An Intro to Variational Inequalities

Leonid Prigozhin

Ben-Gurion University of the Negev, Blaustein Inst. for Desert Research, Israel

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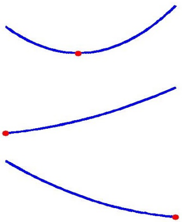
Outlook:

- Introduction: examples, extensions, comments, history.
- Superconductivity: cylinder in a parallel field.
  The primal, dual, and mixed v.i. formulations.
  Electric field calculation.

Suppose f is a smooth function and we solve

$$f(x_0) = \min_{[a,b]} f(x).$$

Three cases are possible:



- if $a < x_0 < b$ then $f'(x_0) = 0$;
- if $x_0 = a$ then $f'(x_0) \geq 0$;
- if $x_0 = b$ then $f'(x_0) \leq 0$;

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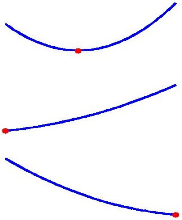


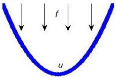
All these conditions can be written as a **variational inequality:**

Find $x_0 \in [a, b]$: $\quad f'(x_0)(x - x_0) \geq 0$ for all $x \in [a, b]$.

If f is strictly convex, the solution is unique.

Example 2: a membrane

Elastic membrane $y = u(\boldsymbol{x})$, $u|_{\partial\Omega} = 0$. The elastic energy $\sim$ to the change of membrane area



$$U = \lambda \int_\Omega \left(\sqrt{1 + u_{x_1}^2 + u_{x_2}^2} - 1 \right) d\Omega \approx \frac{\lambda}{2} \int_\Omega |\nabla u|^2 d\Omega,$$

where λ characterizes the elastic properties.

External force work $A = \int_\Omega \{fu\} d\Omega$. The total potential energy is

$$E(u) = U - A = \int_\Omega \left[\frac{\lambda}{2} |\nabla u|^2 - fu \right] d\Omega.$$

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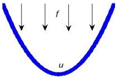

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Two equivalent formulations:

(1) Minimization of energy

$$\min\{E(u) \ : \ u|_{\partial\Omega} = 0\}$$

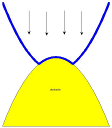
(2) Poisson equation:

$$-\lambda \Delta u = f, \quad u|_{\partial\Omega} = 0.$$

Here (2) is the Euler-Lagrange equation for (1) and is the necessary and sufficient condition for minimum.

Example 2: a membrane and an obstacle

An obstacle: $u(\boldsymbol{x}) \geq g(\boldsymbol{x})$, where $g|_{\partial\Omega} < 0$.



Free boundary problem for Poisson equation?
Better to solve a constr. min problem for energy:

$$\min\{E(u) \ : \ u \in K\}, \tag{1}$$

where $K = \{v \ : \ v \geq g, \ v|_{\partial\Omega} = 0\}$ is the admissible set.

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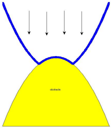

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Optimality conditions: **I. necessary**
Let $u \in K$ be the (unique) solution of (1). If $v \in K$ then
$v^\alpha = (1-\alpha)u + \alpha v \in K$ for any $0 < \alpha < 1$ so $E(u) < E(v^\alpha)$.
Then $\left.\frac{dE(v^\alpha)}{d\alpha}\right|_{\alpha=0+} = \int_\Omega [\lambda \nabla u \cdot \nabla(v-u) - f(v-u)] \, d\Omega \geq 0.$
We arrived at the variational inequality:

$$u \in K \text{ s.t. } a(u, v-u) - (f, v-u) \geq 0 \text{ for any } v \in K, \qquad (2)$$

where $a(\phi, \psi) = \lambda \int_\Omega \nabla\phi \cdot \nabla\psi d\Omega$ and $(\phi, \psi) := \int_\Omega \phi \, \psi d\Omega$.

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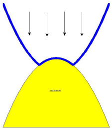

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Optimality conditions: **II. <u>sufficient</u>**
If u solves the v.i.

$$u \in K \text{ s.t. } a(u, v-u) - (f, v-f) \geq 0 \text{ for any } v \in K \qquad (2)$$

i.e. $\lambda(\nabla u, \nabla\{v-u\}) - (f, v-u) \geq 0$ for any $v \in K$ then

$$E(v) = E(u + \{v-u\}) =$$
$$E(u) + \frac{\lambda}{2}|\nabla(v-u)|^2 + \lambda(\nabla u, \nabla\{v-u\}) - (f, v-u) \geq E(u),$$

Hence u solves (1) and v.i. is also a sufficient opt. condition.

Variational inequalities: comments...

1. If the solution is smooth, $a(u, v - u) = (-\lambda \Delta u, v - u)$.
Hence, the v.i. is a weak form of

Find $u \in K$ s.t. $(-\lambda \Delta u - f, v - u) \geq 0$ for any $v \in K$,

However, often solutions of a v.i. are less smooth than that of the b.v.p. without obstacle and weak formulations are employed.
We can still write

Find $u \in K$ s.t. $(Au - f, v - u) \geq 0$ for any $v \in K$,

where Au is defined by: $(Au, v) = \lambda(\nabla u, \nabla v)$ for any v such that $v|_{\partial\Omega} = 0$.

Variational inequalities: comments...

2. Are the constrained optimization problems sufficient?
Not every equation $Au = 0$ is the Euler-Lagrange equation for some functional. A simple example: $du/dt - f(t) = 0$. Another example: the diffusion equation. Hence, there can be no appropriate equivalent optimization problem. However, if a solution has to satisfy an a priory constraint, $u \in K$, the v.i. formulation

$$\text{Find } u \in K \; : (Au, \phi - u) \geq 0 \text{ for any } \phi \in K$$

still makes sense. Such are diffusion problems with a unilateral constraint on the boundary if there is a semi-penetrable membrane.

3. Models for growing sandpiles and type-II superconductors also give rise to evolutionary problems with unilateral constraints.

- After discretization in time the stationary v.i.-s on each time layer are equivalent to constrained optimization problems.
- These evolutionary v.i.-s are also limits of highly nonlinear equations.

Both approaches can be employed for numerical solution. Nevertheless, the v.i. formulations are exact and convenient.

Variational inequalities: comments...

4. In the v.i. Find $u \in K \; : \; (Au, v-u) \geq 0$ for any $v \in K$
K should be a non-empty closed convex set. Its indicator function

$$\Phi(v) = \begin{cases} 0 & v \in K, \\ \infty & v \notin K \end{cases}$$

is convex (and lower semi-continuous, i.e. $\underline{\lim}_{v \to v_0} \Phi(v) \geq \Phi(v_0)$).
The inequality can be written as an unconstrained problem

$$\text{Find } u \; : \; (Au, v-u) + \Phi(v) - \Phi(u) \geq 0 \text{ for any } v. \qquad (3)$$

This is an example of extended variational inequalities which appear also with other non-differentiable functionals Φ; we will use such formulations too.

5. More complicated formulations are quasi-variational inequalities,

$$\text{Find } u \in K(u) \ : \ (Au, v - u) \geq 0 \text{ for any } v \in K(u),$$

Here the set K depends on the unknown solution itself.

Examples:

- The Kim model in superconductivity: the current density in a sc $\boldsymbol{j} \in K$ where K consists of functions satisfying the constraint $|\boldsymbol{u}| \leq j_c(|\boldsymbol{b}|)$. Since the induction $\boldsymbol{b}$ depends on $\boldsymbol{j}$, the formulation is a quasi-variational inequality: $K = K(\boldsymbol{j})$.

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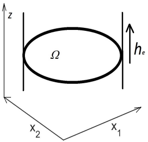
- Let sand be discharged onto a rigid support surface $y = h_0(\boldsymbol{x})$. The slope of a growing sandpile $y = h(\boldsymbol{x}, t)$ should not exceed the repose angle of sand, $|\nabla h| \leq \tan \alpha_{rep}$ wherever $h > h_0$. The uncovered by sand parts of the support, where $h = h_0$, can be steeper. If the support has such steep parts, the inequality is quasi-variational: what part of h_0 is uncovered depends on the solution, i.e. $K = K(h)$.

History

- The study of v.i.-s has started in 60-s and continued in 70-s by Signorini, Fichera, Stampacchia, Lions, Baiocchi, Brezis ...
- Variational inequalities are a natural generalization of the boundary value problems and arise in various problems of Mechanics, Physics, Math. Economy, and Control Theory.
- First quasi-variational inequality was introduced by Bensoussan and Lions in 1973.
- Many problems leading to v.i.-s are considered here:
  G. Duvaut and J.-L. Lions,
  "Inequalities in Mechanics and Physics", 1972.
  J.-F. Rodrigues,
  "Obstacle Problems in Mathematical Physics", 1987.
- The classical book on numerical methods for v.i.-s is
  R. Glowinski, J.-L. Lions, R. Tremolieres,
  "Numerical Analysis of Variational Inequalities", 1981.

Part II. SC cylinder in a parallel field



Let the sc occupy $\{\boldsymbol{x} \in \Omega \subset \mathbb{R}^2,\ -\infty < z < \infty\}$ and $\boldsymbol{h}_e = h_e(t)\boldsymbol{i}_z$. Then
$\boldsymbol{h} = h(\boldsymbol{x}, t)\boldsymbol{i}_z, \quad \boldsymbol{e}, \boldsymbol{j}(\boldsymbol{x}, t) \perp \boldsymbol{i}_z$ and $h|_{\partial\Omega} = h_e(t)$.
In the cylinder cross-section Ω we have:

Faraday law	Ampère law
$\mu_0 \partial_t h + \nabla \times \boldsymbol{e} = 0,$	$\boldsymbol{\nabla} \times h = \boldsymbol{j}$

and $h|_{t=0} = h_0$. The Bean critical-state model:

$$|\boldsymbol{j}| \le j_c, \qquad \boldsymbol{e} \| \boldsymbol{j}, \qquad |\boldsymbol{j}| < j_c \Rightarrow \boldsymbol{e} = \boldsymbol{0}.$$

Set $h = u + h_e(t)$.
Then $u|_{\partial\Omega} = 0$ and $\boldsymbol{j} = \boldsymbol{\nabla} \times u = (\partial_{x_2} u, -\partial_{x_1} u)$. Hence

$$|\boldsymbol{j}| \le j_c \Leftrightarrow |\nabla u| \le j_c.$$

The admissible set: for any t

$$u(.,t) \in K = \{v \ :\ |\nabla v| \le j_c \text{ in } \Omega,\ v|_{\partial\Omega} = 0\}.$$

.

SC cylinder: the primal v.i. formulation

Let $u \in K$, $\boldsymbol{j} = \boldsymbol{\nabla} \times u$ and $\boldsymbol{e}$ satisfy the Bean model, $v \in K$.
From the Faraday law $\mu_0(\partial_t(u + h_e), v - u) = -(\nabla \times \boldsymbol{e}, v - u)$ and

$$(\nabla \times \boldsymbol{e}, v - u) = (\boldsymbol{e}, \boldsymbol{\nabla} \times \{v - u\}) =$$
$$(\boldsymbol{e}, \boldsymbol{\nabla} \times v) - (\boldsymbol{e}, \boldsymbol{j}) \leq (|\boldsymbol{e}|, \{|\nabla v| - j_c\}) \leq 0.$$

We arrived at a variational inequality (the primal v.i. formulation):

Find $u(\boldsymbol{x}, t)$ s.t. $u(., t) \in K$ for all t, $u|_{t=0} = h_0 - h_e(0)$,
and $(\partial_t\{u + h_e\}, v - u) \geq 0$ for any $v \in K$.

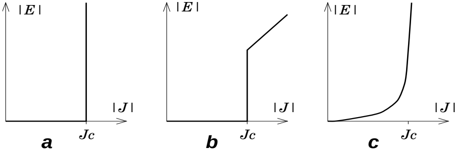
Solution for a simply-connected Ω (Barrett and P., 00):

non-decreasing $h_e(t)$: $u(\boldsymbol{x}, t) = \max\{-dist(\boldsymbol{x}, \partial\Omega), h_0(\boldsymbol{x}) - h_e(t)\}$,
non-increasing $h_e(t)$: $u(\boldsymbol{x}, t) = \min\{dist(\boldsymbol{x}, \partial\Omega), h_0(\boldsymbol{x}) - h_e(t)\}$.

Extension of this solution for multiply-connected domains and the q.v.i. in the case of the Kim model: Barrett and P., 10.

Different current-voltage relations

We assumed $\boldsymbol{e} \| \boldsymbol{j}$ and employed the multivalued $e(j)$ law $\boldsymbol{a}$.



Are there similar formulations for other popular choices?

Bossavit, 94: Let $e(j)$ increase monotonically. Then $\phi(\boldsymbol{j}) := \int_0^{|\boldsymbol{j}|} e(s)ds$ is a convex function and $\Phi(\boldsymbol{j}) := \int_\Omega \phi(\boldsymbol{j}(\boldsymbol{x}))d\boldsymbol{x}$ is a convex functional. For any $\boldsymbol{w}, \boldsymbol{j}$ we obtain:

$$\phi(\boldsymbol{w}) - \phi(\boldsymbol{j}) \geq e(|\boldsymbol{j}|)(|\boldsymbol{w}| - |\boldsymbol{j}|) \ \Rightarrow$$
$$\Phi(\boldsymbol{w}) - \Phi(\boldsymbol{j}) \geq \int_\Omega e(|\boldsymbol{j}|)(|\boldsymbol{w}| - |\boldsymbol{j}|)d\boldsymbol{x}.$$

If we assume $\boldsymbol{e}$ and $\boldsymbol{j}$ have the same direction, then

$$e(|\boldsymbol{j}|)|\boldsymbol{j}| = \boldsymbol{e} \cdot \boldsymbol{j} \text{ while } e(|\boldsymbol{j}|)|\boldsymbol{w}| \geq \boldsymbol{e} \cdot \boldsymbol{w}.$$

Hence for any $\boldsymbol{w}$

$$\Phi(\boldsymbol{w}) - \Phi(\boldsymbol{j}) \geq (\boldsymbol{e}, \boldsymbol{w} - \boldsymbol{j}).$$

General current-voltage relation: $\boldsymbol{e}$ as a subgradient

Let Φ be a convex functional and for any $\boldsymbol{w}$ hold

$$\Phi(\boldsymbol{w}) - \Phi(\boldsymbol{j}) \geq (\boldsymbol{e}, \boldsymbol{w} - \boldsymbol{j}).$$

Then $\boldsymbol{e}$ is a subgradient of Φ at the point $\boldsymbol{j}$, or an element of the subdifferential:

$$\boldsymbol{e} \in \partial\Phi(\boldsymbol{j}).$$

This seems to be the most general form of the current - voltage relation leading to a variational formulation.

Since, from the Faraday law

$$\mu_0(\partial_t(u + h_e), v - u) + (\boldsymbol{e}, \boldsymbol{\nabla} \times \{v - u\}) = 0$$

for any v s.t. $v|_{\partial\Omega} = 0$, we obtain

$$\mu_0(\partial_t(u + h_e), v - u) + \Phi(\boldsymbol{\nabla} \times v) - \Phi(\boldsymbol{\nabla} \times u) \geq 0.$$

If Φ is differentiable, one gets an equation: $\boldsymbol{e}(\boldsymbol{j}) = \delta\Phi/\delta\boldsymbol{j}$. Otherwise a variational inequality should be solved.

Comments

1. Choosing an appropriate functional Φ and setting $\boldsymbol{e} \in \partial\Phi(\boldsymbol{j})$ is the most natural way to account for, e.g., flux-cutting in the double critical-state model (Kashima 08) or material anisotropy.

2. For the critical-state models Φ is the non-differentiable indicator function of the set K. The $e(j)$ relation is multivalued and, although $h = u + h_e$ can be computed or even found analytically, the electric field $\boldsymbol{e}$ remains unknown.

3. For the power law $\boldsymbol{e} = e_0 \left(\frac{|\boldsymbol{j}|}{j_c}\right)^{p-1} \frac{\boldsymbol{j}}{j_c}$ one solves

$$\mu_0 \partial_t h - \frac{e_0}{j_c^p} \nabla \cdot \left(|\nabla h|^{p-1} \nabla h \right) = 0.$$

The solution h tends to that of the Bean model as $p \to \infty$ (Barrett and P. 00). However,

if p is large, accurate calculation of $\boldsymbol{e}$ is difficult.

Dual V.I. in terms of the electric field

Let $\boldsymbol{e}$ and $\boldsymbol{j} = \boldsymbol{\nabla} \times h$ obey the Bean model,

$$|\boldsymbol{j}| \le j_c, \qquad \boldsymbol{e} \| \boldsymbol{j}, \qquad |\boldsymbol{j}| < j_c \Rightarrow \boldsymbol{e} = \boldsymbol{0}.$$

For $h = h_e(t) + u$ and any test field $\boldsymbol{v}$ we obtain

$$(\boldsymbol{\nabla} \times u) \cdot (\boldsymbol{v} - \boldsymbol{e}) = \boldsymbol{j} \cdot (\boldsymbol{v} - \boldsymbol{e}) \le j_c |\boldsymbol{v}| - \boldsymbol{j} \cdot \boldsymbol{e} = j_c |\boldsymbol{v}| - j_c |\boldsymbol{e}|.$$

Integrating the Faraday law $\mu_0 \partial_t h + \nabla \times \boldsymbol{e} = 0$ we obtain $h = u + h_e(t) = u_0 + h_e(0) + \frac{1}{\mu_0} \nabla \times \int_0^t \boldsymbol{e} dt.$ Hence

$$(\boldsymbol{\nabla} \times u, \boldsymbol{v} - \boldsymbol{e}) = (u, \nabla \times \{\boldsymbol{v} - \boldsymbol{e}\}) =$$
$$\left(f + \frac{1}{\mu_0} \nabla \times \int_0^t \boldsymbol{e} dt, \nabla \times \{\boldsymbol{v} - \boldsymbol{e}\} \right),$$

where $f = u_0 - h_e(t) + h_e(0)$. Finally, for any time t and any $\boldsymbol{v}$

$$\left(f + \frac{1}{\mu_0} \nabla \times \int_0^t \boldsymbol{e} dt, \nabla \times \{\boldsymbol{v} - \boldsymbol{e}\} \right) \le (j_c, |\boldsymbol{v}| - |\boldsymbol{e}|)$$

Can be solved. But to use a mixed formulation ($\boldsymbol{e}$ and h) is easier.

Mixed V.I. formulation

We already showed that for any test field $\boldsymbol{v}$

$$(\boldsymbol{\nabla} \times h, \boldsymbol{v} - \boldsymbol{e}) \leq (j_c, |\boldsymbol{v}| - |\boldsymbol{e}|). \tag{1}$$

The Faraday law $\mu_0 \partial_t h + \nabla \times \boldsymbol{e} = 0$ and the Green formula yield

$$\mu_0(\partial_t h, \phi) + (\boldsymbol{e}, \boldsymbol{\nabla} \times \phi) = 0, \tag{2}$$

where ϕ is any smooth enough scalar test function s.t. $\phi|_{\partial\Omega} = 0$.
To complete the formulation: $h|_{t=0} = h_0,\ h|_{\partial\Omega} = h_e(t)$.

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To complete the formulation: $h|_{t=0} = h_0, \ h|_{\partial\Omega} = h_e(t)$.

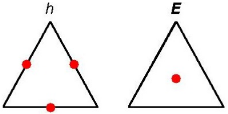
Advantages:

- All variables can be calculated (as for the dual v.i. form.);
- Much simpler finite elements can be used for $\boldsymbol{e}$ (vectorial p-w constant instead of the Raviart-Thomas el).
- Can be extended to the q.v.i. case $j_c(h)$ (as the primal v.i.);

Numerical solution of the mixed v.i.

1. Rotated variables: let $\boldsymbol{E} = [e_2, -e_1]$, $\boldsymbol{V} = [v_2, -v_1]$. Then $|\boldsymbol{E}| = |\boldsymbol{e}|$, $|\boldsymbol{V}| = |\boldsymbol{v}|$ and $\boldsymbol{\nabla} \times \phi \cdot \boldsymbol{e} = -\boldsymbol{\nabla}\phi \cdot \boldsymbol{E}$. Hence
$$(\boldsymbol{\nabla} h, \boldsymbol{V} - \boldsymbol{E}) + (j_c, |\boldsymbol{V}| - |\boldsymbol{E}|) \geq 0,$$
$$\mu_0(\partial_t h, \phi) - (\boldsymbol{\nabla}\phi, \boldsymbol{E}) = 0.$$
2. Smoothing: $|\boldsymbol{E}| \approx \frac{1}{1+\varepsilon}|\boldsymbol{E}|^{1+\varepsilon}$ for $0 < \varepsilon \ll 1$.
The inequality boils down to $\boldsymbol{\nabla} h + j_c|\boldsymbol{E}|^{\varepsilon-1}\boldsymbol{E} = 0.$
3. Time discretization: $\mu_0(h - \hat{h}, \phi) - \Delta t(\boldsymbol{\nabla}\phi, \boldsymbol{E}) = 0.$
4. F.E. discretization:

   nonconforming linear el. for h, ϕ;
   vectorial p-w constant el. for $\boldsymbol{E}$.



5. Iterations to resolve the nonlinearity: on the i-th iteration
$$|\boldsymbol{E}|^{\varepsilon-1}\boldsymbol{E} \approx |\boldsymbol{E}^{i-1}|^{\varepsilon-1}\boldsymbol{E}^{i-1} + (|\boldsymbol{E}^{i-1}|_\delta)^{\varepsilon-1}(\boldsymbol{E}^i - \boldsymbol{E}^{i-1})$$
where $|\boldsymbol{E}|_\delta = \sqrt{|\boldsymbol{E}|^2 + \delta^2}$ with $0 < \delta \ll 1$. Over-relaxation.

Scaling

In the numerical examples we assumed

$$h(\boldsymbol{x},0) = 0 \text{ and } dh_e/dt = const.$$

We used the dimensionless variables:

$$\boldsymbol{x} = \frac{\boldsymbol{x}'}{L}, \quad t = \frac{t'}{t_0}, \quad \boldsymbol{j} = \frac{\boldsymbol{j}'}{j_c}, \quad h = \frac{h'}{Lj_c}, \quad \boldsymbol{e} = \frac{\boldsymbol{e}'\, t_0}{L^2 j_0}.$$

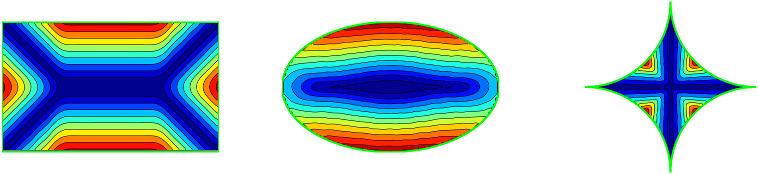
Here

L is the characteristic cross-section size,

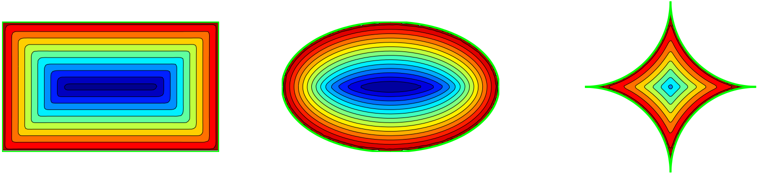
t_0 is chosen to make $dh_e/dt = 1$.

Simulation: homogeneous cylinders in a growing field

Levels of $|\boldsymbol{e}|$:



Levels of h and current contours:

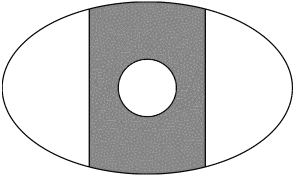


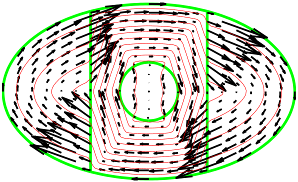
Magnetization of homogeneous cylinders of different cross-sections.
Dimensionless variables, $h_e(t) = t$, shown for $t = 0.3$.

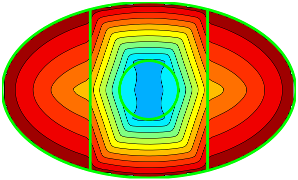
Simulation: inhomogeneous cylinder in a growing field

$$h_e = 0.3$$

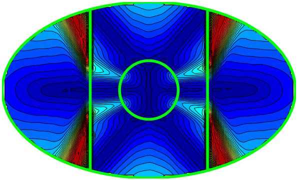
white: $j_c = 1/3$, gray: $j_c = 1$







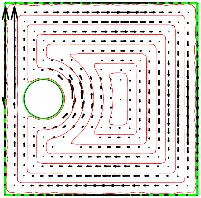
h levels

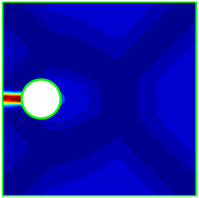


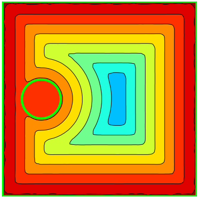
$\boldsymbol{e}$ and $|\boldsymbol{e}|$ levels

Simulation: hollow cylinder in a growing field ($h_e = 0.6$)

Left – $\boldsymbol{e}$ and contours of $\boldsymbol{j}$, right – $|\boldsymbol{e}|$ levels, bottom – h levels



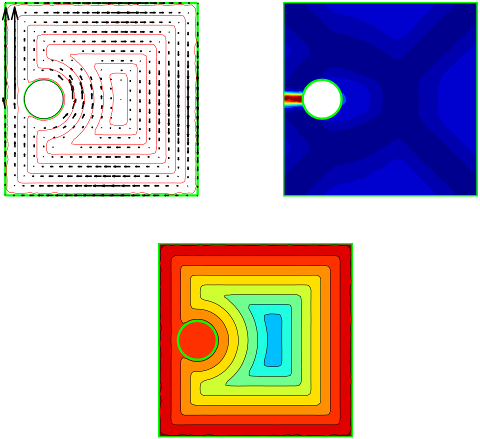




The magnetic field penetrates the hole through a thin channel; the electric field is very strong along this path.

Simulation: hollow cylinder in a growing field ($h_e = 0.6$)

Left – $\boldsymbol{e}$ and contours of $\boldsymbol{j}$, right – $|\boldsymbol{e}|$ levels, bottom – h levels



谢谢！ Thank you! **Спасибо!** **תודה רבה!**