[\Sigma(S)] \prec \mathcal{U}_k \). This shows that \( Y \) has property (B'). By Theorem 3, \( Y \) is an AR. \( \square \)

8. A RELATION TO DUGUNDJI-LEFSCHETZ' THEOREM

A well-known characterization of metric ANR’s, due to Lefschetz [9] and Dugundji [5], is stated in terms of realizations of polytopes. Let \( P \) be a CW-polytope and let \( Q \) be a subpolytope of \( P \). A continuous map \( \varphi : Q \to Y \) is a partial realization of \( P \) relative to a cover \( \mathcal{U} \), provided \( Q \) contains all the vertices of \( P \) and for each simplex \( \sigma \subset P \), \( \varphi[Q \cap \sigma] \prec \mathcal{U} \). If \( Q = P \) then \( \varphi \) is a full realization relative to \( \mathcal{U} \). The theorem of Dugundji-Lefschetz says that a metrizable space \( Y \) is an ANR if and only if every partial realization of \( Y \) relative to \( \mathcal{U} \) can be extended to a full realization of \( Y \) relative to \( \mathcal{U} \).

We show that every metric space with the realization property stated above, has property (B). This provides a proof of the “if” part of Dugundji-Lefschetz’ theorem, in the case of completely metrizable spaces.

Fix a metric space \( Y \) with the mentioned realization property. For each open cover \( \mathcal{U} \) of \( Y \) denote by \( S(\mathcal{U}) \) its open refinement such that for every CW-polytope \( P \), every partial realization of \( P \) relative to \( S(\mathcal{U}) \) can be extended to a full realization of \( P \) relative to \( \mathcal{U} \). Now let \( \mathcal{U}_0 \) be any open cover of \( Y \) with finite mesh and, inductively, let \( \mathcal{U}_{n+1} \) be an open star-refinement of \( S(\mathcal{U}_n) \) with mesh \( \leq 2^{-n} \). Clearly, the sequence \( \{\mathcal{U}_n\}_{n \in \omega} \) satisfies conditions (a), (b) of property (B). It remains to check (c). Fix \( n \in \omega \) and set \( m = n + 6 \). Choose any \( \mathcal{U}_{m+1} \)-dense set \( S \subset Y \). Consider

\[
P_1 = \bigcup \{\Sigma(T) : T \in [S]^{<\omega} \& T \prec \mathcal{U}_m\}.
\]

Clearly, \( P_1 \) is a CW-polytope, \( S \) is a subpolytope of \( P_1 \) and the identity map \( \text{id}_S : S \to Y \) is a partial realization of \( P_1 \) relative to \( \mathcal{U}_m \). As \( \mathcal{U}_m \) is a refinement of \( S(\mathcal{U}_{n+5}) \), there exists a full realization \( \varphi_1 : P_1 \to Y \) of \( P_1 \) relative to \( \mathcal{U}_{n+5} \) with \( \varphi_1|S = \text{id}_S \). Observe that \( \varphi_1 \) is \( (\mathcal{U}_m, \mathcal{U}_{n+5}) \)-compatible. Now define

\[
P_2 = \bigcup \{\Sigma(T) : T \in [S]^{<\omega} \& \varphi_1[\Sigma(T) \cap P_1] \prec S(\mathcal{U}_n)\}.
\]

Now \( \varphi_1 \) is a partial realization of \( P_2 \) relative to \( S(\mathcal{U}_n) \). Let \( \varphi_2 : P_2 \to Y \) be a full realization of \( P_2 \) relative to \( \mathcal{U}_n \) which extends \( \varphi_1 \). Clearly, \( \varphi_2 \) is \( (\mathcal{U}_n, \mathcal{U}_{n+5}) \)-compatible. Fix \( T \in [S]^{<\omega} \) and \( U \in \mathcal{U}_{n+1} \) with \( T \subset U \). Set \( Q = \Sigma(T) \cap P_1 \). For each subsimplex \( \sigma \subset Q \) there is \( W_\sigma \in \mathcal{U}_{n+5} \) with \( \varphi_1[\sigma] \subset W_\sigma \). We have

\[
\varphi_1[Q] \subset U \cup \bigcup \{W_\sigma : \sigma \text{ is a subsimplex of } Q \} \subset \text{ star}(U, \mathcal{U}_{n+5}),
\]

thus \( \varphi_1[Q] \prec S(\mathcal{U}_n) \). Hence \( \Sigma(T) \subset P_2 \) and \( \varphi_2[\Sigma(T)] \prec \mathcal{U}_n \). It follows that \( \varphi_2 \) is \( (\mathcal{U}_{n+1}, \mathcal{U}_n) \)-compatible. This shows condition (c) of property (B).
9. Applications to Hyperspaces

For a topological space $X$ we denote by $CL(X)$ (resp. $K(X)$) the hyperspaces of all nonempty closed, (resp. compact) subsets of $X$. We write $T_V$ for the Vietoris topology on the hyperspace. If $(X, d)$ is a metric space then we denote by $T_{W_d}$ the Wijsman topology on $CL(X)$ induced by $d$, which is the least topology on $CL(X)$ such that all functions $A \mapsto \text{dist}(p, A)$ are continuous ($p \in X$). We denote by $B(p, r)$ and $\overline{B}(p, r)$ the open and closed ball centered at $p$ and with radius $r$. For $p \in X$ and $r > 0$ denote

$$U^-(p, r) = \{ A \in CL(X) : \text{dist}(p, A) < r \},$$

$$U^+(p, r) = \{ A \in CL(X) : \text{dist}(p, A) > r \}.$$

Then the Wijsman topology on $CL(X)$ is generated by all sets $U^-(p, r)$ and $U^+(p, r)$, where $p \in X$ and $r > 0$. Recall that $CL(X)$ with the Wijsman topology is metrizable (Polish) iff $X$ is separable (Polish). Recall also that the Wijsman topology on $CL(X)$ is weaker than the Vietoris one.

It is well-known that the Vietoris hyperspace of a locally connected metric continuum is an AR, which was proved by Wojdyslawski [13]; it is even homeomorphic to the Hilbert cube, see [11]. More generally, if a metric space $X$ is locally path-wise connected then $(CL(X), T_V)$ is an AR, see Curtis [3]. The proof of Curtis’ theorem used Lefschetz-Dugundji’s criterion.

An application of Theorem 3 gives the following result about the Wijsman topology, which probably will be contained in [7].

**Theorem 5.** Let $(X, d)$ be a Polish space with the following property:

$(\ast)$ if $K$ is a finite family of closed balls in $X$ then $X \setminus \bigcup K$ has finitely many open path components.

Then $(CL(X), T_{W_d})$ is an ANR. If moreover $X$ is path-wise connected then $(CL(X), T_{W_d})$ is an AR.

**Proof.** Fix a Polish space $(X, d)$ with property $(\ast)$. Denote by $B$ the collection of all sets of the form $U^-(p_1, r_1) \cap \cdots \cap U^-(p_k, r_k) \cap U^+(q_1, s_1) \cap \cdots \cap U^+(q_l, s_l)$, where $p_1, \ldots, p_k, q_1, \ldots, q_l \in X, r_1, \ldots, r_k, s_1, \ldots, s_l > 0$, each ball $B(p_i, r_i)$ is contained in some path component of $G = X \setminus (\overline{B}(q_1, s_1) \cup \cdots \cup \overline{B}(q_l, s_l))$ and each path component of $G$ contains $B(p_i, r_i)$ for some $i \leq k$.

Property $(\ast)$ implies that $B$ is a base for the Wijsman topology on $CL(X)$.

**Lemma 8.** Let $W \in B$ and let $a, b \in [X]^{<\omega} \cap W$. Then there exists a path in $K(X) \cap W$ joining $a, b$ which is continuous with respect to the Vietoris topology.

**Proof.** Let $W = U^-(p_1, r_1) \cap \cdots \cap U^-(p_k, r_k) \cap U^+(q_1, s_1) \cap \cdots \cap U^+(q_l, s_l)$, where $p_i, q_i, r_i, s_i$ are as above, and denote $G = X \setminus (\overline{B}(q_1, s_1) \cup \cdots \cup \overline{B}(q_l, s_l))$. Let $\Lambda$ denote the set of all pairs $(x, y)$ such that $x \in a$, $y \in b$ and $x, y$ lie in the same path component of $G$. For each $(x, y) \in \Lambda$ choose a path $\gamma_{x,y} : [0, 1] \to G$ with $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(1) = y$. Define $\Gamma : [0, 1] \to K(X)$ by

$$\Gamma(t) = \begin{cases} 
\bigcup_{(x,y) \in \Lambda} \gamma_{x,y}[0, 2t], & t \leq 1/2, \\
\bigcup_{(x,y) \in \Lambda} \gamma_{x,y}[1 - 2t, 1], & t \geq 1/2.
\end{cases}$$

Clearly, $\Gamma(t) \in W$ for every $t \in [0, 1]$. By the definition of $B$, $\Gamma(0) = a$ and $\Gamma(1) = b$. A routine verification shows that $\Gamma$ is Vietoris continuous.

\[\square\]
Lemma 9. Let \( \varphi : \text{bd} \sigma \rightarrow K(X) \) be a Vietoris continuous map from the boundary of a simplex \( \sigma \). If the dimension of \( \sigma \) is at least 2 then there exists a Vietoris continuous extension \( \psi : \sigma \rightarrow K(X) \) of \( \varphi \) such that for each \( W \in B \) we have \( \psi[\sigma] \subset W \) whenever \( \varphi[\text{bd} \sigma] \subset W \).

Proof. The natural injection \( i : \text{bd} \sigma \rightarrow K(\text{bd} \sigma) \) can be extended to a Vietoris continuous map \( r : \sigma \rightarrow K(\text{bd} \sigma) \), since \( \text{bd} \sigma \) is a Peano continuum and the Vietoris hyperspace of such a space is homeomorphic to the Hilbert cube. Now the formula

\[
\psi(p) = \bigcup_{x \in r(p)} \varphi(x), \quad p \in \sigma,
\]

defines the desired extension of \( \varphi \). \( \square \)

Fix a complete metric \( g \) in \( (CL(X), T_{W_{d}}) \). We show that \( (CL(X), g) \) has property (B). Let \( \{U_n\}_{n \in \omega} \) be a sequence of covers of \( CL(X) \) such that for each \( n \in \omega, U_n \subset B \), mesh\( U_n \leq 2^{-n} \) and \( U_{n+1} \) is a star-refinement of \( U_n \). We show that condition (c) of property (B) is fulfilled. Fix \( n \in \omega \) and set \( m = n + 6 \). As \( [X]^{< \omega} \) is dense in \( (CL(X), T_{W_{d}}) \), we can find a set \( S \subset [X]^{< \omega} \) which is \( U_{m+1} \)-dense. Define

\[
P_1 = \bigcup \{\Sigma(T) : T \in [S]^{< \omega} \& T \prec U_m\},
\]

where \( \Sigma(T) \) denotes the abstract simplex spanned by \( T \), i.e. the collection of all formal convex combinations of the form \( \sum_{t \in T} \lambda_t t \). Then \( P_1 \) is a CW-polytope. By Lemma 8, the identity map \( id : S \rightarrow S \) can be extended to a Vietoris continuous map \( \varphi^1 : P_1^{(1)} \rightarrow K(X) \), such that for each \( T \in [S]^2 \) we have \( \varphi^1[\Sigma(T)] \subset W \) for some \( W \in U_m \), whenever \( T \prec U_m \). Now by Lemma 9, \( \varphi^1 \) can be extended to a Vietoris continuous map \( \varphi_1 : P_1 \rightarrow K(X) \), which is a \( (U_m, U_{n+5}) \)-compatible singular polytope. Next define

\[
P_2 = \bigcup \{\Sigma(T) : T \in [S]^{< \omega} \& T \prec U_{m+1}\}.
\]

Again by Lemma 8, \( \varphi_1 \) can be extended to a Vietoris continuous map \( \varphi^2 : P_1 \cup P_2 \rightarrow K(X) \) with a property analogous to \( \varphi^1 \). Finally, by Lemma 9, \( \varphi^2 \) can be extended to a Vietoris continuous map \( \varphi_2 : P_2 \rightarrow K(X) \). The last map is a \( U_{m+1} \)-dense singular polytope in \( (CL(X), T_{W_{d}}) \) which is both \( (U_m, U_{n+5}) \)- and \( (U_{n+1}, U_n) \)-compatible.

This shows that \( (CL(X), T_{W_{d}}) \) has property (B). If, moreover, \( X \) is path connected then \( \varphi_2 \) can be continuously extended onto \( \bigcup \{\Sigma(T) : T \in [S]^{< \omega}\} \), which gives a convex polytope with the same properties as \( \varphi_2 \). This shows that \( (CL(X), T_{W_{d}}) \) has property (B*). This completes the proof. \( \square \)

There is a well-known fact which says that given a finite collection \( \mathcal{K} \) of compact convex subsets of \( \mathbb{R}^N \), the complement \( \mathbb{R}^N \setminus \bigcup \mathcal{K} \) has finitely many components. On the other hand, a finite collection of closed bounded convex sets in infinite dimensional normed space does not disconnect this space. Thus we obtain the following consequence of Theorem 5.

Corollary 6. If \( (X, \| \cdot \|) \) is a separable Banach space then \( (CL(X), T_{W_{\| \cdot \|}}) \) is an absolute retract.
REFERENCES


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