On the Bean critical-state model in superconductivity

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Abstract

We show that the critical-state problem in type-II superconductivity is equivalent to an evolutionary quasivariational inequality. In a special case, where the inequality becomes variational, the existence and uniqueness of the solution are proved.

1 Introduction

The Bean critical-state model [1, 2] provides a phenomenological description for the hysteretic magnetization of type-II superconductors in a temporally varying external magnetic field. The magnetic field penetrates into these superconductors in the form of superconductive electron current vortices around the extremely thin filaments of normal material. Each of the magnetic vortices carries the same amount (one quantum) of magnetic flux, and so the magnetization depends on the vortex distribution. According to the Bean model, the distribution of vortices in a type-II superconductor is determined by the balance between electromagnetic driving forces and forces pinning the vortices to material inhomogeneities. Whenever the external magnetic field is changed, magnetic vortices start to enter or leave the superconductor through its boundary. If a region appears where the driving forces overcome the pinning, the system of vortices rearranges itself into another metastable state such that all vortices are pinned again and the equilibrium with the external field at the boundary is re-established. Since the unpinned vortices move rapidly, the system quickly adjusts itself to the changing external conditions, and thus a quasistationary model with instantaneous interactions is justified.

In terms of macroscopic quantities, these assumptions may be summarized by the statement that the current density never exceeds some critical value determined by the density of pinning forces and, as long as this threshold is not reached, the magnetic induction remains unchanged. There is also a caveat that if the current density is not orthogonal to the magnetic field, the equilibrium of the vortices depends on the component of current density normal to the magnetic field, and we will discuss this further below.

Together with Maxwell equations, these rules form a mathematical model of magnetization. The model, however, consists of a complicated system of equations and inequalities, and leads to a difficult free boundary problem, because the boundary between the regions of critical and subcritical currents is unknown.

To solve the problem numerically, Bossavit [3] has recently proposed an interesting variational formulation for this free boundary problem based on the generalization of Ohm's law: the electric field was represented as the subdifferential of a non-smooth convex functional of current density. In fact, Bossavit generalized also the Bean model: he included into his formulation the material transition from the superconductive into the normal state at a point, where the effective resistivity, characterizing the energy dissipation due to the movement of vortices, reaches the value of resistivity in the normal state.

This transition has not been previously considered in the frame of Bean model, presumably because when the resistivity is high, the heat generation may become uncontrollable and cause catastrophic jumps of magnetic flux [4], undesirable in practical applications. Implicitly, it was assumed in the Bean model that no transition into the normal state occurs and, in particular, the effective resistivity is less than the resistivity in the normal state. Such regimes are of main interest in applications.

In this work we restrict our consideration to those cases, in which the standard assumptions of Bean model are justified. We will prove that this model is equivalent to a quasivariational inequality, which is a reformulation of the free boundary problem convenient for both theoretical and numerical purposes. We will also show that the effective resistivity in this model is a Lagrange multiplier, related to the current density constraint. The existence and uniqueness of a solution will be proved in the special case when the critical current does not depend on the magnetic field and the model reduces to a variational inequality.

2 Model of the critical state

Let a superconductor occupy an open bounded domain $\Omega \subset R^3$ with a Lipschitz boundary Γ and $\omega = R^3 \setminus \overline{\Omega}$ be the space exterior to this domain. We start with Maxwell equations with the displacement current omitted, because the time scale for electromagnetic wave propagation is short compared to the magnetization time scale:

$$\frac{\partial \boldsymbol{B}}{\partial t} + \mathrm{curl}\boldsymbol{E} = \boldsymbol{0}, \tag{1}$$

$$\boldsymbol{J} = \mathrm{curl}\boldsymbol{H}. \tag{2}$$

In the exterior ω, the constitutive equation reads $\boldsymbol{B} = \mu_0 \boldsymbol{H}$, where μ_0 is the permeability of vacuum, and the current density is given:

$$\mathrm{curl}\boldsymbol{H} = \boldsymbol{J}_e \quad \text{in } \omega. \tag{3}$$

Here $\boldsymbol{J}_e(x,t)$ is the density of external currents. It is supposed that $\mathrm{div}\boldsymbol{J}_e = 0$ and $\mathrm{supp}\boldsymbol{J}_e$ is a bounded subset of ω.

For the superconducting medium a nonlinear dependence $\boldsymbol{B} = \mu(|\boldsymbol{H}|)\boldsymbol{H}$ is supposed to be known. In the presence of electrical current flowing through the superconductor, vortices respond to the action of a Lorentz force which we average into a body force with density

$$\boldsymbol{F}_L = \boldsymbol{J} \wedge \boldsymbol{B}.$$

Whenever the vortices become unpinned and move, they move in the direction of this force, and so their velocity $\boldsymbol{v}$ is parallel to $\boldsymbol{F}_L$. The movement of vortices induces the electric field

$$\boldsymbol{E} = \boldsymbol{B} \wedge \boldsymbol{v},$$

which is thus parallel to $\boldsymbol{B} \wedge (\boldsymbol{J} \wedge \boldsymbol{B})$. If $\boldsymbol{B}$ is perpendicular to $\boldsymbol{J}$, as is always the case for two-dimensional problems and also for some three-dimensional, e.g., those with axial symmetry, the vectors of current density and electric field are co-linear and

$$\boldsymbol{E} = \rho \boldsymbol{J} \qquad \text{in } \Omega, \tag{4}$$

where

$$\rho(x, t) \geq 0 \tag{5}$$

is an unknown nonnegative function. Only the case $\boldsymbol{B} \perp \boldsymbol{J}$ will be considered in this work.

Equation (4) may be regarded as Ohm's law with an effective resistivity ρ. However, since the resistivity is an auxiliary unknown, this relation, for a given current density, fixes only the possible direction of the electric field but not its magnitude. The next two conditions are postulates of the critical state theory:

1) The current density cannot exceed some critical value, J_c. In the Bean model of the critical state, J_c is a constant determined by the properties of the superconductive material. However, Kim et al. [4, 5] found that generally the critical current density depends on the magnetic field and various relations of the type $J_c = J_c(|\boldsymbol{H}|)$ have been proposed (see, e.g., [6]). The constraint on the current density may be written as

$$|\text{curl}\boldsymbol{H}| \leq J_c(|\boldsymbol{H}|) \qquad \text{in } \Omega. \tag{6}$$

We assume that $\exists m, M: \; 0 < m < J_c(r) < M, \; \forall r \geq 0$.

2) Magnetic vortices do not move in the regions where the current density is less than critical. The current in these regions is purely superconductive and the electric field must be zero. Mathematically, this can be formulated as

$$|\text{curl}\boldsymbol{H}| < J_c(|\boldsymbol{H}|) \implies \rho = 0. \tag{7}$$

In addition, an initial distribution of magnetic induction,

$$\boldsymbol{B}|_{t=0} = \boldsymbol{B}_0(x), \tag{8}$$

should be specified satisfying the condition $\mathrm{div}\boldsymbol{B}_0 = 0$.

On the boundary dividing the two media, the tangential component of electric field $\boldsymbol{E}$ is continuous,

$$[\boldsymbol{E}_\tau] = \mathbf{0} \qquad \text{on } \Gamma,$$

where $[\,.\,]$ denotes the jump across the boundary. We neglect the surface current[1], and so the tangential component of magnetic field $\boldsymbol{H}$ on this boundary is also assumed to be continuous:

$$[\boldsymbol{H}_\tau] = \mathbf{0} \qquad \text{on } \Gamma. \tag{9}$$

We also suppose that $|\boldsymbol{H}| \to 0$ as $|x| \to \infty$. This completes the mathematical model. A more convenient variational reformulation of this model is derived below.

3 Quasivariational inequality

Let us define a Hilbert space of vector functions

$$V = \left\{ \boldsymbol{\varphi} \in L^2(R^3; R^3) \;\middle|\; \begin{array}{c} \mathrm{curl}\,\boldsymbol{\varphi} \in L^2(R^3; R^3), \\ \mathrm{curl}\,\boldsymbol{\varphi}|_\omega = \mathbf{0} \end{array} \right\}$$

with the norm $\|\boldsymbol{\varphi}\|_V = \|\boldsymbol{\varphi}\|_{L^2} + \|\mathrm{curl}\boldsymbol{\varphi}\|_{L^2}$. Note that for any $\boldsymbol{\varphi} \in V$, the boundary values $\boldsymbol{\varphi}_\tau$ on both sides of Γ are defined in $H^{-1/2}(\Gamma; R^3)$ and $[\boldsymbol{\varphi}_\tau]|_\Gamma = \mathbf{0}$; see, e.g., [7].

Multiplying (1) by $\boldsymbol{\varphi} \in V$ and integrating over ω, we obtain

$$\begin{aligned} 0 = \int_\omega \left(\frac{\partial \boldsymbol{B}}{\partial t} + \mathrm{curl}\boldsymbol{E} \right) \cdot \boldsymbol{\varphi} &= \int_\omega \frac{\partial \boldsymbol{B}}{\partial t} \cdot \boldsymbol{\varphi} + \int_\omega [\boldsymbol{E} \cdot \mathrm{curl}\,\boldsymbol{\varphi} - \mathrm{div}(\boldsymbol{E} \wedge \boldsymbol{\varphi})] \\ &= \int_\omega \frac{\partial \boldsymbol{B}}{\partial t} \cdot \boldsymbol{\varphi} + \oint_{\Gamma_+} (\boldsymbol{E} \wedge \boldsymbol{\varphi}) \cdot \boldsymbol{n}, \end{aligned}$$

since $\mathrm{curl}\,\boldsymbol{\varphi} = \mathbf{0}$ in ω. Here the normal $\boldsymbol{n}$ is directed towards the domain ω. Similarly, in Ω (1),(2), and (4) yield

$$0 = \int_\Omega \frac{\partial \boldsymbol{B}}{\partial t} \cdot \boldsymbol{\varphi} + \int_\Omega \rho\, \mathrm{curl}\boldsymbol{H} \cdot \mathrm{curl}\,\boldsymbol{\varphi} - \oint_{\Gamma_-} (\boldsymbol{E} \wedge \boldsymbol{\varphi}) \cdot \boldsymbol{n}.$$

[1]This is the Meissner current [2] which is much less than the total current in most applications of type-II superconductors.

Adding the two last equations and taking into account that the tangential components of $\boldsymbol{E}$ and $\boldsymbol{\varphi}$ are continuous on Γ, we obtain the variational relation

$$\int_{R^3} \frac{\partial \boldsymbol{B}}{\partial t} \cdot \boldsymbol{\varphi} + \int_{\Omega} \rho \, \mathrm{curl} \boldsymbol{H} \cdot \mathrm{curl} \, \boldsymbol{\varphi} = 0, \qquad \forall \boldsymbol{\varphi} \in V. \tag{10}$$

The magnetic field is now determined by (3), (5)-(10) and the boundary condition at infinity. Since $\boldsymbol{B}$ is a known function of $\boldsymbol{H}$, the model contains only two unknowns: the magnetic field $\boldsymbol{H}$ and effective resistivity ρ. As shown below, we will be able to regard the latter function a Lagrange multiplier related to the inequality constraint (6).

Let us define $\boldsymbol{J}_e = \boldsymbol{0}$ in Ω and consider first an auxiliary problem in R^3:

$$\begin{aligned} \mathrm{curl} \widetilde{\boldsymbol{H}} &= \boldsymbol{J}_e, \\ \mathrm{div} \widetilde{\boldsymbol{H}} &= 0, \\ |\widetilde{\boldsymbol{H}}| \to 0 & \text{ as } |x| \to \infty. \end{aligned} \tag{11}$$

We assume that for all t, $\boldsymbol{J}_e$ is a distribution with the compact support $\mathrm{supp} \boldsymbol{J}_e \subset \omega$ and $\mathrm{div} \boldsymbol{J}_e = 0$ in the space of distributions $\mathcal{D}'(R^3)$. Therefore, (11) has a unique solution which may be represented by means of convolution of two distributions,

$$\widetilde{\boldsymbol{H}} = \mathrm{curl}(\mathcal{G} * \boldsymbol{J}_e),$$

where $\mathcal{G} = 1/(4\pi|x|)$ is the Green function of Laplace equation. For $\boldsymbol{J}_e \in H^1(0,T; H^{-1}(R^3; R^3))$ this solution belongs to $H^1(0,T; L^2(R^3; R^3))$ and is an infinitely smooth function in $R^3 \setminus \mathrm{supp} \boldsymbol{J}_e$ [8]. Hence, introducing a new variable $\boldsymbol{h} = \boldsymbol{H} - \widetilde{\boldsymbol{H}}$ and using (3),(6), and (9), we obtain

$$\mathrm{curl} \boldsymbol{h} = \boldsymbol{0} \ \text{ in } \omega, \tag{12}$$

$$|\mathrm{curl} \boldsymbol{h}| \le J_c(|\boldsymbol{h} + \widetilde{\boldsymbol{H}}|) \ \text{ in } \Omega. \tag{13}$$

$$[\boldsymbol{h}_\tau] = \boldsymbol{0} \qquad \text{on } \Gamma. \tag{14}$$

The only difference in the two-dimensional case is that the Green function $\mathcal{G} = -1/(2\pi)\ln(|x|)$.

Below we use the following notations: $Q = \Omega \times (0,T)$, $\mathcal{H} = L^\infty(Q)$, $\mathcal{V} = L^2(0,T;V)$, and $\mathcal{L} = L^2(R^3 \times (0,T); R^3)$. By $\mathcal{X}'$ we denote the space dual to $\mathcal{X}$, $(.,.)$ means the natural pairing of the elements of $\mathcal{X}$ and $\mathcal{X}'$, and we make no distinction between Hilbert space $\mathcal{L}$ and its dual. We also introduce the partial ordering on the space $\mathcal{H}$: $\varphi \geq \psi$ if this inequality holds almost everywhere. This relation induces a partial ordering on the dual space, so that $\chi \in \mathcal{H}'$, $\chi \geq 0$ if and only if $(\chi, \psi) \geq 0$ for all $\psi \in \mathcal{H}$, $\psi \geq 0$.

Let us define a family of closed convex sets of vector functions,

$$\mathcal{K}(\boldsymbol{h}) = \left\{ \boldsymbol{\varphi} \in \mathcal{V} \middle| \ |\mathrm{curl}\ \boldsymbol{\varphi}| \leq J_c(|\boldsymbol{h} + \widetilde{\boldsymbol{H}}|) \ \text{ a.e. in } Q \right\},$$

and consider the quasivariational inequality

$$\begin{gathered} \text{find } \ \boldsymbol{h} \in \mathcal{K}(\boldsymbol{h}) \ \text{ such that} \\ \partial \widetilde{\boldsymbol{B}}(\boldsymbol{h})/\partial t \in \mathcal{L}, \\ \left(\partial \widetilde{\boldsymbol{B}}(\boldsymbol{h})/\partial t, \boldsymbol{\varphi} - \boldsymbol{h} \right) \geq 0, \ \forall \boldsymbol{\varphi} \in \mathcal{K}(\boldsymbol{h}), \\ \widetilde{\boldsymbol{B}}(\boldsymbol{h})|_{t=0} = \boldsymbol{B}_0, \end{gathered} \tag{15}$$

where $\widetilde{\boldsymbol{B}}(\boldsymbol{h}) = \boldsymbol{B}(\boldsymbol{h} + \widetilde{\boldsymbol{H}})$.

THEOREM 1. *The function $\boldsymbol{h}(x, t)$ is a solution of the quasivariational inequality* (15) *if and only if there exists a functional $\rho \in \mathcal{H}'$ such that the pair $\{\boldsymbol{H}, \rho\}$, where $\boldsymbol{H} = \boldsymbol{h} + \widetilde{\boldsymbol{H}}$, is a weak solution of the critical-state problem* (3), (5)-(10).

PROOF. For any function $\boldsymbol{h} \in \mathcal{V}$ such that $\partial \widetilde{\boldsymbol{B}}(\boldsymbol{h})/\partial t \in \mathcal{L}$ we define the linear functional

$$F_h(\boldsymbol{\varphi}) = \left(\partial \widetilde{\boldsymbol{B}}(\boldsymbol{h})/\partial t, \boldsymbol{\varphi} \right)$$

and the nonlinear operator $G_h : \mathcal{A} \longrightarrow \mathcal{H}$,

$$G_h(\boldsymbol{\varphi}) = \frac{1}{2} \left(|\mathrm{curl}\, \boldsymbol{\varphi}|^2 - J_c^2(|\boldsymbol{h} + \widetilde{\boldsymbol{H}}|) \right) \Big|_Q ,$$

where

$$\mathcal{A} = \{ \boldsymbol{\varphi} \in \mathcal{V} \mid \ |\mathrm{curl} \boldsymbol{\varphi}| \leq 2M \text{ a.e. in } Q \}$$

is a closed convex set. Now the inequality (15) can be formally written as an optimization problem

$$\boldsymbol{h} \in \mathop{\arg\min}_{\substack{G_h(\boldsymbol{\varphi}) \leq 0 \\ \boldsymbol{\varphi} \in \mathcal{A}}} F_h(\boldsymbol{\varphi}). \tag{16}$$

This representation allows us to introduce a Lagrange multiplier into the quasivariational inequality. To do this, let us fix $\boldsymbol{h}$ in F_h and G_h. The continuous functional F_h is linear, and the mapping G_h is convex in the sense of the partial ordering on $\mathcal{H}$ defined above. The cone $\mathcal{C}$ of nonnegative elements in $\mathcal{H}$ has a non-empty interior. Since $J_c > m > 0$ for any $\boldsymbol{h}$, the constraint qualification hypothesis [9]

$$\exists \boldsymbol{\varphi}_0 \in \mathcal{A}: \ -G_h(\boldsymbol{\varphi}_0) \in \operatorname{int} \mathcal{C}$$

is satisfied with $\boldsymbol{\varphi}_0 \equiv \mathbf{0}$.

If the functional F_h is bounded from below in $\mathcal{K}(\boldsymbol{h})$, and we will check this later, the condition of optimality for (16) may be obtained using the Lagrange multiplier technique ([9], Ch. 3, Th. 5.1):

$\boldsymbol{u}$ is a point of minimum if and only if there exists a Lagrange multiplier $\rho \in \mathcal{H}', \ \rho \geq 0$, such that the pair $\{\boldsymbol{u}, \rho\}$ is a saddle point of the Lagrangian, i.e.,

$$F_h(\boldsymbol{u}) + (\rho^*, G_h(\boldsymbol{u})) \leq F_h(\boldsymbol{u}) + (\rho, G_h(\boldsymbol{u})) \leq F_h(\boldsymbol{h}^*) + (\rho, G_h(\boldsymbol{h}^*))$$

for all $\boldsymbol{h}^* \in \mathcal{A}, \ \rho^* \in \mathcal{H}', \ \rho^* \geq 0$.

In our case both functional and constraint depend on the unknown solution. However, if this solution exists, it should satisfy the optimality condition above. Therefore, substituting $\boldsymbol{u} = \boldsymbol{h}$, we obtain the condition of optimality for the implicit optimization problem (16), and are now able to formulate the following saddle-point condition for the quasivariational inequality:

$\boldsymbol{h}$ is a solution of (15) if and only if it satisfies the initial condition, $\partial \widetilde{\boldsymbol{B}}(\boldsymbol{h})/\partial t \in \mathcal{L}$, and there exists a Lagrange multiplier $\rho \in \mathcal{H}', \ \rho \geq 0$, such that the pair $\{\boldsymbol{h}, \rho\}$ is a saddle point of the Lagrangian, i.e.,

$$F_h(\boldsymbol{h}) + (\rho^*, G_h(\boldsymbol{h})) \leq F_h(\boldsymbol{h}) + (\rho, G_h(\boldsymbol{h})) \leq F_h(\boldsymbol{h}^*) + (\rho, G_h(\boldsymbol{h}^*)) \tag{17}$$

for all $\boldsymbol{h}^* \in \mathcal{A}, \ \rho^* \in \mathcal{H}', \ \rho^* \geq 0$. Note that from the first inequality in (17) follows that $G_h(\boldsymbol{h}) \leq 0$ and

$$(\rho, G_h(\boldsymbol{h})) = 0. \tag{18}$$

This is the complementary slackness condition, which means that the Lagrange multiplier may be nonzero only where the constraint is active.

Let $\boldsymbol{h}$ be a solution of (15). Then the functional F_h is bounded from below in $\mathcal{K}(\boldsymbol{h})$, and so there exists a saddle point $\{\boldsymbol{h}, \rho\} \in \mathcal{V} \times \mathcal{H}'$. Let us set $\boldsymbol{H} = \boldsymbol{h} + \widetilde{\boldsymbol{H}}$ and show that $\{\boldsymbol{H}, \rho\}$ is a weak solution to the critical-state problem. Since $\boldsymbol{h} \in \mathcal{K}(\boldsymbol{h})$, (3),(6), and (9) hold in the sense of distributions or almost everywhere. The inequality (5) is true in the sense of the partial ordering on $\mathcal{H}'$.

Like any nonnegative functional in $\mathcal{H}'$, ρ may be represented as

$$(\rho, \psi) = \int_Q \psi d\mu,$$

where μ is a nonnegative additive function defined on the Lebesgue-measurable subsets of Q and such that $\mu(Q) < \infty$ and $\mathrm{mes}(\widetilde{Q}) = 0 \Longrightarrow \mu(\widetilde{Q}) = 0, \ \forall \widetilde{Q} \subset Q$ [10]. Define

$$Q^{-} = \mathrm{supp} G_h(\boldsymbol{h}) = \{(x,t) \in Q \mid |\mathrm{curl}\boldsymbol{H}| < J_c(|\boldsymbol{H}|) \ a.e.\},$$

the complementarity slackness condition (18) yields

$$\int_{Q^-} G_h(\boldsymbol{h}) d\mu = 0.$$

Since $G_h(\boldsymbol{h}) < 0$ a.e. in Q^-, we conclude that $\mu(Q^-) = 0$, and so (7) holds in the following weak sense:

$$\forall \psi \in \mathcal{H}, \ \ \mathrm{supp}\, \psi \subset Q^- \Longrightarrow (\rho, \psi) = 0.$$

The second inequality in (17) means that the functional $F_h(\boldsymbol{h}^*) + (\rho, G_h(\boldsymbol{h}^*))$ attains a minimum on $\mathcal{A}$ at the point $\boldsymbol{h}^* = \boldsymbol{h}$. We now define

$$\mathcal{W} = \{\boldsymbol{\varphi} \in \mathcal{V} \mid \ |\mathrm{curl}\boldsymbol{\varphi}| \in \mathcal{H}\}.$$

Since $|\mathrm{curl}\boldsymbol{h}| < M$ a.e. in Q, for any function $\boldsymbol{\varphi} \in \mathcal{W}$ there exists $\epsilon_0 > 0$ such that $\boldsymbol{h} + \epsilon\boldsymbol{\varphi} \in \mathcal{A}$ for all $\epsilon \in (-\epsilon_0, \epsilon_0)$. Therefore,

$$F_h(\boldsymbol{\varphi}) + (\rho, \mathrm{curl}\boldsymbol{h} \cdot \mathrm{curl}\boldsymbol{\varphi}) = 0, \ \forall \boldsymbol{\varphi} \in \mathcal{W}.$$

The functions $\mathrm{curl}\boldsymbol{h}$ and $\mathrm{curl}\boldsymbol{H}$ coincide in Ω, so this is equivalent to a weak form of (10):

$$(\partial \boldsymbol{B}(\boldsymbol{H})/\partial t, \boldsymbol{\varphi}) + (\rho, \mathrm{curl}\boldsymbol{H} \cdot \mathrm{curl}\, \boldsymbol{\varphi}) = 0, \ \forall \boldsymbol{\varphi} \in \mathcal{W}. \tag{19}$$

This proves that $\{\boldsymbol{H}, \rho\}$ is a weak solution to problem (3), (5)-(10).

Now let $\{\boldsymbol{H}, \rho\} \in \mathcal{L} \times \mathcal{H}'$ be a weak solution of the critical-state model (in the sense clarified in the first part of this proof) and $\partial \boldsymbol{B}(\boldsymbol{H})/\partial t \in \mathcal{L}$. We need to show that $\boldsymbol{h} = \boldsymbol{H} - \widetilde{\boldsymbol{H}}$ is a solution of the quasivariational inequality (15).

Equation (14), understood in the sense of distributions, and (12), (13), satisfied almost everywhere, yield $\mathrm{curl}\boldsymbol{h} \in L^2(R^3; R^3)$ for almost every t and $\boldsymbol{h} \in \mathcal{K}(\boldsymbol{h})$. It follows from the weak version of (7) that $(\rho, G_h(\boldsymbol{h})) = 0$, and hence

$$(\rho, |\mathrm{curl}\boldsymbol{h}|^2) = (\rho, J_c^2(\boldsymbol{h} + \widetilde{\boldsymbol{H}})). \tag{20}$$

Let $\boldsymbol{\varphi} \in \mathcal{K}(\boldsymbol{h})$. Using equations (19), which is the weak form of (10), and (20), we obtain

$$\begin{aligned}\left(\partial \widetilde{\boldsymbol{B}}(\boldsymbol{h})/\partial t, \boldsymbol{\varphi} - \boldsymbol{h}\right) &= -(\rho, \mathrm{curl}\boldsymbol{h} \cdot \mathrm{curl}\{\boldsymbol{\varphi} - \boldsymbol{h}\}) \\ &= (\rho, J_c^2(\boldsymbol{h} + \widetilde{\boldsymbol{H}}) - \mathrm{curl}\boldsymbol{h} \cdot \mathrm{curl}\boldsymbol{\varphi}) \geq 0,\end{aligned}$$

since $\boldsymbol{\varphi}, \boldsymbol{h} \in \mathcal{K}(\boldsymbol{h})$. Thus the theorem is proved.

4 Variational inequality

If, as is assumed in the Bean model, the critical current density J_c does not depend on the magnetic field, we have $\mathcal{K}(\boldsymbol{h}) \equiv \mathcal{K}$ and the inequality (15) becomes a variational one.

Also, another simplifying assumption, $\mu(|\boldsymbol{H}|) \approx \mu_0$, is justified for strong fields [2]. Under these two assumptions, the variational inequality may be written as follows:

$$\begin{gathered}\boldsymbol{h}(.,t) \in K: \ (\partial \boldsymbol{h}/\partial t - \boldsymbol{f}, \boldsymbol{\varphi} - \boldsymbol{h}) \geq 0, \ \forall \boldsymbol{\varphi} \in K, \\ \boldsymbol{h}|_{t=0} = \boldsymbol{h}_0.\end{gathered} \tag{21}$$

Here $\boldsymbol{f} = -\partial \widetilde{\boldsymbol{H}}/\partial t \in \mathcal{L}$, $\boldsymbol{h}_0 = \boldsymbol{B}_0/\mu_0 - \widetilde{\boldsymbol{H}}|_{t=0}$, and

$$K = \{\boldsymbol{\varphi} \in V \mid |\mathrm{curl}\boldsymbol{\varphi}| \leq J_c \ \text{ a.e. in } \Omega\}.$$

THEOREM 2. *Let* $\boldsymbol{h}_0 \in K$. *Then the variational inequality* (21) *has a unique solution* $\boldsymbol{h} \in C([0,T]; L^2(R^3; R^3))$ *such that* $\boldsymbol{h}(.,t) \in K$ *for almost all* t *and* $\partial \boldsymbol{h}/\partial t \in \mathcal{L}$. *Also,* $\mathrm{div}\boldsymbol{h} = 0$ *a.e. and* $\boldsymbol{h}(.,t) \in H^1(R^3; R^3)$ *for almost all* t.

Proof. The set K is a closed convex subset of the Hilbert space $L^2(R^3;R^3)$ and we can rewrite the variational inequality as a Cauchy problem

$$\begin{aligned} d\boldsymbol{h}/dt + \partial I_K(\boldsymbol{h}) \ni \boldsymbol{f}, \\ \boldsymbol{h}|_{t=0} = \boldsymbol{h}_0, \end{aligned} \tag{22}$$

where ∂I_K is the subdifferential of the indicator function

$$I_K(\boldsymbol{\psi}) = \begin{cases} 0 & \text{if } \boldsymbol{\psi} \in K, \\ \infty & \text{otherwise.} \end{cases}$$

Since I_K is a lower-semicontinuous convex function defined on $L^2(R^3;R^3)$, the first part of the theorem follows now immediately from the known results on differential equations with maximal monotone operators (see [11], Ch. 4, Th. 2.1).

Let us check that $\text{div}\boldsymbol{h} = 0$. For any $\boldsymbol{\chi} \in \partial I_K(\boldsymbol{h}(.,t))$,

$$(\boldsymbol{\chi}, \boldsymbol{\varphi} - \boldsymbol{h}) \le 0, \quad \forall \boldsymbol{\varphi} \in K. \tag{23}$$

By definition, $(\text{div}\boldsymbol{\chi}, \psi) = (\boldsymbol{\chi}, -\text{grad}\psi)$, $\forall \psi \in \mathcal{D}(R^3)$. Since $\text{curl}\,\text{grad}\psi$ is zero, function $\boldsymbol{\varphi} = -\text{grad}\psi$ belongs to K. Using (23) we obtain

$$(\text{div}\boldsymbol{\chi}, \psi) \le (\boldsymbol{\chi}, \boldsymbol{h}), \quad \forall \psi \in \mathcal{D}(R^3).$$

This means that $\text{div}\boldsymbol{\chi} = 0$ a.e. for any $\boldsymbol{\chi} \in \partial I_K(\boldsymbol{h})$. Since $\widetilde{\boldsymbol{H}}$ is a solenoidal vector field, also $\text{div}\boldsymbol{f} = 0$, and (22) yields $\text{div}\boldsymbol{h} \equiv \text{div}\boldsymbol{h}_0$. We have assumed that $\text{div}\boldsymbol{B}_0 = 0$, hence $\text{div}\boldsymbol{h}_0 = 0$. We proved that $\text{div}\boldsymbol{h} = 0$ and, since $\boldsymbol{h}$ and $\text{curl}\boldsymbol{h}$ belong to $L^2(R^3;R^3)$ for almost all t, we have also $\boldsymbol{h}(.,t) \in H^1(R^3;R^3)$ [7], which completes the proof.

Corollary. *The inequality* (21) *is equivalent to the variational inequality*

$$\begin{aligned} \boldsymbol{h}(.,t) \in K_0 : \ (\partial\boldsymbol{h}/\partial t - \boldsymbol{f}, \boldsymbol{\varphi} - \boldsymbol{h}) \ge 0, \ \forall \boldsymbol{\varphi} \in K_0, \\ \boldsymbol{h}|_{t=0} = \boldsymbol{h}_0, \end{aligned} \tag{24}$$

with the set

$$K_0 = \left\{ \boldsymbol{\varphi} \in H^1(R^3;R^3) \;\middle|\; \begin{array}{c} |\text{curl}\boldsymbol{\varphi}| \le J_c \ \text{ a.e. in } \Omega, \\ \text{curl}\boldsymbol{\varphi} = \boldsymbol{0} \ \text{ a.e. in } \omega, \\ \text{div}\boldsymbol{\varphi} = 0 \ \text{ a.e. in } R^3 \end{array} \right\}.$$

PROOF. We have proved that $\boldsymbol{h}$, the unique solution of (21), belongs to K_0 for almost all t. Clearly, $\boldsymbol{h}$ is a solution of (24) as well, because $K_0 \subset K$. Since (24) also has only one solution, the two inequalities are equivalent.

The numerical solution of variational inequalities (21) or (24) can be obtained by means of discretization and solution of convex programming problems, arising at each time layer [12]. Solution of the quasivariational inequality would need an additional level of iterations. However, the realization of this procedure is difficult because the unknown magnetic field must be calculated in the whole space. To avoid this difficulty, we now derive a variational formulation in terms of the current density.

5 Obstacle problem

Let us define a closed convex set

$$K_1 = \left\{ \boldsymbol{\psi} \in L^2(R^3; R^3) \,\middle|\, \begin{array}{c} |\boldsymbol{\psi}| \leq J_c \text{ a.e. in } \Omega, \\ \boldsymbol{\psi} = \mathbf{0} \text{ a.e. in } \omega, \\ \text{div}\boldsymbol{\psi} = 0 \text{ a.e. in } R^3 \end{array} \right\}.$$

For any $\boldsymbol{\psi} \in K_1$, the function $\mathcal{R}\boldsymbol{\psi} = \text{curl}(\mathcal{G} * \boldsymbol{\psi})$, where $\mathcal{G}$ is the Green function, is the only solution of the problem

$$\begin{gathered} \text{curl}\boldsymbol{\varphi} = \boldsymbol{\psi}, \\ \text{div}\boldsymbol{\varphi} = 0, \\ |\boldsymbol{\varphi}| \to 0 \text{ as } |x| \to \infty. \end{gathered}$$

Therefore, $\mathcal{R}\boldsymbol{\psi} \in K_0$. On the other hand, $\text{curl}\boldsymbol{\varphi} \in K_1$ for any $\boldsymbol{\varphi} \in K_0$. The linear operator $\mathcal{R}$ is inverse to the operator curl and establishes a one-to-one correspondence between the sets K_0 and K_1. Since $\widetilde{\boldsymbol{H}} = \mathcal{R}\boldsymbol{J}_e$, we can now rewrite (24) as

$$\begin{gathered} \boldsymbol{J}(.,t) \in K_1 : (\mathcal{R}\{\partial\boldsymbol{J}/\partial t + \partial\boldsymbol{J}_e/\partial t\}, \mathcal{R}\boldsymbol{\psi} - \mathcal{R}\boldsymbol{J}), \ \forall\boldsymbol{\psi} \in K_1, \\ \mathcal{R}\boldsymbol{J}|_{t=0} = \boldsymbol{h}_0, \end{gathered}$$

or, equivalently,

$$\begin{gathered} \boldsymbol{J}(.,t) \in K_1 : (\mathcal{R}^*\mathcal{R}\{\partial\boldsymbol{J}/\partial t + \partial\boldsymbol{J}_e/\partial t\}, \boldsymbol{\psi} - \boldsymbol{J}), \ \forall\boldsymbol{\psi} \in K_1, \\ \boldsymbol{J}|_{t=0} = \text{curl}\boldsymbol{h}_0, \end{gathered}$$

where $\boldsymbol{J} = \mathrm{curl}\boldsymbol{h}$ is the current density and $\mathcal{R}^*$ is ajoint to $\mathcal{R}$.

Let $\boldsymbol{\Phi} \in H^{-1}(R^3; R^3)$ be a distribution with the compact support, $\boldsymbol{\Psi} \in \mathcal{D}(R^3; R^3)$, and $\mathrm{div}\boldsymbol{\Phi} = \mathrm{div}\boldsymbol{\Psi} = 0$. Making use of Green theorem, we obtain

$$\begin{gathered}(\mathcal{R}^*\mathcal{R}\boldsymbol{\Phi}, \boldsymbol{\Psi}) = (\mathcal{R}\boldsymbol{\Phi}, \mathcal{R}\boldsymbol{\Psi}) = \textstyle\int_{R^3} \mathrm{curl}(\mathcal{G} * \boldsymbol{\Phi}) \cdot \mathrm{curl}(\mathcal{G} * \boldsymbol{\Psi}) = \\ \textstyle\int_{R^3} \mathcal{G} * \boldsymbol{\Phi} \cdot \mathrm{curl} \wedge \mathrm{curl}(\mathcal{G} * \boldsymbol{\Psi}) = \int_{R^3} \mathcal{G} * \boldsymbol{\Phi} \cdot [\mathrm{grad}\,\mathrm{div}(\mathcal{G} * \boldsymbol{\Psi}) - \triangle(\mathcal{G} * \boldsymbol{\Psi})] = \\ = \textstyle\int_{R^3} \mathcal{G} * \boldsymbol{\Phi} \cdot [\mathcal{G} * (\mathrm{grad}\,\mathrm{div}\boldsymbol{\Psi}) - (\triangle\mathcal{G}) * \boldsymbol{\Psi}] = \int_{R^3} \mathcal{G} * \boldsymbol{\Phi} \cdot \boldsymbol{\Psi},\end{gathered}$$

since $\mathrm{div}\boldsymbol{\Psi} = 0$ and $-\triangle\mathcal{G}$ is the delta function. Thus $\mathcal{R}^*\mathcal{R}\boldsymbol{\Phi} = \mathcal{G} * \boldsymbol{\Phi}$, which is the magnetic vector potential of current $\boldsymbol{\Phi}$. Using this formula, we arrive at the variational inequality

$$\begin{gathered}\boldsymbol{J} \in K_1 : (\mathcal{G} * \{\partial(\boldsymbol{J} + \boldsymbol{J}_e)/\partial t\}, \boldsymbol{\varphi} - \boldsymbol{J}) \geq 0,\ \forall \boldsymbol{\varphi} \in K_1, \\ \boldsymbol{J}|_{t=0} = \mathrm{curl}\boldsymbol{h}_0.\end{gathered} \tag{25}$$

Note. A variational inequality with the similar pseudodifferential operator arises in elasticity ([12], the stamp problem).

In two-dimensional problems, $\boldsymbol{H} = (H_1(x_1, x_2, t), H_2(x_1, x_2, t), 0)$ and so $\boldsymbol{J} = (0, 0, J(x_1, x_2, t))$. The divergence of current density is automatically zero, the variational inequality (25) becomes scalar and can be written as

$$\begin{gathered}J(.,t) \in K_2 :\ (\mathcal{G} * \{\partial(J + J_e)/\partial t\}, \varphi - J) \geq 0,\ \forall \varphi \in K_2, \\ J|_{t=0} = J_0,\end{gathered}$$

where

$$K_2 = \{\varphi \in L^2(\Omega)\,|\ \ |\varphi| \leq J_c\ a.e.\}.$$

A similar scalar variational inequality arises in three-dimensional problems with axial symmetry. The formulations obtained can serve a basis for effective numerical method for solving the critical-state problems, and we are going to describe the realization of this approach in another publication.

6 Conclusion

We have proved that the critical-state model in type-II superconductivity is equivalent to an evolutionary quasivariational inequality. Superconductors of this type may be considered an example of spatially extended open dissipative systems, which have recently attracted much interest among physicists [13].

These systems have usually infinitely many metastable states. However, under the action of external forces, they often tend to organize themselves into a critical state which is only marginally stable and are then able to demonstrate almost instantaneous interaction of their distantly separated parts. Although various modifications of cellular-automaton model [14] have been used for simulating such systems' behaviours, continuous models can also be proposed for sandpiles, river networks, and ferromagnets [15, 16]. Like the Bean model for superconductors, these continuous models are quasistationary models of equilibrium and are equivalent to variational or quasivariational evolutionary inequalities, similar to the inequalities (15) and (21).

To derive these models, one has to specify only the direction of system's evolution, and which changes of external conditions make the state of the system unstable. The rate, with which a dissipative system driven by the external forces rearranges itself, is determined implicitly by some conservation law coupled with a condition of equilibrium. This rate appears in the model as a Lagrange multiplier related to the equilibrium constraint. Since the multiplier depends on the system's state and the varying external conditions in a nonlocal way, the mathematical models are able to account for the long-ranged interactions characteristic of dissipative systems in the critical state. The specific form of equilibrium constraints and conservation laws may differ for different dissipative systems. However, the multiplicity of metastable states usually implies a unilateral condition of equilibrium. This makes the variational or quasivariational inequalities a suitable tool for modelling such systems.

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