Scaling in modeling sand surface evolution

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To account for the multiplicity of metastable pile shapes and occurrence of avalanches upon pile surface it is better to write the model of pile surface dynamics not as an evolutionary equation for the surface alone but to use an additional variable for the flux of granular material upon this evolving surface. Two models:

- **Variational model** (P. 86,94,96; Aronsson, Evans, Wu 96)
- **BCRE-type models** (Bouchaud, Cates, Ravi Prakash, Edwards 94,95; de Gennes 95)

**The aim of this talk:** to show that Variational model is the long-scale limit of BCRE model and to explain why different models should be used for small and large piles, e.g., Aeolian sand ripples and dunes.
Part I: Variational model

Mean shape of a growing pile: we assume that
1. sand flow is confined to a thin surface layer
2. avalanches are small fluctuations of pile surface

Traces of avalanches on the slip face of a small dune

Large conical pile below a point source
Mass balance

\[ \partial_t h + \nabla \cdot \mathbf{q} = w \quad (x \in \Omega \subset \mathbb{R}^2, \ t > 0) \]

\[ h|_{t=0} = h_0, \ h|_{\partial \Omega} = h_0|_{\partial \Omega} \]

where

\[ y = h(x, t) \] – pile surface

\[ w(x, t) \geq 0 \] – source intensity

\[ h_0(x) \] – support surface,

\[ \mathbf{q}(x, t) \] – the horizontal projection of sand surface flux

A constitutive relation for the flux \( \mathbf{q} \) is needed.

Suppose the support has no steep slopes:

\[ |\nabla h_0| \leq k = \tan \alpha_r, \]

where \( \alpha_r \) is the sand angle of repose.
Surface flux: constitutive relation

We assume:

- flux $q$ is towards the steepest decent of $h$, i.e. $q \parallel -\nabla h$
- there is no oversteep slopes, $|\nabla h| \leq k$
- no flux upon subcritical slopes, $|\nabla h| < k \rightarrow q = 0$
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This is a multivalued relation: we can write

$$\mathbf{q} = -m \nabla h,$$

where $m \in f(|\nabla h|)$ is an auxiliary unknown.
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Hence

$$\partial_t h - \nabla \cdot \{m(|\nabla h|)\nabla h\} = w, \quad m \in f(|\nabla h|)$$

Different variational formulations can be derived:

- primal variational inequality for sand surface $h$
- dual variational inequality for sand surface flux $q$
- mixed for both $h$ and $q$
Primal variational inequality formulation

Assume, for simplicity, \( h_0|_{\partial \Omega} = 0 \). Then the set

\[
K = \{ \eta(x) : |\nabla \eta| \leq k \text{ in } \Omega, \quad \eta = 0 \text{ on } \partial \Omega \}
\]

is the set of admissible sand surfaces: \( h(.,t) \in K \) for all \( t \).
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Let \( \eta \in K \). We multiply \( \partial_t h - w = \nabla(m\nabla h) \) by \( \eta - h \), integrate by parts and arrive at a variational inequality:

\[
(\partial_t h - w, \eta - h) = - (m \nabla h, \nabla(\eta - h)) = (m, k^2 - \nabla h \cdot \nabla \eta) \geq 0.
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Primal variational inequality formulation

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$$(\partial_t h - w, \eta - h) = - (m \nabla h, \nabla (\eta - h)) = (m, k^2 - \nabla h \cdot \nabla \eta) \geq 0.$$  

This eliminates $m$ and the dual variable $q$.

The problem for $h$ becomes (P. 86,94,96; Evans et al. 96,97):

Find $h(x, t)$ such that $h|_{t=0} = h_0$ and for all $t$

$h(., t) \in K$, $(\partial_t h - w, \eta - h) \geq 0$ for any $\eta \in K$.

Existence, uniqueness. Equivalency: if $h$ solves the v.i. problem there exists $m$ (a Lagrange multiplier related to the constraint $|\nabla h| \leq k$) s.t. $(h, m)$ is a weak solution of the sand model.
Analytical solutions vs experiment

Build up of piles on flat open platforms: pile shapes in the experiment (Puhl 92) are described by the model.

1. Point source, $w = \delta(x - x_0)$. A cone with critical slopes grows until its base touches the support boundary. Then a runway appears and the pile growth stops: all additional sand follows the runway and leaves the system. Both $h$ and $q$ are known; $q \sim 1/r$ before the runway appears, then is a measure along the runway.
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2. **Distributed positive source,** $w(x) > 0$:
   The final **stationary** shape of the pile is $h(x) = k\text{dist}(x, \partial\Omega)$. 

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Sand Surface Evolution
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   The final **stationary** shape of the pile is \( h(x) = k \text{dist}(x, \partial \Omega) \).

3. **General distributed source**: \( w \geq 0 \):
   The **stationary** solution is given by an integral representation formula for \( m \); \( h \) is unique in \( \text{spt}(m) \): there \( h = k \text{dist}(x, \partial \Omega) \) (Cannarsa and Cardaliaguet, 04.)
Time-discretized var. inequalities

\[ h \in K : \left( \frac{h - \hat{h}}{\tau} - w, \eta - h \right) \geq 0 \forall \eta \in K \]

are equivalent to the constraint optimization problems

\[ \min_{h \in K} \left\{ \frac{1}{2} (h, h) - (f, h) \right\}, \]

where \( \tau \) is the time step, \( \hat{h} = h(t - \tau) \), \( f = \hat{h} + \tau w \).

On each time layer one can solve the constraint optimization problem directly using different methods (P. 93, Finzi Vita and Falcone 05,06) or approximate the multivalued \( m(\nabla h) \) relation by the power law, \( m = |\nabla h|^{p-2} \) and then solve the p-Laplacian equations with \( p \to \infty \) (Evans et al. 99, Barrett and P. 00).

**Results:** reasonable convergence for the primal variable, \( h \).
Three examples

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Sand Surface Evolution
Disadvantage of the primal formulation:
difficult to find the dual variable, the flux

\[ q = -m(|\nabla h|)\nabla h, \]

because \( m \) is a multivalued function.
The dual variable is of interest in various applications.

Such a situation is typical also of other critical state problems:

- **The Bean model** in type-II superconductivity: knowing the magnetic field \( h \) one finds the current density \( j = \nabla \times h \) but it can be difficult to find the electric field \( e \);

- **The ideal plasticity**: it is easier to compute stress (the primal variable) than strain (the dual variable).
The variational inequality formulation for the flux:

Find \( q(x, t) \) such that for all \( t \)

\[
\left( \nabla \cdot \left\{ \int_0^t q \, dt \right\} - f, \nabla \cdot \{ p - q \} \right) + k \int_{\Omega} |p| - k \int_{\Omega} |q| \geq 0
\]

for any vector field \( p(x, t) \).

Here \( f = h_0 + \int_0^t w \, dt \).

**Advantage of the dual formulation:** both variables can be found.

If the problem for \( q \) is solved one computes the primal variable as

\[
h = h_0 + \int_0^t w \, dt - \nabla \cdot \left\{ \int_0^t q \, dt \right\}.
\]

Numerical examples: point source
Numerical examples: impermeable wall and obstacle

Pile on a support partly surrounded by a wall. **Left:** the domain (white region) and the impermeable wall (grey regions); $w = 1$ inside the white circle and $w = 0$ outside. Computed are **middle:** the final stationary pile surface $h$; **right:** levels of the corresponding stationary surface flux, $|q|$. 
Variational model of sand surface evolution: what else?

Generalizations:
- Supports with steep slopes (quasivariational inequality)
- Lakes and river networks on a given relief (qvi, \( k = 0 \))
- Model for inhomogeneous materials (segregation in size)
- Avalanches as instantaneous slides (local fluctuation of \( k \))

Similar mathematical problems:
- Critical-state models in type-II superconductivity
- Optimal transportation (Monge-Kantorovich problems)

Limitations:
- Not all avalanches are slides
- Small pile shapes are somewhat different
- Could not be used in a model for aeolian ripples
- ...
Part II: BCRE model for sand surface dynamics

- Derived by BCRE (Bouchaud, Cates, Ravi Prakash, and Edwards 94,95) as a simple model for avalanches upon a close-to-critical slope.

- Simplified further and studied by de Gennes et al. 95,97.

BCRE equations are written for two variables, the sand surface $h$ and the effective thickness of rolling layer $R$:

$$\partial_t h = \Gamma[h, R],$$

$$\partial_t R + \nabla \cdot (v R) = w - \Gamma[h, R],$$

where $\Gamma[h, R]$ accounts for mass exchange between rolling layer and the pile, $w \geq 0$ - source intensity, and $v$ - horizontal projection of rolling grains velocity assumed constant by BCRE who wanted to model avalanches upon a close-to-critical slope.
Generalized BCRE equations

Simple constitutive relations for an arbitrary slope angle (P. 99):

$$\mathbf{v} = -\mu \nabla h, \quad \Gamma[h, R] = \gamma R \left(1 - \frac{|
abla h|^2}{k^2}\right).$$

Here $\mu$ and $\gamma$ characterize, resp., the mobility of a dislodged grain on the pile surface and the rate at which these grains are trapped and absorbed into the pile; $k = \tan \alpha_r$; the inertia is neglected.
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What are the correct relations? Which model to use?
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**Relation details are less important than scaling!**
Scaling

Three length scales:

- characteristic thickness of the rolling grains layer,
  \[ L_R = \frac{\langle w \rangle}{\gamma}, \]  where \( \langle w \rangle \) is the mean source rate;

- characteristic path of a rolling grain before it is trapped,
  \[ L_P = \frac{\mu}{\gamma} \]

- the pile size \( L \).

Dimensionless variables:

\[
x' = \frac{1}{L} x, \quad h' = \frac{1}{L} h, \quad R' = \frac{1}{L_R} R, \\

w' = \frac{1}{\langle w \rangle} w, \quad t' = \frac{\langle w \rangle}{L} t
\]
Dimensionless BCRE model

\[ \partial_t h = R \left( 1 - \frac{|
abla h|^2}{k^2} \right), \]

\[ \frac{L_R}{L} \partial_t R - \frac{L_P}{L} \nabla \cdot (R \nabla h) = w - R \left( 1 - \frac{|
abla h|^2}{k^2} \right) \]

Typically, \( L_R \ll L_P < L_c \). Two cases:

1. Large piles, \( L_R \ll L_P \ll L_c \).
2. Small piles, \( L_R \ll L_P, L_P \approx L_c \).

Quasistationary eq. for \( R \): used in a model of sand ripples (P. 99, Manukyan and P. 09); Sde Boker, Nov. 2017
Dimensionless BCRE model

\[ \partial_t h = R \left( 1 - \frac{\left| \nabla h \right|^2}{k^2} \right), \]

\[ \frac{L_R}{L} \partial_t R - \frac{L_P}{L} \nabla \cdot (R \nabla h) = w - R \left( 1 - \frac{\left| \nabla h \right|^2}{k^2} \right) \]

Typically, \( L_R \ll L_P < L \). Two cases:

1. Large piles, \( L_R \ll L_P \ll L \).
   Long-scale limit of BCRE equations: the variational model (P. and Zaltzman 01,03).

2. Small piles, \( L_R \ll L_P, \quad L_P \simeq L \).
   Quasistationary eq. for \( R \): used in a model of sand ripples (P. 99, Manukyan and P. 09);
Although BCRE model permits grains to roll down any slope, the particles are quickly stopped except for the close-to-critical slopes. Their path $L_P$ is short comparing to the pile size $L$.

In the variational model these paths are neglected: no surface flow upon subcritical slopes.

Physically, it is clear that for large piles the BCRE model should be similar to the variational model.

Mathematically, the convergence is not easy to prove. It was proved only for time-discretized versions of the two problems, P. and Zaltzman 03.
The long scale limit of BCRE model

$L_R \ll L_P \ll L$. Suppose $R(x, 0) \geq 0$ and $h(x, 0)$ are given, $|\nabla h(x, 0)| \leq k$, the boundary conditions specified.

Let us

- denote $\nu = L_P/L$ and $m = \nu R$;
- assume $L_R/L = \nu \lambda(\nu)$, where $\lambda(\nu) \to 0$ as $\nu \to 0$;
- add small diffusion vanishing as $\nu \to 0$ to regularize the model;
- discretize in time.

**Theorem:** Solution $h$ of the discretized in time BCRE problem converges to the solution of the discretized primal variational inequality as $\nu \to 0$. 
Asymptotic behavior of BCRE equations

Let \( \frac{L_P}{L} = \nu, \frac{L_R}{L} = 0.1\nu \).

(a) small pile, \( \nu = 0.2 \). (b) large pile, \( \nu = 0.01 \);
dashed lines – solution of variational inequality.
For small \( \nu \) the problem is stiff (10^5 time steps).
"Tails" of a small conical pile

Growth of a small polenta pile, Alonso and Herrmann, 96. Note the curved tails near the pile bottom. As the pile grows, these tails remain about tens of grain diameters long. Hence the tails of a large pile is difficult to notice.
Asymptotic behavior of BCRE equations

Filling a 2D silo from a point source:
Steadily translated solution to BCRE equations

\[ 0 \leq x \leq 2L, \text{ point source: } w = \delta(x - L) \]

for \( 0 < x < L/2 \), we find

\[ h_x' = \frac{k}{\nu k x + \sqrt{(\nu k x)^2 + 1}}, \]

- \( h_x' \to 0 \) as \( x \to 0 \)
  for any \( \nu \): a tail;
- \( h_x' \to k \) as \( \nu \to 0 \)
  for all \( 0 < x < L/2 \)
  solution to v.i.
Two coexisting spatiotemporal scales: sand dunes and ripples.
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- Modeling dunes (large scale) one neglects the small features, like ripples on the dune surface. Here one can use the variational model for sand surface evolution with the source \( w = \nabla \cdot Q \), where \( Q \) is the flux of sand transported by the wind perturbed by the forming dune.

- Modeling sand ripples (small scale) needs a different approach: one can use the modified BCRE model with the quasistationary eq. for the effective thin rolling layer. Here the source term is determined by interaction between saltating and reptating sand grains with the sand bed.
Modeling Aeolian ripples

Two types of sand grain movement when the wind is sufficiently strong (Bagnold 41): saltation and reptation.

A 1D model (P. 99) was built upon the modified BCRE model:

\[ \partial_t h = \Gamma[h, R] - f, \quad \nabla \cdot (\mathbf{v} R) = Q - \Gamma[h, R], \]

where \( \Gamma[h, R] = \gamma R \left(1 - \frac{|\nabla h|^2}{k^2}\right) \), \( \mathbf{v} = -\mu \nabla h \) as before, \( f \) is the erosion rate, proportional to saltation intensity and depending on the surface orientation, and \( Q(x, t) \) is the integral term describing the “rain” of falling reptating particles which can roll upon the surface before they are stopped and incorporated into the sand bed.
Results:
1. Linear stability analysis predicts flat surface instability and a realistic initial ripple wavelength (several mean reptation jumps, as in the Anderson 90 model);
2. On the nonlinear stage: a non-trivial mechanism of the wavelength coarsening.

Numerical simulation of ripple growth:

Left: initial instability and coarsening (x is in the mean reptation jump units); Right: the coarsening mechanism.
Ripples and megaripples:

- Ripples made of homogeneous sand are small and flat: their growth saturates because the surface grains, destabilized by saltation impacts, are carried away by the wind, stronger at ripple tops. Their final wavelength is correlated with the grain size and is about 3-6 cm.

- Ripples of inhomogeneous sand become much bigger (megaripples): coarser grains form an armoring layer on the ripple crests and stoss slopes; significantly less crude particles are found in troughs. The armoring layer can be very thin (almost a monolayer) or can become thick near the ripple crests.

The model P. 99 had been extended to describe formation of megaripples and sand sorting.
Numerical simulation of ripples for bidisperse sand mixture containing 10% of crude grains; size ratio $d_1/d_2 = 3$. The lengths are in centimeters. The bottom plot shows the color scale for the concentration of fine particles.
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Thank you!