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SOLUTE DISPERSION IN A PULSATING FLOW

L. B. Prigozhin

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The one-dimensional model proposed by Taylor [1] of the dispersion of soluble matter describes approximately the distribution of the solute concentration averaged over the tube section in Poiseuille flow. Aris [2] obtained more accurately the effective diffusion coefficient in Taylor's model and solved the problem for the general case of steady flow in a channel of arbitrary section. Many papers have been published in the meanwhile devoted to particular applications of this theory (for example, [3-5]). Various dispersion models have been constructed [6-8] that make the Taylor—Aris model more accurate at small times and agree with it at large times. The acceleration of the mixing of the solute considered in these models in the presence of the simultaneous influence of molecular diffusion and convective transport also operates in unsteady flows. In particular, the presence of velocity pulsations influences the growth of the dispersion even if the mean flow velocity is equal to zero at every point of the flow. In the present paper, the Taylor—Aris theory is extended to the case of laminar flows with periodically varying flow velocity.

1. Dispersion Model

We consider laminar flow of a fluid in an infinite straight tube with section Ω of area S. We assume that the flow velocity v is everywhere directed along the tube axis, has period T_0 with respect to the time, and is the same at every section. The coordinate origin is chosen to move with the mean flow velocity, so that

$$\int_{t_0}^{t_0+T_0}\int_{\Omega} v\,d\Omega\,dt=0 \tag{1.1}$$

In t dimensionless coordinates, the equation of convective diffusion of the solute in the flow takes the form

$$\frac{\partial c}{\partial t}+v\frac{a}{D}\frac{\partial c}{\partial z}=\frac{\partial^2 c}{\partial x^2}+\frac{\partial^2 c}{\partial y^2}+\frac{\partial^2 c}{\partial z^2},\quad (x,\ y,\ z)\in\Omega\times R^1,\ t>0$$
$$\left(x=\frac{x_0}{a},\ y=\frac{y_0}{a},\ z=\frac{z_0}{a},\ t=\frac{Dt_0}{a^2}\right) \tag{1.2}$$

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On the solution c(x, y, z, t) we impose the conditions that it be bounded as $z \to \pm\infty$ and also the boundary condition

$$\frac{\partial c}{\partial n}=0, \quad (x, y, z) \in \partial\Omega \times R^1 \tag{1.3}$$

Here, $a=\sqrt{S}$ is the characteristic linear dimension of the region Ω, D is the molecular diffusion coefficient, c is the concentration, n is the normal to the surface of the tube, and $\partial\Omega$ is the boundary of the region Ω.

In the dimensionless coordinates (z, t), the one-dimensional Taylor—Aris dispersion model can be written in the form

$$\frac{\partial c^*}{\partial t}=\frac{D^*}{D}\frac{\partial^2 c^*}{\partial z^2}, \quad z\in R^1, \quad t>0 \tag{1.4}$$

In this equation, D^* is the effective diffusion coefficient, and c^* is the concentration, bounded as $z \to \pm\infty$, of the solute averaged over the section. The initial values of c and c^* at t = 0 in the problems (1.2)-(1.3) and (1.4) are assumed to be consistent:

$$c^*(z,0)=\int_\Omega c(x,y,z,0)\,d\Omega$$

(the area of Ω in the variables (x, y) is equal to unity).

The Taylor effective diffusion coefficient is chosen in the case of steady flows to satisfy the condition of asymptotic equality of the dispersions determined in accordance with the convective model and the simplified model.

In the case of pulsating flows, one is usually interested in the rate of change of the concentration averaged over a period. It is natural to define the effective diffusion coefficient for flows of this type in the same way as the coefficient in the Taylor—Aris model, for which the ratio of the dispersion of the solute to the dispersion in the convective model averaged over an oscillation period tends to unity as $t \to \infty$.

2. The Method of Moments

We consider the axial moments of the distribution of the concentration and their mean values over the section Ω [2]:

$$\mu_p(x,y,t)=\int_{R^1} z^p c(x,y,z,t)\,dz, \quad m_p(t)=\int_\Omega \mu_p(x,y,t)\,d\Omega, \quad p=0,1,2,\ldots$$

assuming that at t = 0 these quantities exist. Using Eqs. (1.2)-(1.3), we obtain a sequence of initial—boundary-value problems (2.1) and Cauchy problems for the ordinary differential equations (2.2):

$$\frac{\partial\mu_p}{\partial t}+\Lambda\mu_p=\frac{pa}{D}v\mu_{p-1}+p(p-1)\mu_{p-2}, \quad \left.\frac{\partial\mu_p}{\partial n}\right|_{\partial\Omega}=0, \quad \mu_p|_{t=0}=\mu_p{}^0(x,y) \quad \Lambda=-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2} \tag{2.1}$$

$$\frac{dm_p}{dt}=p(p-1)m_{p-2}+\frac{pa}{D}\int_\Omega v\mu_{p-1}\,d\Omega, \quad m_p(0)=\int_\Omega \mu_p{}^0\,d\Omega \tag{2.2}$$

We also assume that the boundary $\partial\Omega$ of Ω consists of a finite number of smooth curves. Then in $L_2(\Omega)$ there exists a complete orthonormal system of eigenfunctions $\{1, u, u_2, \ldots\}$ with eigenvalues $0=\lambda_0<\lambda_1\leqslant\lambda_2\leqslant\ldots$ of the operator Λ such that $\partial u_i/\partial n|_{\partial\Omega}=0$.

The solution of the problem (2.1) can be expressed in terms of the Green's function G and the coefficients of the expansion of the function μ_p^0 in a Fourier series with respect to the system $\{u_i\}$:

$$\mu_p=\mu_{p0}+\sum_{i=1}^{\infty}\mu_{pi}e^{-\lambda_i t}u_i(x,y)+p\int_0^t\int_\Omega G(x,\xi,y,\eta,t-\tau)\left\{\frac{a}{D}v\mu_{p-1}+\mu_{p-2}(p-1)\right\}d\xi\,d\eta\,d\tau$$

$$G(x,\xi,y,\eta,t-\tau)=1+\sum_{i=1}^{\infty}e^{-\lambda_i(t-\tau)}u_i(x,y)u_i(\xi,\eta),\quad \mu_{pi}=\int_{\Omega}\mu_p{}^0u_i\,d\Omega \tag{2.3}$$

For p = 0,

$$\mu_0(x,y,t)=\mu_{00}+\sum_{i=1}\mu_{0i}e^{-\lambda_i t}u_i(x,y) \tag{2.4}$$

$$m_0(t)\equiv\mu_{00} \tag{2.5}$$

The identity (2.5) expresses the constancy of the amount of the solute in the system.

The coefficients $v_i(t)$ of the series expansion

$$v=v_0(t)+\sum_{i=1}^{\infty}v_i(t)u_i(x,y) \tag{2.6}$$

of the function v can, by virtue of its periodicity, be expanded in the trigonometric Fourier series

$$v_i=v_{i0}+\sum_{j=1}^{\infty}(p_{ij}\cos(\omega_j t)+q_{ij}\sin(\omega_j t)),\quad \omega_j=\frac{2\pi j}{T},\quad T=\frac{DT_0}{a^2} \tag{2.7}$$

Here, T is the period of the function v with respect to t, and $v_{00}=0$ by virtue of the condition (1.1).

Calculating μ_1 from (2.3) using (2.4), (2.6), and (2.7) and ignoring the terms that vanish as $t\to\infty$, we obtain

$$\mu_1\sim\frac{a}{D}(\mu_{00}f(x,y,t)+M_1)$$

$$f=\sum_{i=1}^{\infty}u_i\left\{\sum_{j=1}^{\infty}(p_{ij}(\lambda_i\cos(\omega_j t)+\omega_j\sin(\omega_j t))+q_{ij}(\lambda_i\sin(\omega_j t)-\omega_j\cos(\omega_j t)))\frac{1}{\omega_j{}^2+\lambda_i{}^2}+\frac{v_{i0}}{\lambda_i}\right\}+$$
$$\sum_{j=1}^{\infty}\frac{p_{0j}\sin(\omega_j t)-q_{0j}\cos(\omega_j t)}{\omega_j},\quad M_1=\text{const} \tag{2.8}$$

Here, the symbol ~ denotes equality to accuracy $O(te^{-\lambda_1 t})$, and the function f is periodic.

It follows from the relation (2.8) that the "center of gravity" of the solute at large t executes periodic oscillations about a certain point that moves with the mean velocity of the flow.

We now turn to determination of the dispersion of the solute. In the Taylor—Aris model (1.4), the dispersion m_2^* increases at a constant rate:

$$\frac{dm_2{}^*}{dt}=2\mu_{00}\frac{D^*}{D} \tag{2.9}$$

The change in the dispersion of the solute in the framework of the convective model is described by Eq. (2.2) for p = 2. Using (2.5) and (2.8), we have

$$\frac{dm_2}{dt}\sim2\left(\mu_{00}+\frac{a^2}{D^2}\int_{\Omega}v(\mu_{00}f+M_1)\,d\Omega\right)$$

Let $\langle\ \rangle$ be the averaging operator

$$\langle\varphi\rangle(t)=\frac{1}{T}\int_t^{t+T}\varphi(\tau)\,d\tau$$

Since the operators $\langle\ \rangle$ and d/dt commute, the averaged dispersion, which is equal to the dispersion of the averaged concentration distribution, varies at the rate

$$\frac{d\langle m_2\rangle}{dt}\sim2\left(\mu_{00}+\frac{a^2}{TD^2}\int_t^{t+T}\int_{\Omega}v(\mu_{00}f+M_1)\,d\Omega\,d\tau\right)$$

Calculating the integral on the right-hand side, we obtain

$$\frac{d\langle m_2\rangle}{dt} \sim 2\mu_{00}\left(1+\frac{a^2}{D^2}\sigma\right) \tag{2.10}$$

$$\sigma=\frac{1}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{p_{ij}^2+q_{ij}^2}{\lambda_i^2+\omega_j^2}\lambda_i+\sum_{i=1}\frac{v_{i0}^2}{\lambda_i} \tag{2.11}$$

It follows from (2.9) and (2.10) that irrespective of the initial distribution of the solute

$$\lim_{t\to\infty}\frac{m_2^*}{\langle m_2\rangle}=1$$

if

$$D^*=D+\frac{a^2\sigma}{D} \tag{2.12}$$

Note that the coefficients of the Fourier series of the function $v_0(t)$ do not occur in the expression (2.11). It is only the velocity components leading to flow stratification that accelerate the mixing.

3. Associated Problem

The practical calculation of the effective diffusion coefficient (2.12) by means of the series (2.11) is difficult. We show that the coefficient σ introduced in Sec. 2 can be expressed in terms of the solution of the problem

$$\frac{\partial u}{\partial t}+\Lambda u=v, \quad (x,y)\in\Omega, \quad 0\leqslant t\leqslant T, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega}=0, \quad u|_{t=0}=u|_{t=T} \tag{3.1}$$

Using Fourier's method, we can prove the following proposition.

If v is a continuously differentiable function with respect to t with values $L_2(\Omega)$ defined on a circle S^1 of length T, and the condition

$$\int_0^T\int_\Omega v\,d\Omega\,dt=0$$

is satisfied, a solution to the problem (3.1) exists and is unique up to a constant in the class of functions continuously differentiable with respect to t and defined in the Sobolev space $H_1(\Omega)$ on S^1.

The solution can be represented in the form of the series

$$u=U_0+\int_0^t v_0\,d\tau+\sum_{i=1}^{\infty}\left\{g_i e^{-\lambda_i t}+\int_0^t e^{-\lambda_i(t-\tau)}\,v_i\,d\tau\right\}u_i, \qquad g_i=\int_0^T e^{-\lambda_i(T-\tau)}\,v_i\,d\tau/(1-e^{-\lambda_i t}) \tag{3.2}$$

which converges in $H_1(\Omega)$ uniformly with respect to t. Here, $v_i(t)$ are the coefficients of the series (2.6) and U_0 is an arbitrary constant. It is easy to show that from (3.2) there follows the equation

$$\sigma=\frac{1}{T}\int_0^T\int_\Omega(\nabla u,\nabla u)\,d\Omega\,dt$$

which holds for any value of U_0.

Since u is a solution of the problem (3.1), this equation can be written differently:

$$\sigma=\frac{1}{T}\int_0^T\int_\Omega uv\,d\Omega\,dt \tag{3.3}$$

Using (3.3) and returning to the variables x_0, y_0, t_0, from (2.12) we obtain the final expression for the effective diffusion coefficient:

$$D^{*}=D+\frac{1}{D}\left(\frac{1}{ST_0}\int_0^{T_0}\int_\Omega u_0 v\,d\Omega\,dt_0\right) \tag{3.4}$$

where $u_0(x_0, y_0, t_0)$ is a T_0-periodic solution of the problem

$$\frac{1}{D}\frac{\partial u_0}{\partial t_0}-\Delta u_0=v(x_0, y_0, t_0) \tag{3.5}$$

$$\left.\frac{\partial u_0}{\partial n}\right|_{\partial\Omega}=0 \tag{3.6}$$

In the stationary case the well-known solution of [2] can be obtained from (3.4)-(3.6).

4. Example

We consider the flow of a Newtonian incompressible fluid in a circular tube of radius R established under the influence of a harmonically varying pressure gradient

$$-\frac{dP}{dz}=\rho(\chi_0+\chi_1\cos(\omega_0 t_0))$$

where ρ is the density of the fluid.

We introduce characteristic scales for the steady and pulsating components of the velocity equal to the mean flow velocities in steady flows with pressure gradients $-dP_i/dz_0 = \rho\chi_i$ $(i = 0, 1)$. In the case of a circular tube, these quantities are equal to $V_i = \chi_i R^2/8\nu$, where ν is the viscosity of the fluid. The velocity of the flow in the pulsating case with respect to the chosen coordinate origin is given by [9]

$$v=V_0\left(1-2\left(\frac{r_0}{R}\right)^2\right)+\frac{8V_1}{H^2}\left(B\left(\frac{r_0}{R}\right)\cos(\omega_0 t_0)+\left(1-A\left(\frac{r_0}{R}\right)\right)\sin(\omega_0 t_0)\right)$$

$$A(r)=A_1\operatorname{bei}(Hr)+B_1\operatorname{ber}(Hr),\quad B(r)=A_1\operatorname{ber}(Hr)-B_1\operatorname{bei}(Hr)$$

$$A_1=\frac{\operatorname{bei}H}{\operatorname{bei}^2H+\operatorname{ber}^2H},\qquad B_1=\frac{\operatorname{ber}H}{\operatorname{bei}^2H+\operatorname{ber}^2H}$$

where ber and bei are Kelvin functions, and $H = R\sqrt{\omega_0/\nu}$ is the dimensionless oscillation frequency. The velocity component $8V_1\sin(\omega_0 t_0)/H^2$, which does not affect the average dispersion, can be ignored. We seek a solution of Eq. (3.5) in the form

$$u_0(r_0, t_0)=R^2\left(V_0E\left(\frac{r_0}{R}\right)+\frac{8V_1}{H^2}\left\{F\left(\frac{r_0}{R}\right)\cos(\omega_0 t_0)+G\left(\frac{r_0}{R}\right)\sin(\omega_0 t_0)\right\}\right)$$

Then the expression (3.4) becomes

$$D^{*}=D+\frac{V_0^2R^2}{D}k_0+\frac{V_1^2R^2}{D}k_1 \tag{4.1}$$

$$k_0=2\int_0^1 rE(r)(1-2r^2)\,dr \tag{4.2}$$

$$k_1=\frac{64}{H^4}\int_0^1 r(B(r)F(r)-A(r)G(r))\,dr \tag{4.3}$$

Here, the functions E(r), F(r), and G(r) are solutions of the system of ordinary differential equations

$$-\frac{1}{r}\frac{d}{dr}\left(r\frac{dE}{dr}\right)=1-2r^2,\quad W^2F+\frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)=A(r),\quad W^2G-\frac{1}{r}\frac{d}{dr}\left(r\frac{dF}{dr}\right)=B(r)$$

$$\left.\frac{dE}{dr}\right|_0=\left.\frac{dE}{dr}\right|_1=\left.\frac{dF}{dr}\right|_0=\left.\frac{dF}{dr}\right|_1=\left.\frac{dG}{dr}\right|_0=\left.\frac{dG}{dr}\right|_1=0$$

and we have the dimensionless diffusion frequency $W = R\sqrt{\omega_0/D} = H\sqrt{Sc}$, where $Sc = \nu/D$ is the Schmidt number. The determination of the function E(r) and the calculation of the integral (4.2) does not present difficulties. The obtained coefficient $k_0 = 1/48$ is the Taylor coefficient.

For Sc ≠ 1, we find

$$F=a_1 \operatorname{bei}(Hr)+a_2 \operatorname{ber}(Hr)+a_3 \operatorname{bei}(Wr)+a_4 \operatorname{ber}(Wr), \quad G=-a_2 \operatorname{bei}(Hr)+a_1 \operatorname{ber}(Hr)-a_4 \operatorname{bei}(Wr)+a_3 \operatorname{ber}(Wr)$$

$$a_1=\frac{A_1}{W^2-H^2}, \quad a_2=\frac{B_1}{W^2-H^2}$$

and the coefficients a_3 and a_4 can be found from the boundary conditions at the point r = 1. Omitting the lengthy calculations, we give the result of the calculation of the integral (4.3):

$$k_1=\frac{64}{H^3(H^4-W^4)}\left\{\frac{H}{W}\frac{(\operatorname{ber}' H)^2+(\operatorname{bei}' H)^2}{(\operatorname{ber}' W)^2+(\operatorname{bei}' W)^2}(\operatorname{ber} W \operatorname{ber}' W+\operatorname{bei} W \operatorname{bei}' W)-\operatorname{ber} H \operatorname{ber}' H-\operatorname{bei} H \operatorname{bei}' H\right\} \quad (4.4)$$

where the prime denotes differentiation.

The expression (4.4) has a removable singularity at H = W and in the case Sc = 1 the coefficient k_1 can be calculated by LHôpital's rule. Expanding the Kelvin functions in (4.4) in a Taylor series at small values of H and W, we find that in quasisteady regimes k_1 is near 1/96.

Using (2.11), we can show that for harmonic variation of the pressure gradient the analogous relation $k_1 \approx k_0/2$ holds for quasisteady flows in channels with any section if in Eq. (4.1) we understand by R the characteristic linear dimension of the region, and V_0 and V_1 are chosen as in the considered example. When H and W are increased, the coefficient k_1 tends to zero, which is explained by the decrease in the amplitude of the oscillations and the deformation of the velocity profile with increasing frequency [9].

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