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Sandpiles and superconductors: nonconforming linear finite element approximations for mixed formulations of quasi-variational inequalities

John W. Barrett*
Department of Mathematics, Imperial College London, London SW7 2AZ, UK
*Corresponding author: jwb@imperial.ac.uk

and

Leonid Prigozhin
Department of Solar Energy and Environmental Physics, Blaustein Institutes for Desert Research, Ben-Gurion University of the Negev, Sede Boqer Campus 84990, Israel
leonid@math.bgu.ac.il

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Similar evolutionary variational and quasi-variational inequalities with gradient constraints arise in the modelling of growing sandpiles and type-II superconductors. Recently, mixed formulations of these inequalities were used for establishing existence results in the quasi-variational inequality case. Such formulations, and this is an additional advantage, made it possible to determine numerically not only the primal variables, e.g., the evolving sand surface and the magnetic field for sandpiles and superconductors, respectively, but also the dual variables, the sand flux and the electric field. Numerical approximations of these mixed formulations in previous studies employed the Raviart–Thomas element of the lowest order. Here we introduce simpler numerical approximations of these mixed formulations based on the nonconforming linear finite element. We prove (subsequence) convergence of these approximations and illustrate their effectiveness by numerical experiments.

Keywords: quasi-variational inequalities; critical state problems; power laws; primal and mixed formulations; nonconforming finite elements; convergence analysis.

1. Introduction

Recently, the present authors have introduced mixed formulations of variational and quasi-variational inequality problems arising in the mathematical modelling of (i) growing sandpiles, (ii) cylindrical superconductors in a parallel external field, and (iii) thin film superconductors in a perpendicular external field in Barrett & Prigozhin (2010, 2013a,b). In each of these papers, a numerical approximation, based on the lowest order Raviart–Thomas element, of the corresponding mixed formulation was introduced, and (subsequence) convergence was proved as the mesh parameters and the power-law regularization parameter, $r-1$, tended to zero. Hence, the existence of a solution to these mixed formulations was established. In this paper, we introduce simpler numerical approximations based on a nonconforming linear finite element approximation of these mixed formulations. In addition, we prove (subsequence) convergence of these approximations as the mesh and regularization parameters tend to zero.

We first briefly describe these mixed formulations. Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain with a Lipschitz boundary $\partial\Omega$.

1.1 *Mathematical models and their mixed formulations*

1.1.1 *(i) Growing sandpiles.* Let a cohesionless granular material (sand), characterized by its angle of repose α, be poured out onto a rigid surface $y = w_0(\underline{x})$, where y is vertical and $\underline{x} \in \Omega$. The support surface $w_0 \in W_0^{1,\infty}(\Omega)$ and the non-negative density of the distributed source $f \in L^2(0,T;L^2(\Omega))$ are given. We consider the growing sandpile $y = w(\underline{x},t)$ and set an open boundary condition $w|_{\partial\Omega} = 0$. Denoting by $\underline{q}(\underline{x},t)$ the horizontal projection of the flux of material pouring down the evolving pile surface, we can write the mass balance equation as

$$\frac{\partial w}{\partial t} + \underline{\nabla} \, . \, \underline{q} = f. \tag{1.1}$$

The quasi-stationary model of sand surface evolution (see Prigozhin, 1986, 1994, 1996) assumes that the flow of sand is confined to a thin surface layer and directed towards the steepest descent of the pile surface. Wherever the support surface is covered by sand, the pile slope should not exceed the critical value; that is, $w > w_0 \;\Rightarrow\; |\underline{\nabla} w| \leqslant k_0$, where $k_0 = \tan\alpha$ is the internal friction coefficient. Of course, the uncovered parts of the support can be steeper. This model does not allow for any flow on the subcritical parts of the pile surface; that is, $|\underline{\nabla} w| < k_0 \Rightarrow \underline{q} = \underline{0}$. These constitutive relations can be conveniently reformulated for a.e. $(\underline{x},t) \in \Omega \times (0,T)$ as

$$|\underline{\nabla} w| \leqslant M(w) \quad \text{and} \quad M(w)|\underline{q}| + \underline{\nabla} w \, . \, \underline{q} = 0, \tag{1.2}$$

where, for any $\eta \in C(\bar{\Omega})$,

$$M(\eta)(\underline{x}) := \begin{cases} k_0 & \eta(\underline{x}) > w_0(\underline{x}), \\ \max(k_0, |\underline{\nabla} w_0(\underline{x})|) & \eta(\underline{x}) \leqslant w_0(\underline{x}), \end{cases} \quad \forall \underline{x} \in \bar{\Omega}. \tag{1.3}$$

Let us define, for any $\eta \in C(\bar{\Omega})$, the closed convex nonempty set

$$K(\eta) := \{\varphi \in W_0^{1,\infty}(\Omega) \;:\; |\underline{\nabla}\varphi| \leqslant M(\eta) \text{ a.e. in } \Omega\}. \tag{1.4}$$

As $M(w)|\underline{q}| + \underline{\nabla}\varphi \, . \, \underline{q} \geqslant 0$ for any $\varphi \in K(w)$, we have, on noting (1.2), that $w \in K(w)$ and $\underline{\nabla}(\varphi - w) \, . \, \underline{q} \geqslant 0$. A weak form of the latter inequality is: for a.a. $t \in (0,T)$

$$\int_\Omega \underline{\nabla} \, . \, \underline{q} \, (w - \varphi) \, \mathrm{d}\underline{x} \geqslant 0 \quad \forall \varphi \in K(w). \tag{1.5}$$

Combining (1.5) and (1.1) yields an evolutionary quasi-variational inequality for the evolving pile surface: Find $w \in K(w)$ such that for a.a. $t \in (0,T)$

$$\int_\Omega \left(\frac{\partial w}{\partial t} - f\right)(\varphi - w) \, \mathrm{d}\underline{x} \geqslant 0 \quad \forall \varphi \in K(w). \tag{1.6}$$

Assuming there is no sand on the support initially, we set

$$w(\cdot, 0) = w_0(\cdot). \tag{1.7}$$

We note that with the open boundary condition $w|_{\partial\Omega} = 0$ an uncontrollable influx of material from outside can occur through the parts of the boundary where $\underline{\nabla} w \, . \, \underline{\nu} \geqslant k_0$, with $\underline{\nu}$ being the outward unit normal to $\partial\Omega$. This makes the solution nonunique and, possibly, discontinuous. Such an influx is prevented in our model by assuming that

$$\underline{\nabla} w_0 \,.\, \underline{\nu} < k_0 \quad \text{on } \partial\Omega, \tag{1.8}$$

which implies, see Barrett & Prigozhin (2013a), that $\underline{\nabla} w \,.\, \underline{\nu} < k_0$ on $\partial\Omega$ also for $t > 0$.

If $|\underline{\nabla} w_0| \leqslant k_0$ a.e. in Ω, then $K(\eta) \equiv K := \{\varphi \in W_0^{1,\infty}(\Omega) : |\underline{\nabla}\varphi| \leqslant k_0 \text{ a.e. in } \Omega\}$ and the quasi-variational inequality (1.6) becomes simply a variational inequality; this case was studied in Prigozhin (1986, 1996) and Aronsson *et al.* (1996).

Here we will use a mixed variational formulation of the growing sandpile model involving both variables. Such formulations are often advantageous, because they allow one to determine, not only the evolving sand surface w, but also the surface flux $\underline{q}$, which too is of interest in various applications; see Prigozhin (1993, 1994) and Barrett & Prigozhin (2010). In such formulations, and this is their additional advantage, the difficulty to deal with gradient constraint in (1.6) is replaced by a simpler, although nonsmooth, nonlinearity. Therefore, instead of excluding the surface flux $\underline{q}$ from the model formulation, as in the transition to (1.6) above, we reformulate the conditions (1.2) for a.a. $t \in (0, T)$ as

$$\int_\Omega [M(w)(|\underline{v}| - |\underline{q}|) + \underline{\nabla} w \,.\, (\underline{v} - \underline{q})]\, \mathrm{d}\underline{x} \geqslant 0 \tag{1.9}$$

for any test flux $\underline{v}$, and consider a mixed formulation of the sand model as (1.1) and (1.9).

The natural function space for the flux $\underline{q}$ is the space of vector-valued bounded Radon measures having L^2 divergence. If $\underline{q}$ is such a measure, the discontinuity of $M(w)$ makes it difficult to give a sense to the term $\int_\Omega M(w)|\underline{q}|\,\mathrm{d}\underline{x}$ in the inequality (1.9) of the mixed formulation. Existence of a solution was recently proved in Barrett & Prigozhin (2013a), for a regularized version of the growing sandpile model with a continuous operator $M_\varepsilon : C(\bar{\Omega}) \to C(\bar{\Omega})$, determined as follows. For a fixed small $\varepsilon > 0$, we approximate the initial data $w_0 \in W_0^{1,\infty}(\Omega)$ by $w_0^\varepsilon \in W_0^{1,\infty}(\Omega) \cap C^1(\bar{\Omega})$, and $M(\cdot)$ by the continuous function $M_\varepsilon(\cdot)$ such that for any $\underline{x} \in \bar{\Omega}$

$$M_\varepsilon(\eta)(\underline{x}) := \begin{cases} k_0, & \eta(\underline{x}) \geqslant w_0^\varepsilon(\underline{x}) + \varepsilon, \\ k_1^\varepsilon(\underline{x}) + (k_0 - k_1^\varepsilon(\underline{x}))\left(\dfrac{\eta(\underline{x}) - w_0^\varepsilon(\underline{x})}{\varepsilon}\right), & \eta(\underline{x}) \in [w_0^\varepsilon(\underline{x}), w_0^\varepsilon(\underline{x}) + \varepsilon], \\ k_1^\varepsilon(\underline{x}) := \max(k_0, |\underline{\nabla} w_0^\varepsilon(\underline{x})|), & \eta(\underline{x}) \leqslant w_0^\varepsilon(\underline{x}). \end{cases} \tag{1.10}$$

Below, we also adopt such a regularization. We note that the existence of a solution for the regularized primal quasi-variational inequality (1.6) follows also from a recent result of Rodrigues & Santos (2012).

Obviously, if $|\nabla w^0| \leqslant k_0$ no regularization is needed as $M \equiv k_0$. In this variational inequality, case the mixed formulations of the growing sandpile problem, and its numerical approximation by the lowest order Raviart–Thomas element, were studied in Barrett & Prigozhin (2006) and Dumont & Igbida (2009).

1.1.2 *(ii) Cylindrical superconductors in a parallel external field.* Let us consider an infinite type-II superconducting cylinder having a cross-section Ω and placed in a given parallel nonstationary uniform external magnetic field $b_{\mathrm{e}}(t)$. In this case, the magnetic field of a current induced in the superconductor also has only one nonzero component and can be regarded as a scalar function $w(\underline{x}, t)$, which vanishes on $\partial\Omega$. The electric field, $\underline{e}$, inside the superconductor is the same in each cross-section of the cylinder and is orthogonal to the magnetic field. A similar statement holds for the current density, $\underline{j}$, inside the superconductor. With $\underline{e}(\underline{x},t) \equiv [e_1(\underline{x},t), e_2(\underline{x},t)]^{\mathrm{T}}$, Faraday's law can be rewritten as (1.1) with

$$f = -\frac{\mathrm{d}b_{\mathrm{e}}}{\mathrm{d}t} \quad \text{and} \quad \underline{q} = [e_2, -e_1]^{\mathrm{T}} \Rightarrow \nabla \times \underline{e} = \frac{\partial e_2}{\partial x_1} - \frac{\partial e_1}{\partial x_2} = \underline{\nabla} \,.\, \underline{q}. \tag{1.11}$$

Here, and throughout this paper, we use scaled dimensionless electromagnetic variables. In particular, we do not distinguish between the magnetic induction and the magnetic field on assuming that the magnetic permeability of the superconductor is equal to that of a vacuum, and is scaled to unity.

Ampère's law yields that the current density $\underline{j} = \underline{\nabla} \times w = [\partial w/\partial x_2, -\partial w/\partial x_1]^{\mathrm{T}}$, and so $|\underline{j}| = |\underline{\nabla} w|$. Let $\underline{j}$ and $\underline{e}$ satisfy the critical state model relations:

$$|\underline{j}| \leqslant j_{\mathrm{c}}, \quad |\underline{j}| < j_{\mathrm{c}} \Rightarrow \underline{e} = \underline{0}, \quad \underline{e} \neq \underline{0} \Rightarrow \underline{e} \parallel \underline{j} \Rightarrow \underline{q} \parallel -\underline{\nabla} w, \tag{1.12}$$

where j_{c} is the critical current density, which may be constant or depend only on $\underline{x}$ (the Bean model, see Bean, 1964) or depend also on the total magnetic field, $w + b_{\mathrm{e}}$ (the Kim model, see Kim *et al.*, 1962). Similarly to the growing sandpile problem, one can show that w satisfies the quasi-variational inequality problem (1.6) with f as in (1.11) and $K(w)$ replaced by $\hat{K}(w + b_{\mathrm{e}})$, where

$$\hat{K}(\psi) := \{\eta \in W_0^{1,\infty}(\Omega) \, : \, |\underline{\nabla}\eta| \leqslant j_{\mathrm{c}}(\psi) \text{ a.e. in } \Omega\}. \tag{1.13}$$

This is supplemented with $w(\cdot,0) = w_0(\cdot)$, where $w_0 \in \hat{K}(w_0 + b_{\mathrm{e}}(0))$. Once again, if j_{c} is independent of the total magnetic field, i.e., the Bean model, this quasi-variational inequality problem collapses to a variational inequality problem. Similarly, the conditions (1.12) can be reformulated as (1.9) with $M(w)$ replaced by $j_{\mathrm{c}}(w + b_{\mathrm{e}})$, and this supplemented with (1.1) yields the mixed formulation of this cylindrical superconductor problem, see Barrett & Prigozhin (2010) for further details. We note that $\underline{q} = [-e_2, e_1]^{\mathrm{T}}$ in Barrett & Prigozhin (2010, p. 684).

In Barrett & Prigozhin (2010), and in this paper, we assume for the critical state model that

$$j_{\mathrm{c}}(w + b_{\mathrm{e}})(\underline{x},t) = k(\underline{x})\hat{M}(w(\underline{x},t) + b_{\mathrm{e}}(t)), \tag{1.14}$$

where $\hat{M} \in C(\mathbb{R}, [\hat{M}_0, \hat{M}_1])$ with $\hat{M}_0, \hat{M}_1 \in \mathbb{R}_{>0}$, and $k \in C(\bar{\Omega})$ with $k(\underline{x}) \geqslant k_{\min} > 0$ for all $\underline{x} \in \bar{\Omega}$. In Barrett & Prigozhin (2010) we exploited the fact that $|\underline{\nabla}(w + b_{\mathrm{e}})| \leqslant k\hat{M}(w + b_{\mathrm{e}})$ can be rewritten as $|\underline{\nabla}[\hat{F}(w + b_{\mathrm{e}})]| \leqslant k$, where $\hat{F}'(\cdot) = [\hat{M}(\cdot)]^{-1}$ and $\hat{F}(0) = 0$. Clearly, such a reformulation is not applicable to $M(\cdot)$, (1.3), or $M_\varepsilon(\cdot)$, (1.10), for the growing sandpile problem.

Engineers often describe the current–voltage relation of type-II superconductors by a power law

$$\underline{e} = \left(\frac{|\underline{j}|}{j_{\mathrm{c}}}\right)^{p-2} \frac{\underline{j}}{j_{\mathrm{c}}} \Rightarrow \underline{\nabla} w = -j_{\mathrm{c}} \, |\underline{q}|^{r-2} \underline{q} \quad \text{where } \frac{1}{r} + \frac{1}{p} = 1, \tag{1.15}$$

with the power p typically between 10 and 100. As is well known, the critical state model relations (1.12) can be regarded as the $p \to \infty$ ($r \to 1$) limit of the power law (1.15); see Barrett & Prigozhin (2000) in the case of the homogeneous Bean model, $j_{\mathrm{c}} \in \mathbb{R}_{>0}$, and Theorem 3.8 for (1.14).

1.1.3 *(iii) Thin film superconductors in a perpendicular external field.* Here we consider an infinitely thin film superconductor occupying the two-dimensional domain Ω in the $x_3 = 0$ plane. With $b_{\mathrm{e}}(t)$ the normal to the film component of the given nonstationary uniform external magnetic field, the normal to the film component of the total magnetic field can then be expressed by the Biot–Savart law as

$$b_3(\underline{x},t) = b_{\rm e}(t) + \frac{1}{4\pi}\nabla \times \int_{\Omega} \frac{\underline{j}(\underline{y},t)}{|\underline{x}-\underline{y}|}\,{\rm d}\underline{y}, \tag{1.16}$$

where $\underline{j}$ is the sheet current density in the film. Using Faraday's law with $\underline{e}$, the component of the electric field tangential to the film, and the change of variable $\underline{q}$ in (1.11), we obtain that

$$\frac{\partial b_3}{\partial t} = -\nabla \times \underline{e} = -\underline{\nabla}\,.\,\underline{q}. \tag{1.17}$$

As $\underline{\nabla}\,.\,\underline{j}=0$ in Ω, which is simply connected, we can introduce a stream (magnetization) function w, which vanishes on $\partial\Omega$, such that $\underline{j}=\underline{\nabla}\times w$ in Ω. Substituting this and (1.17) into the time derivative of (1.16), we obtain that

$$\frac{1}{4\pi}\nabla \times \int_{\Omega} \frac{1}{|\underline{x}-\underline{y}|}\underline{\nabla}\times\frac{\partial w(\underline{y},t)}{\partial t}\,{\rm d}\underline{y} + \underline{\nabla}\,.\,\underline{q}(\underline{x},t) = -\frac{{\rm d}b_{\rm e}(t)}{{\rm d}t}. \tag{1.18}$$

The critical state model relations are given, as before, by (1.12). However, in this problem we limit our considerations to the variational inequality case and assume the Bean model with a field independent sheet critical current density $j_{\rm c}=k\in C(\bar{\Omega})$ and, as in (ii) above, $k(\underline{x})\geqslant k_{\min}>0$ for all $\underline{x}\in\bar{\Omega}$. The model relations can be reformulated as (1.9) with $M(w)$ replaced by k, and this supplemented with (1.18) yields the mixed formulation of this thin film superconductor problem. For the initial data, we take $w(\cdot,0)=w_0(\cdot)$ with $|\underline{\nabla}w_0|\leqslant k$. Similarly, one can show that w satisfies a primal variational inequality problem, see Theorem 3.12. In addition, one can approximate the critical state model relations by the power-law model (1.15), see Barrett & Prigozhin (2013b) for further details and Section 1.3. Similarly to Barrett & Prigozhin (2010), we note that the sign of $\underline{q}$ is changed in Barrett & Prigozhin (2013b) ($\underline{v}$ in the notation there).

1.2 *Notation*

Above, and throughout, we adopt the standard notation for Sobolev spaces on a bounded domain $D\subset\mathbb{R}^d$ with a Lipschitz boundary, denoting the norm of $W^{m,s}(D)$ ($m\in\mathbb{N}$, $s\in[1,\infty]$) by $\|\cdot\|_{m,s,D}$ and the semi-norm by $|\cdot|_{m,s,D}$. Of course, we have that $|\cdot|_{0,s,D}\equiv\|\cdot\|_{0,s,D}$. We extend these norms and semi-norms in the natural way to the corresponding spaces of vector functions. For $s=2$, $W^{m,2}(D)$ will be denoted by $H^m(D)$ with the associated norm and semi-norm written as, respectively, $\|\cdot\|_{m,D}$ and $|\cdot|_{m,D}$. We set $W^{1,s}_0(D):=\{\eta\in W^{1,s}(D):\eta=0 \text{ on } \partial D\}$ and $H^1_0(D)\equiv W^{1,2}_0(D)$. We recall the Poincaré inequality for any $s\in[1,\infty]$

$$|\eta|_{0,s,D}\leqslant C_\star(D)|\underline{\nabla}\eta|_{0,s,D}\quad \forall\eta\in W^{1,s}_0(D), \tag{1.19}$$

where the constant $C_\star(D)$ depends on D, but is independent of s; see e.g., Gilbarg & Trudinger (1983, p. 164). In addition, $|D|$ will denote the measure of D. We also require $H^{1/2}(D)$ for $D\subset\mathbb{R}^2$ and

$$H^{1/2}_{00}(D):=\left\{\eta\in H^{1/2}(D):\tilde{\eta}:=\begin{cases}\eta & \text{in } D\\ 0 & \text{in } \mathbb{R}^2\setminus D\end{cases}\in H^{1/2}(\mathbb{R}^2)\right\}. \tag{1.20}$$

For any Banach space $\mathscr{B}$, we denote its dual by $\mathscr{B}^\star$. Then we recall that

$$\|\eta\|_{[H^{1/2}(D)]^\star} \leqslant [\|\eta\|_{[H^1(D)]^\star}\,\|\eta\|_{L^2(D)}]^{1/2} \quad \forall \eta \in L^2(D). \tag{1.21}$$

For $m \in \mathbb{N}$, let (i) $C^m(\bar{D})$ denote the Banach space of continuous functions with all derivatives up to order m continuous on $\bar{D}$, (ii) $C_0^m(D)$ denote the space of continuous functions with compact support in D with all derivatives up to order m continuous on D, and (iii) $C_0^m(\bar{D})$ denote the Banach space $\{\eta \in C^m(\bar{D}) : \eta = 0 \text{ on } \partial D\}$. In the case $m = 0$, we drop the superscript 0 for all three spaces.

As one can identify $L^1(D)$ as a closed subspace of the Banach space of bounded Radon measures, $\mathscr{M}(\bar{D}) \equiv [C(\bar{D})]^\star$, it is convenient to adopt the notation

$$\int_{\bar{D}} |\mu| \equiv \|\mu\|_{\mathscr{M}(\bar{D})} := \sup_{\substack{\eta\in C(\bar{D})\\ |\eta|_{0,\infty,D}\leqslant 1}} \langle \mu, \eta\rangle_{C(\bar{D})} < \infty, \tag{1.22}$$

where $\langle\cdot,\cdot\rangle_{\mathscr{B}}$ denotes the duality pairing on $\mathscr{B}^\star \times \mathscr{B}$ for any Banach space $\mathscr{B}$. We note that if $\{\mu_n\}_{n\geqslant 0}$ is a bounded sequence in $\mathscr{M}(\bar{D})$, then there exist a subsequence $\{\mu_{n_j}\}_{n_j\geqslant 0}$ and a $\mu \in \mathscr{M}(\bar{D})$ such that as $n_j \to \infty$

$$\mu_{n_j} \to \mu \quad \text{weakly in } \mathscr{M}(\bar{D}); \quad \text{i.e., } \langle \mu_{n_j} - \mu, \eta\rangle_{C(\bar{D})} \to 0 \quad \forall \eta \in C(\bar{D}). \tag{1.23}$$

In addition, we have that

$$\liminf_{n_j\to\infty} \int_{\bar{D}} |\mu_{n_j}| \geqslant \int_{\bar{D}} |\mu|; \tag{1.24}$$

see e.g., Folland (1984, p. 223). For $D \subset \mathbb{R}^2$, we require the following Banach spaces:

$$\underline{V}^s(D) := \{\underline{v} \in [L^s(D)]^2 : \underline{\nabla}\,.\,\underline{v} \in L^2(D)\} \quad \text{for a given } s \in [1,\infty] \tag{1.25a}$$

and

$$\underline{V}^{\mathscr{M}}(D) := \{\underline{v} \in [\mathscr{M}(\bar{D})]^2 : \underline{\nabla}\,.\,\underline{v} \in L^2(D)\}; \tag{1.25b}$$

$$\underline{Z}^s(D) := \{\underline{v} \in [L^s(D)]^2 : \underline{\nabla}\,.\,\underline{v} \in [H_{00}^{1/2}(D)]^\star\} \quad \text{for a given } s \in [1,\infty] \tag{1.25c}$$

and

$$\underline{Z}^{\mathscr{M}}(D) := \{\underline{v} \in [\mathscr{M}(\bar{D})]^2 : \underline{\nabla}\,.\,\underline{v} \in [H_{00}^{1/2}(D)]^\star\}. \tag{1.25d}$$

We recall the Aubin–Lions–Simon compactness theorem, see Simon (1987, Corollary 4). Let $\mathscr{B}_0$, $\mathscr{B}$ and $\mathscr{B}_1$ be Banach spaces, $\mathscr{B}_i$, $i = 0, 1$, reflexive, with a compact embedding $\mathscr{B}_0 \hookrightarrow \mathscr{B}$ and a continuous embedding $\mathscr{B} \hookrightarrow \mathscr{B}_1$. Then, for $\alpha > 1$, the embedding

$$\left\{\eta \in L^\infty(0,T;\mathscr{B}_0) : \frac{\partial \eta}{\partial t} \in L^\alpha(0,T;\mathscr{B}_1)\right\} \hookrightarrow C([0,T];\mathscr{B}) \tag{1.26}$$

is compact. We write $(\cdot,\cdot)$ for the standard inner product on $L^2(\Omega)$. Finally, throughout C denotes a generic positive constant independent of the power parameters, $r \in (1,2)$ and $p \in (2,\infty)$, recall (1.15), the mesh parameter h and the time step parameter τ. Whereas, $C(s)$ denotes a positive constant dependent on the parameter s.

1.3 *Outline*

We introduce

$$c(\underline{\psi}, \underline{v}) := \frac{1}{4\pi} \int_{\Omega} \int_{\Omega} \frac{\underline{\psi}(\underline{x}) \,.\, \underline{v}(\underline{y})}{|\underline{x} - \underline{y}|} \, \mathrm{d}\underline{x} \, \mathrm{d}\underline{y} \tag{1.27a}$$

and

$$a(\phi, \eta) := c(\underline{\nabla}\phi, \underline{\nabla}\eta) = \frac{1}{4\pi} \int_{\Omega} \int_{\Omega} \frac{\underline{\nabla}\phi(\underline{x}) \,.\, \underline{\nabla}\eta(\underline{y})}{|\underline{x} - \underline{y}|} \, \mathrm{d}\underline{x} \, \mathrm{d}\underline{y}. \tag{1.27b}$$

It follows that $c(\cdot\cdot)$ and $a(\cdot,\cdot)$ are symmetric, continuous and coercive bilinear forms on $[[H^{1/2}(\Omega)]^{\star}]^2 \times [[H^{1/2}(\Omega)]^{\star}]^2$ and $H^{1/2}_{00}(\Omega) \times H^{1/2}_{00}(\Omega)$, respectively, see Barrett & Prigozhin (2000, Lemma 2.1). Then we introduce for all $\chi,\ \eta \in W^{1,p}_0(\Omega)$

$$\mathscr{A}(\chi, \eta) := \begin{cases} (\chi, \eta) & \text{in cases (i) and (ii),} \\ a\,(\chi, \eta) & \text{in case (iii);} \end{cases} \tag{1.28}$$

and set $\|\cdot\|_{\mathscr{A}} = [\mathscr{A}(\cdot,\cdot)]^{1/2}$. In addition, we introduce for all $\eta \in L^2(0,T;C(\bar{\Omega}))$

$$\mathfrak{M}(\eta)(\underline{x},t) := \begin{cases} M_{\varepsilon}(\eta(\cdot,t))(\underline{x}) & \text{in case (i),} \\ k(\underline{x})\hat{M}(\eta(\underline{x},t) + b_{\mathrm{e}}(t)) & \text{in case (ii),} \\ k(\underline{x}) & \text{in case (iii),} \end{cases} \quad \text{for a.e. } (\underline{x},t) \in \Omega \times (0,T); \tag{1.29}$$

where $M_{\varepsilon}(\cdot)$ is given by (1.10), $\hat{M}(\cdot)$ satisfies (1.14) and $k \in L^{\infty}(\Omega)$ with $k(\underline{x}) \geqslant k_{\min} > 0$ for a.e. $\underline{x} \in \Omega$. We note that the assumption $\hat{M}_0 \in \mathbb{R}_{>0}$ does allow for any continuous $\hat{M}(\cdot)$ that is strictly positive on any bounded interval of $\mathbb{R}$, but such that $\hat{M}(s) \to 0$ as $|s| \to \infty$. This follows as any solution of the critical state model will be bounded, and hence $\hat{M}(\cdot)$ can be modified to satisfy $\hat{M}_0 \in \mathbb{R}_{>0}$ without changing the problem; see Barrett & Prigozhin (2010) for details.

Furthermore, we set

$$w^0 := \begin{cases} w_0^{\varepsilon} \in C^1_0(\bar{\Omega}) & \text{in case (i),} \\ w_0 \in W^{1,\infty}_0(\Omega) \quad \text{s.t. } |\underline{\nabla} w_0(\cdot)| \leqslant \mathfrak{M}(w_0)(\cdot,0) & \text{in cases (ii) and (iii)} \end{cases} \tag{1.30a}$$

and

$$\mathscr{F} := \begin{cases} \text{non-negative } f \in L^2(0,T;L^2(\Omega)) & \text{in case (i),} \\ -\dfrac{\mathrm{d}b_{\mathrm{e}}}{\mathrm{d}t} \in L^2(0,T) & \text{in cases (ii) and (iii).} \end{cases} \tag{1.30b}$$

It follows from (1.1), (1.18), (1.15), (1.28), (1.27a,1.27b), (1.29) and (1.30a,1.30b) that the formal weak mixed formulation of the power-law approximation of our three quasi-variational inequality problems can be written in a unified way for a given $r \in (1,2)$:

(Q_r) Find $w_r \in H^1(0,T;W_0^{1,p}(\Omega))$ and $\underline{q}_r \in L^r(0,T;[L^r(\Omega)]^2)$ such that for a.a. $t \in (0,T)$

$$\mathscr{A}\left(\frac{\partial w_r}{\partial t},\eta\right) - (\underline{q}_r, \underline{\nabla}\eta) = (\mathscr{F},\eta) \quad \forall \eta \in W_0^{1,p}(\Omega), \tag{1.31a}$$

$$(\mathfrak{M}(w_r)|\underline{q}_r|^{r-2}\underline{q}_r, \underline{v}) + (\underline{\nabla} w_r, \underline{v}) = 0 \quad \forall \underline{v} \in [L^r(\Omega)]^2, \tag{1.31b}$$

where $w_r(\cdot,0) = w^0(\cdot)$.

We will be more precise about the function spaces of this weak formulation with respect to the different problems (i), (ii) and (iii) in Section 3. In Barrett & Prigozhin (2010, 2013a,b), we introduced a finite element approximation of (1.31a,1.31b) based on the lowest order Raviart–Thomas element for $\underline{q}_r$ for cases (i), (ii) and (iii); with piecewise constants for w_r in cases (i) and (ii), and continuous piecewise linears in case (iii). There, integration by parts was performed on the second terms on the left-hand sides of (1.31a,1.31b) as the Raviart–Thomas element is a conforming approximation of the divergence operator. In addition, in case (ii) we exploited (1.14) and based our finite element approximation on the following rewrite of (1.31b)

$$(k|\underline{q}_r|^{r-2}\underline{q}_r, \underline{v}) - (\hat{F}(w_r + b_e) - \hat{F}(b_e), \underline{\nabla}\,.\,\underline{v}) = 0 \quad \forall \underline{v} \in \underline{V}^r(\Omega). \tag{1.32}$$

In Barrett & Prigozhin (2010, 2013a), we proved (subsequence) convergence of these finite element approximations in cases (i) and (ii), to the corresponding weak mixed formulation of the critical state model, (Q), as the mesh parameters tend to zero and $r \to 1$. In Barrett & Prigozhin (2013b) we proved convergence of the finite element approximation in case (iii) to the corresponding weak mixed formulation of the power-law model, (Q_r), as the mesh parameters tend to zero. We note that in case (iii), one can show that the solution of (Q_r) is unique as $\mathfrak{M}$ depends only on $\underline{x}$, recall (1.29). We also proved in Barrett & Prigozhin (2013b) (subsequence) convergence of the solution to (Q_r) to a solution of the corresponding weak mixed formulation of the critical state model, (Q), as $r \to 1$. Finally, we remark that the power-law model, (Q_r), is of interest in its own right in the superconductivity context, cases (ii) and (iii), as it is a popular choice among engineers for a current–voltage relation for some superconducting materials.

In this paper, we consider a simpler finite element approximation of (1.31a,1.31b) based on a nonconforming linear approximation of w_r and a piecewise constant approximation of $\underline{q}_r$. Of course, for linear second-order elliptic problems the nonconforming linear approximation is a computationally inexpensive way of obtaining the lowest order Raviart–Thomas approximation, see Marini (1985); but this does not carry across to nonlinear problems. We note that in Barrett & Prigozhin (2013a) for case (i), in addition to considering the Raviart–Thomas approximation of (Q_r), (1.31a,1.31b), we also considered an approximation based on continuous piecewise linears for w_r and a piecewise constant approximation of $\underline{q}_r$. Once again, we showed (subsequence) convergence of this finite element approximation to the corresponding weak mixed formulation of the sandpile model, (Q), as the mesh parameters tend to zero and $r \to 1$. Although this finite element approximation leads to a good approximation of the surface w in practice, the approximation of the sand flux $\underline{q}$ is poor. We note that all the convergence results stated above for the sand flux (rotated electric field) variable are weak convergence results. Hence, there is no guarantee that this flux approximation will be useful in practice. Nevertheless, the Raviart–Thomas sand flux (rotated electric field) approximations for (i), (ii) and (iii) converged strongly in practice for the numerical experiments in Barrett & Prigozhin (2010, 2013a,b), respectively; see also Barrett & Prigozhin (2012) for case (iii). Similarly, strong convergence is also observed in practice for the sand flux (rotated electric field) approximation resulting from the nonconforming linear approximation of

w_r and constant approximation $\underline{q}_r$ studied in this paper. For case (iii), see also Barrett *et al.* (2013) where thin film problems involving transport currents, which lead to nonhomogenous time-dependent boundary data for w_r and singular time-dependent forcing data $\mathscr{F}$, are solved using this nonconforming approximation.

The outline of this paper is as follows. In the next section we introduce our nonconforming linear finite element approximation, $(Q_r^{h,\tau})$, of the power-law mixed formulation (Q_r), (1.31a,1.31b), and prove well-posedness and stability bounds. Here h and τ are the spatial and temporal discretization parameters, respectively. In Section 3, we first prove (subsequence) convergence of $(Q_r^{h,\tau})$ to (Q_r^{τ}), a discrete time approximation of (Q_r), as $h \to 0$. Then under various assumptions, and appealing to results in Barrett & Prigozhin (2010, 2013a,b) as much as possible, we prove (subsequence) convergence of (Q_r^{τ}) to (Q), as $\tau \to 0$ and $r \to 1$, for case (i); and (subsequence) convergence of (Q_r^{τ}) to (Q_r), as $\tau \to 0$, and then (subsequence) convergence of (Q_r) to (Q), as $r \to 1$, in cases (ii) and (iii). The full sequence converges in case (iii) in the first two convergence results, as in this case one can prove uniqueness of the solution to problems (Q_r^{τ}) and (Q_r). Finally, in Section 4 we state an algorithm for solving the resulting nonlinear algebraic equations arising from the approximation $(Q_r^{h,\tau})$ at each time level, and present some numerical experiments.

2. Finite element approximation

We make the following assumptions on the data.

(A1) $\Omega \subset \mathbb{R}^2$ is simply connected and has a Lipschitz boundary $\partial\Omega$ with outward unit normal $\underline{\nu}$. The conditions stated on the data in (1.30a,1.30b) and (1.29) hold. In addition, in case (i) the initial data $w_0^{\varepsilon} \in C_0^1(\bar{\Omega})$ is such that $\underline{\nabla} w_0^{\varepsilon} \,.\, \underline{\nu} < k_0$.

For ease of exposition, we shall assume that Ω is a polygonal domain to avoid perturbation of domain errors in the finite element approximation. We make the following standard assumption on the partitioning.

(A2) Ω is polygonal. Let $\{\mathscr{T}^h\}_{h>0}$ be a regular family of partitionings of Ω into disjoint open triangles σ with $h_\sigma := \operatorname{diam}(\sigma)$ and $h := \max_{\sigma\in\mathscr{T}^h} h_\sigma$, so that $\bar{\Omega} = \bigcup_{\sigma\in\mathscr{T}^h} \bar{\sigma}$. Moreover, $k|_\sigma$ can be extended to $k \in C(\bar{\sigma})$ for all $\sigma \in \mathscr{T}^h$; that is, k is piecewise continuous and its discontinuities occur only along the internal edges of $\mathscr{T}^h$.

Let $\underline{\nu}_{\partial\sigma}$ be the outward unit normal to $\partial\sigma$, the boundary of σ. We then introduce the following finite element spaces:

$$S^h := \{\eta^h \in L^\infty(\Omega) : \eta^h|_\sigma = a_\sigma \in \mathbb{R}\ \forall \sigma \in \mathscr{T}^h\}, \quad \underline{S}^h := [S^h]^2, \tag{2.1a}$$

$$U^h := \{\eta^h \in C(\bar{\Omega}) : \eta^h|_\sigma = \underline{a}_\sigma \,.\, \underline{x} + b_\sigma,\ \underline{a}_\sigma \in \mathbb{R}^2,\ b_\sigma \in \mathbb{R}\ \forall \sigma \in \mathscr{T}^h\}, \quad U_0^h := U^h \cap W_0^{1,\infty}(\Omega), \tag{2.1b}$$

$$N^h := \{\eta^h \in L^\infty(\Omega) : \eta^h|_\sigma = \underline{a}_\sigma \,.\, \underline{x} + b_\sigma,\ \underline{a}_\sigma \in \mathbb{R}^2,\ b_\sigma \in \mathbb{R} \quad \forall \sigma \in \mathscr{T}^h,$$
$$\text{and } \eta^h \text{ is continuous at the midpoints of the edges of neighbouring triangles}\}, \tag{2.1c}$$

$$N_0^h := \{\eta^h \in N^h : \eta^h = 0 \text{ at the midpoints of the edges on } \partial\Omega\}. \tag{2.1d}$$

Let $\pi_U^h : C(\bar{\Omega}) \to U^h$ denote the U^h interpolation operator such that $\pi_U^h \eta(\underline{x}_j^v) = \eta(\underline{x}_j^v)$, $j = 1, \ldots, J^v$, where $\{\underline{x}_j^v\}_{j=1}^{J^v}$ are the vertices of the partitioning $\mathscr{T}^h$. Let $\pi_N^h : C(\bar{\Omega}) \to N^h$ denote the N^h interpolation operator such that $\pi_N^h \eta(\underline{x}_j^e) = \eta(\underline{x}_j^e)$, $j = 1, \ldots, J^e$, where $\{\underline{x}_j^e\}_{j=1}^{J^e}$ are the midpoints of the edges of the partitioning $\mathscr{T}^h$. We note for $m=0$ and 1 and any $s \in [1,2]$ that

$$|(I-\pi_U^h)\eta|_{m,\sigma} + |(I-\pi_N^h)\eta|_{m,\sigma} \leqslant C h_\sigma^{3-m-2/s} |\eta|_{2,s,\sigma} \quad \forall \sigma \in \mathscr{T}^h, \tag{2.2a}$$

$$\lim_{h\to 0}[\|(I-\pi_U^h)\eta\|_{m,\infty,\Omega} + |(I-\pi_N^h)\eta|_{0,\infty,\Omega} + m|\underline{\nabla}\eta - \underline{\nabla}_h(\pi_N^h\eta)|_{0,\infty,\Omega}] = 0 \quad \forall \eta \in C^m(\bar{\Omega}); \tag{2.2b}$$

where I is the identity operator and

$$\underline{\nabla}_h \eta^h|_\sigma = \underline{\nabla}\eta^h \quad \forall \sigma \in \mathscr{T}^h,\ \forall \eta^h \in N^h. \tag{2.3}$$

Let $\underline{P}^h : [L^1(\Omega)]^2 \to \underline{S}^h$ be such that

$$\underline{P}^h \underline{v}|_\sigma = \fint_\sigma \underline{v} \quad \forall \sigma \in \mathscr{T}^h, \tag{2.4}$$

where $\fint_\sigma \cdot := (1/|D|)\int_D \cdot \,\mathrm{d}\underline{x}$. We note that

$$|\underline{P}^h\underline{v}|_{0,s,\sigma} \leqslant |\underline{v}|_{0,s,\sigma} \quad \forall \underline{v} \in [L^s(\sigma)]^2,\ s \in [1,\infty],\ \forall \sigma \in \mathscr{T}^h, \tag{2.5a}$$

$$\lim_{h\to 0} ||\underline{v}| - |\underline{P}^h\underline{v}||_{0,\infty,\Omega} \leqslant \lim_{h\to 0} |\underline{v} - \underline{P}^h\underline{v}|_{0,\infty,\Omega} = 0 \quad \forall \underline{v} \in [C(\bar{\Omega})]^2. \tag{2.5b}$$

In addition, one can show by mapping to a reference element, applying a trace inequality and the Poincaré inequality (1.19), and then mapping back that for any $s \in [1,\infty]$ and for all $\sigma \in \mathscr{T}^h$

$$|(\underline{I} - \underline{P}^h)\underline{v}|_{0,s,\partial_i\sigma} \leqslant C h_\sigma^{1-1/s} |\underline{v}|_{1,s,\sigma} \quad \forall \underline{v} \in [W^{1,s}(\sigma)]^2, \quad i=1,2,3, \tag{2.6}$$

where $\partial_i\sigma$ is one of the three edges of $\partial\sigma$; that is $\partial\sigma = \sum_{i=1}^3 \partial_i\sigma$. Similarly, we define $P^h : L^1(\Omega) \to S^h$ with the equivalent to (2.5a,2.5b) and (2.6) holding. In addition, we have that for any $s \in [1,\infty]$ and for all $\sigma \in \mathscr{T}^h$

$$\left| \left(I - \fint_{\partial_i\sigma} \right) \eta^h \right|_{0,s,\partial_i\sigma} \leqslant C h_\sigma |\underline{\nabla}\eta^h|_{0,s,\partial\sigma} \leqslant C h_\sigma^{1-1/s} |\underline{\nabla}\eta^h|_{0,s,\sigma} \quad \forall \eta^h \in N^h, \quad i=1,2,3. \tag{2.7}$$

We recall for $r>1$ and for all $\underline{c}$, $\underline{d} \in \mathbb{R}^d$ that

$$\frac{1}{r}\frac{\partial |\underline{c}|^r}{\partial c_i} = |\underline{c}|^{r-2} c_i \ \Rightarrow\ |\underline{c}|^{r-2}\underline{c}\,.\,(\underline{c}-\underline{d}) \geqslant \frac{1}{r}[|\underline{c}|^r - |\underline{d}|^r] \geqslant |\underline{d}|^{r-2}\underline{d}\,.\,(\underline{c}-\underline{d}) \tag{2.8a}$$

and

$$(|\underline{c}|^{r-2}\underline{c} - |\underline{d}|^{r-2}\underline{d})\,.\,\underline{c} \geqslant \frac{r-1}{r}[|\underline{c}|^r - |\underline{d}|^r]. \tag{2.8b}$$

Let $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ be a partitioning of $[0,T]$ into possibly variable time steps $\tau_n := t_n - t_{n-1}$, $n=1,\ldots,N$. We set $\tau := \max_{n=1,\ldots,N} \tau_n$ and, on recalling (1.30b), we introduce

$$\mathscr{F}^n(\cdot) := \frac{1}{\tau_n}\int_{t_{n-1}}^{t_n} \mathscr{F}(\cdot,t)\,\mathrm{d}t \in L^2(\Omega), \quad n=1,\ldots,N. \tag{2.9}$$

We note that

$$\sum_{n=1}^N \tau_n |\mathscr{F}^n|_{0,s,\Omega}^s \leqslant \int_0^T |\mathscr{F}|_{0,s,\Omega}^s \,\mathrm{d}t \quad \text{for any } s \in [1,2]. \tag{2.10}$$

On setting

$$w_0^{\varepsilon,h} = P^h[\pi_N^h w_0^\varepsilon], \tag{2.11}$$

we introduce $M_\varepsilon^h : S^h \to S^h$ approximating $M_\varepsilon : C(\bar{\Omega}) \to C(\bar{\Omega})$, defined by (1.10), for any $\sigma \in \mathscr{T}^h$ as

$$M_\varepsilon^h(\eta^h)|_\sigma := \begin{cases} k_0, & \eta_\sigma^h \geqslant w_{0,\sigma}^{\varepsilon,h} + \varepsilon, \\ k_{1,\sigma}^{\varepsilon,h} + (k_0 - k_{1,\sigma}^{\varepsilon,h}) \left(\dfrac{\eta_\sigma^h - w_{0,\sigma}^{\varepsilon,h}}{\varepsilon} \right), & \eta_\sigma^h \in [w_{0,\sigma}^{\varepsilon,h}, w_{0,\sigma}^{\varepsilon,h} + \varepsilon], \\ k_{1,\sigma}^{\varepsilon,h} := \max(k_0, |\underline{\nabla}_h(\pi_N^h w_0^\varepsilon)\,|_\sigma\,|), & \eta_\sigma^h \leqslant w_{0,\sigma}^{\varepsilon,h}, \end{cases} \tag{2.12}$$

where $\eta_\sigma^h = \eta^h|_\sigma$ and $w_{0,\sigma}^{\varepsilon,h} = w_0^{\varepsilon,h}|_\sigma$ for all $\sigma \in \mathscr{T}^h$.

We note that M_ε is also well defined on S^h with $M_\varepsilon : S^h \to L^\infty(\Omega)$, and we have the following result.

LEMMA 2.1 For any $\eta^h \in S^h$, we have that

$$|M_\varepsilon(\eta^h) - M_\varepsilon^h(\eta^h)|_{0,\infty,\Omega} \leqslant C(\varepsilon^{-1})[|(I - P^h)w_0^\varepsilon|_{0,\infty,\Omega} + |\underline{\nabla} w_0^\varepsilon - \underline{\nabla}_h(\pi_N^h w_0^\varepsilon)|_{0,\infty,\Omega}]. \tag{2.13}$$

Proof. See the proof of Lemma 2.1 in Barrett & Prigozhin (2013a). □

Finally, it follows from (2.12), (2.2b) and Assumption (A1) that

$$k_{1,\infty}^{\varepsilon,h} := \max_{\sigma \in \mathscr{T}^h} k_{1,\sigma}^{\varepsilon,h} \leqslant C. \tag{2.14}$$

2.1 *Approximation* $(Q_r^{h,\tau})$

On recalling (1.28) and (1.27a,1.27b), we introduce for all $\chi^h,\ \eta^h \in N_0^h$

$$\mathscr{A}^h(\chi^h, \eta^h) := \begin{cases} (\chi^h, \eta^h) & \text{in cases (i) and (ii),} \\ c(\underline{\nabla}_h \chi^h, \underline{\nabla}_h \eta^h) & \text{in case (iii);} \end{cases} \tag{2.15}$$

and set $\|\cdot\|_{\mathscr{A}^h} = [\mathscr{A}^h(\cdot,\cdot)]^{1/2}$. In addition, on recalling (1.29), we introduce for all $\eta^h \in S^h$

$$\mathfrak{M}^{h,n}(\eta^h)|_\sigma := \begin{cases} M_\varepsilon^h(\eta^h)|_\sigma & \text{in case (i),} \\ k(\underline{x}_\sigma)\hat{M}(\eta^h|_\sigma + b_e(t_n)) & \text{in case (ii),} \quad \forall \sigma \in \mathscr{T}^h, \\ k(\underline{x}_\sigma) & \text{in case (iii)} \end{cases} \tag{2.16}$$

where $\underline{x}_\sigma$ is the centroid of σ. We note from (2.16), (2.12), (2.14) and Assumption (A1) that $\mathfrak{M}^{h,n} : S^h \to S^h$ and there exist $\mathfrak{M}_{\min},\ \mathfrak{M}_{\max} \in \mathbb{R}$ such that for $n = 1, \ldots, N$

$$0 < \mathfrak{M}_{\min} \leqslant \mathfrak{M}^{h,n}(\eta^h)|_\sigma \leqslant \mathfrak{M}_{\max} \quad \forall \eta^h \in S^h, \ \ \forall \sigma \in \mathscr{T}^h. \tag{2.17}$$

We now define our finite element approximation of (Q_r), (1.31a,1.31b), for a given $r > 1$:

$(Q_r^{h,\tau})$ For $n = 1, \ldots, N$, find $W_r^n \in N_0^h$ and $\underline{Q}_r^n \in \underline{S}^h$ such that

$$\mathscr{A}^h\left(\frac{W_r^n - W_r^{n-1}}{\tau_n}, \eta^h\right) - (\underline{Q}_r^n, \underline{\nabla}_h \eta^h) = (\mathscr{F}^n, \eta^h) \quad \forall \eta^h \in N_0^h, \tag{2.18a}$$

$$(\mathfrak{M}^{h,n}(P^h W_r^n)|\underline{Q}_r^n|^{r-2}\underline{Q}_r^n, \underline{v}^h) + (\underline{\nabla}_h W_r^n, \underline{v}^h) = 0 \quad \forall \underline{v}^h \in \underline{S}^h; \tag{2.18b}$$

where $W_r^0 = \pi_N^h w^0$.

Associated with $(Q_r^{h,\tau})$ is the corresponding approximation of a generalized p-Laplacian problem for $p > 1$, where we recall that $1/r + 1/p = 1$:

$(P_p^{h,\tau})$ For $n = 1, \ldots, N$, find $W_r^n \in N_0^h$ such that

$$\mathscr{A}^h\left(\frac{W_r^n - W_r^{n-1}}{\tau_n}, \eta^h\right) + ([\mathfrak{M}^{h,n}(P^h W_r^n)]^{-(p-1)}|\underline{\nabla}_h W_r^n|^{p-2}\underline{\nabla}_h W_r^n, \underline{\nabla}_h \eta^h)$$
$$= (\mathscr{F}^n, \eta^h) \quad \forall \eta^h \in N_0^h, \tag{2.19}$$

where $W_r^0 = \pi_N^h w^0$.

THEOREM 2.2 Let the Assumptions (A1) and (A2) hold. Then for all $r \in (1,2)$, for all regular partitionings $\mathscr{T}^h$ of Ω and for all $\tau_n > 0$, there exists a solution, $W_r^n \in N_0^h$ and $\underline{Q}_r^n \in \underline{S}^h$ to the nth step of $(Q_r^{h,\tau})$, (2.18a,2.18b). This solution is unique in case (iii). In addition, we have that

$$\max_{n=0,\ldots,N} \|W_r^n\|_{\mathscr{A}^h} + \sum_{n=1}^{N} \|W_r^n - W_r^{n-1}\|_{\mathscr{A}^h}^2 + \sum_{n=1}^{N} \tau_n\, |\underline{Q}_r^n|_{0,r,\Omega}^r + \left(\sum_{n=1}^{N} \tau_n |\underline{\nabla}_h W_r^n|_{0,p,\Omega}^p\right)^{1/p} \leqslant C, \tag{2.20}$$

where $1/r + 1/p = 1$. Moreover, $(Q_r^{h,\tau})$, (2.18a,2.18b), is equivalent to $(P_p^{h,\tau})$, (2.19).

Proof. The proof is similar to the proof of Theorem 2.2 in Barrett & Prigozhin (2013a) with U_0^h replaced by N_0^h. It follows immediately from (2.18b) that for all $\sigma \in \mathscr{T}^h$

$$\underline{\nabla} W_r^n = -\mathfrak{M}^{h,n}(P^h W_r^n)|\underline{Q}_r^n|^{r-2}\underline{Q}_r^n \;\Leftrightarrow\; \underline{Q}_r^n = -[\mathfrak{M}^{h,n}(P^h W_r^n)]^{-(p-1)}|\underline{\nabla} W_r^n|^{p-2}\underline{\nabla} W_r^n \quad \text{on } \sigma. \tag{2.21}$$

Substituting this expression for $\underline{Q}_r^n$ into (2.18a) yields (2.19). Hence $(P_p^{h,\tau})$, with (2.21), is equivalent to $(Q_r^{h,\tau})$.

Consider the strictly convex minimization problem:

$$\min_{\eta^h \in N_0^h} E_p^{h,n}(\eta^h), \tag{2.22a}$$

where, for a given $\varphi^h \in N_0^h$, $E_p^{h,n} : N_0^h \to \mathbb{R}$ is defined by

$$E_p^{h,n}(\eta^h) := \frac{1}{2\tau_n}\|\eta^h - W_r^{n-1}\|_{\mathscr{A}^h}^2 + \frac{1}{p}\int_\Omega [\mathfrak{M}^{h,n}(P^h \varphi^h)]^{-(p-1)}|\underline{\nabla}_h \eta^h|^p \,\mathrm{d}\underline{x} - (\mathscr{F}^n, \eta^h). \tag{2.22b}$$

In case (iii), as, on recalling (2.16), $\mathfrak{M}^{h,n}(\cdot)$ only depends on $\underline{x}$, (2.19) is the Euler–Lagrange system associated with the strictly convex minimization problem (2.22a,2.22b). Hence, in case (iii) there exists a unique solution to $(\mathrm{P}_p^{h,\tau})$, (2.19), and therefore to $(Q_r^{h,\tau})$, (2.18a,2.18b).

We now apply the Brouwer fixed point theorem to prove existence of a solution to $(\mathrm{P}_p^{h,\tau})$, and therefore to $(Q_r^{h,\tau})$ in cases (i) and (ii). Let $F^h : N_0^h \to N_0^h$ be such that for any $\varphi^h \in N_0^h$, $F^h\varphi^h \in N_0^h$ solves

$$\begin{aligned}&\mathscr{A}^h\left(\frac{F^h\varphi^h - W_r^{n-1}}{\tau_n}, \eta^h\right) + ([\mathfrak{M}^{h,n}(P^h\varphi^h)]^{-(p-1)}|\underline{\nabla}_h F^h\varphi^h|^{p-2}\underline{\nabla}_h F^h\varphi^h, \underline{\nabla}_h\eta^h)\\ &= (\mathscr{F}^n, \eta^h) \quad \forall \eta^h \in N_0^h. \end{aligned} \tag{2.23}$$

The well-posedness of the mapping F^h follows from noting that (2.23) is the Euler–Lagrange system associated with the strictly convex minimization problem (2.22a,2.22b), that is, there exists a unique element $F^h\varphi^h \in N_0^h$ solving (2.23). It follows immediately from (2.22a,2.22b), as $\|\cdot\|_{\mathscr{A}^h} \equiv |\cdot|_{0,\Omega}$ in cases (i) and (ii), that

$$\frac{1}{2\tau_n}|F^h\varphi^h - W_r^{n-1}|_{0,\Omega}^2 - (\mathscr{F}^n, F^h\varphi^h) \leqslant E_p^{h,n}(F^h\varphi^h) \leqslant E_p^{h,n}(0) = \frac{1}{2\tau_n}|W_r^{n-1}|_{0,\Omega}^2. \tag{2.24}$$

It is easily deduced from (2.24) that

$$F^h\varphi^h \in B_\gamma := \{\eta^h \in N_0^h : |\eta^h|_{0,\Omega} \leqslant \gamma\}, \tag{2.25}$$

where $\gamma \in \mathbb{R}_{>0}$ depends on $|W_r^{n-1}|_{0,\Omega}$, $|\mathscr{F}^n|_{0,\Omega}$ and τ_n. Hence $F^h : B_\gamma \to B_\gamma$. In addition, it is easily verified that the mapping F^h is continuous, as $\mathfrak{M}^{h,n} : S^h \to S^h$ is continuous with respect to S^h on recalling (2.16), (2.12) and (1.14). Therefore, the Brouwer fixed point theorem yields that the mapping F^h has at least one fixed point in B_γ. Hence, there exists a solution to $(\mathrm{P}_p^{h,\tau})$, (2.19), and therefore to $(Q_r^{h,\tau})$, (2.18a,2.18b), in cases (i) and (ii).

It follows from (2.21) and (2.17) that for $n = 1, \ldots, N$

$$|\underline{\nabla}_h W_r^n|_{0,p,\Omega}^p = |[\mathfrak{M}^{h,n}(P^h W_r^n)]^{p-1}\underline{Q}_r^n|_{0,r,\Omega}^r \leqslant (\mathfrak{M}_{\max})^{p-1}(\mathfrak{M}^{h,n}(P^h W_r^n), |\underline{Q}_r^n|^r). \tag{2.26}$$

Choosing $\eta^h = W_r^n$, $\underline{v}^h = \underline{Q}_r^n$ in (2.18a,2.18b), combining and noting the simple identity

$$(c-d)c = \tfrac{1}{2}[c^2 + (c-d)^2 - d^2] \quad \forall c,\, d \in \mathbb{R}, \tag{2.27}$$

we obtain for $n = 1, \ldots, N$, on applying Young's inequality and (1.19), that for all $\delta > 0$

$$\begin{aligned}&\|W_r^n\|_{\mathscr{A}^h}^2 + \|W_r^n - W_r^{n-1}\|_{\mathscr{A}^h}^2 + 2\tau_n(\mathfrak{M}^{h,n}(P^h W_r^n), |\underline{Q}_r^n|^r)\\ &= \|W_r^{n-1}\|_{\mathscr{A}^h}^2 + 2\tau_n(\mathscr{F}^n, W_r^n)\\ &\leqslant \|W_r^{n-1}\|_{\mathscr{A}^h}^2 + 2\tau_n\left[\frac{1}{r}\delta^{-r}|\mathscr{F}^n|_{0,r,\Omega}^r + \frac{1}{p}\delta^p|W_r^n|_{0,p,\Omega}^p\right]\\ &\leqslant \|W_r^{n-1}\|_{\mathscr{A}^h}^2 + 2\tau_n\left[\frac{1}{r}\delta^{-r}|\mathscr{F}^n|_{0,r,\Omega}^r + \frac{1}{p}[\delta\, C_\star(\Omega)]^p|\underline{\nabla} W_r^n|_{0,p,\Omega}^p\right]. \end{aligned} \tag{2.28}$$

It follows on summing (2.28) from $n=1$ to m, with $\delta = 1/(C_\star(\Omega)[\mathfrak{M}_{\max}]^{1/r})$, and noting (2.26) and (2.17) that for $m=1,\ldots,N$

$$\|W_r^m\|^2_{\mathscr{A}^h} + \sum_{n=1}^{m} \|W_r^n - W_r^{n-1}\|^2_{\mathscr{A}^h} + \sum_{n=1}^{m} \tau_n (\mathfrak{M}^{h,n}(P^h W_r^n), |\underline{Q}_r^n|^r)$$
$$\leqslant \|W_r^0\|^2_{\mathscr{A}^h} + 2\,[C_\star(\Omega)]^r \mathfrak{M}_{\max} \sum_{n=1}^{m} \tau_n |\mathscr{F}^n|^r_{0,r,\Omega}. \tag{2.29}$$

The desired result (2.20) follows immediately from (2.29), (2.10), (2.17) and (2.26). □

We end this section with the following discrete Poincaré and compactness results for N_0^h, which are extensions of Proposition 4.13 in Chapter 1 and Theorem 2.4 in Chapter 2 of Temam (1984). In addition, we are more precise about the domain Ω and the subsequent elliptic regularity.

LEMMA 2.3 Let $s \in (1,\infty)$ and Assumption (A2) hold. Then we have that

$$|(\eta^h, \underline{\nabla}\,.\,\underline{v}) + (\underline{\nabla}_h \eta^h, \underline{v})| \leqslant Ch|\underline{\nabla}_h \eta^h|_{0,s,\Omega} |\underline{v}|_{1,s',\Omega} \quad \forall \eta^h \in N_0^h, \ \ \forall \underline{v} \in [W^{1,s'}(\Omega)]^2; \tag{2.30}$$

where, here and throughout the paper, $1/s + 1/s' = 1$. Hence, it follows that

$$|\eta^h|_{0,s,\Omega} \leqslant C|\underline{\nabla}_h \eta^h|_{0,s,\Omega} \quad \forall \eta^h \in N_0^h. \tag{2.31}$$

Proof. First on splitting $\partial\sigma$ into its three edges, i.e., $\partial\sigma = \sum_{i=1}^{3} \partial_i\sigma$, it follows from (2.7) and (2.6) that for all $\eta^h \in N_0^h$ and for all $\underline{v} \in [W_0^{1,s'}(\Omega)]^2$

$$\begin{aligned}
|(\eta^h, \underline{\nabla}\cdot\underline{v}) + (\underline{\nabla}_h \eta^h, \underline{v})| &= \left| \sum_{\sigma\in\mathscr{T}^h} \sum_{i=1}^{3} \int_{\partial_i\sigma} \eta^h \underline{v}\,.\,\underline{\nu}_{\partial_i\sigma}\,\mathrm{d}s \right| \\
&= \left| \sum_{\sigma\in\mathscr{T}^h} \sum_{i=1}^{3} \int_{\partial_i\sigma} \left[\left(I - \fint_{\partial_i\sigma} \right) \eta^h \right] \underline{v}\,.\,\underline{\nu}_{\partial_i\sigma}\,\mathrm{d}s \right| \\
&= \left| \sum_{\sigma\in\mathscr{T}^h} \sum_{i=1}^{3} \int_{\partial_i\sigma} \left[\left(I - \fint_{\partial_i\sigma} \right) \eta^h \right] [(\underline{I} - \underline{P}^h)\underline{v}]\,.\,\underline{\nu}_{\partial_i\sigma}\,\mathrm{d}s \right| \\
&\leqslant \sum_{\sigma\in\mathscr{T}^h} \sum_{i=1}^{3} \left| \left(I - \fint_{\partial_i\sigma} \right) \eta^h \right|_{0,s,\partial_i\sigma} |(\underline{I} - \underline{P}^h)\underline{v}|_{0,s',\partial_i\sigma} \\
&\leqslant C \sum_{\sigma\in\mathscr{T}^h} h_\sigma |\underline{\nabla}\eta^h|_{0,s,\sigma} |\underline{v}|_{1,s',\sigma} \leqslant Ch|\underline{\nabla}_h \eta^h|_{0,s,\Omega} |\underline{v}|_{1,s',\Omega};
\end{aligned} \tag{2.32}$$

and hence the desired result (2.30).

It immediately follows from (2.30) that

$$|(\eta^h, \underline{\nabla}\cdot\underline{v})| \leqslant C|\underline{\nabla}_h \eta^h|_{0,s,\Omega} \|\underline{v}\|_{1,s',\Omega} \quad \forall \eta^h \in N_0^h, \ \ \forall \underline{v} \in [W_0^{1,s'}(\Omega)]^2. \tag{2.33}$$

Given any $\theta \in L^{s'}(\Omega)$, then there exists a $\underline{v} \in [W^{1,s'}(\Omega)]^2$ such that

$$\underline{\nabla} \cdot \underline{v} = \theta \quad \text{a.e. in } \Omega, \quad \|\underline{v}\|_{1,s',\Omega} \leqslant C\, |\theta|_{0,s',\Omega}. \tag{2.34}$$

The result (2.34) is easily achieved by choosing $\underline{v} = -\underline{\nabla} z$, where $-\Delta z = \theta'$ a.e. in $\Omega' \supset \Omega$ and $z = 0$ on $\partial\Omega'$, where θ' is the extension of θ from Ω to Ω' by zero and $\partial\Omega' \in C^\infty$. Combining (2.33) and (2.34) yields the desired result (2.31). □

LEMMA 2.4 Given $\{\eta^h\}_{h>0}$, with $\eta^h \in N_0^h$, such that for an $s \in (4, \infty)$

$$|\underline{\nabla}_h \eta^h|_{0,s,\Omega} \leqslant C; \tag{2.35}$$

then there exists a subsequence of $\{\eta^h\}_{h>0}$ (not indicated), and an $\eta \in W_0^{1,s}(\Omega)$ such that as $h \to 0$

$$\underline{\nabla}_h \eta^h \to \underline{\nabla} \eta \quad \text{weakly in } [L^s(\Omega)]^2, \tag{2.36a}$$

$$\eta^h \to \eta \quad \text{strongly in } L^s(\Omega), \tag{2.36b}$$

$$\underline{\nabla}_h \eta^h \to \underline{\nabla} \eta \quad \text{strongly in } [[H^{1/2}(\Omega)]^\star]^2. \tag{2.36c}$$

Proof. It follows immediately from (2.35) and (2.31) that there exist an $\eta \in L^s(\Omega)$ and a $\underline{d} \in [L^s(\Omega)]^2$, and a subsequence of $\{\eta^h\}_{h>0}$ (not indicated) such that as $h \to 0$

$$\eta^h \to \eta \quad \text{weakly in } L^s(\Omega), \quad \underline{\nabla}_h \eta^h \to \underline{d} \quad \text{weakly in } [L^s(\Omega)]^2. \tag{2.37}$$

Passing to the limit $h \to 0$ in (2.30) for the subsequence we deduce that

$$\eta \in W_0^{1,s}(\Omega) \quad \text{and} \quad \underline{d} = \underline{\nabla} \eta. \tag{2.38}$$

Hence, the desired result (2.36a) follows from combining (2.37) and (2.38).

We now introduce $\hat{\eta}^h \in U_0^h$ such that

$$(\underline{\nabla} \hat{\eta}^h, \underline{\nabla} \chi^h) = (\underline{\nabla}_h \eta^h, \underline{\nabla} \chi^h) \quad \forall \chi^h \in U_0^h. \tag{2.39}$$

It follows from (1.19), (2.39) and (2.35) that

$$\|\hat{\eta}^h\|_{1,\Omega} \leqslant C |\underline{\nabla} \hat{\eta}^h|_{0,\Omega} \leqslant C |\underline{\nabla}_h \eta^h|_{0,\Omega} \leqslant C. \tag{2.40}$$

We deduce from (2.40) that there exists a further subsequence of $\{\eta^h\}_{h>0}$ (not indicated) such that as $h \to 0$

$$\underline{\nabla} \hat{\eta}^h \to \underline{\nabla} \hat{\eta} \quad \text{weakly in } [L^2(\Omega)]^2, \tag{2.41a}$$

$$\hat{\eta}^h \to \hat{\eta} \quad \text{strongly in } L^\kappa(\Omega) \quad \forall \kappa \in [1, \infty); \tag{2.41b}$$

where $\hat{\eta} \in H_0^1(\Omega)$.

As Ω is polygonal, it follows from Grisvard (1985, Chapter 4) that given $\theta \in L^{s'}(\Omega)$ for some $s' \in (1, \frac{4}{3})$, then there exists a unique $z \in W^{2,s'}(\Omega)$ such that

$$-\Delta z = \theta \quad \text{a.e. in } \Omega, \quad z = 0 \quad \text{on } \partial\Omega \quad \text{and} \quad \|z\|_{2,s',\Omega} \leqslant C|\theta|_{0,s',\Omega}. \tag{2.42}$$

It follows from (2.42) and (2.39) that

$$(\hat{\eta}^h - \eta^h, \theta) = [(\underline{\nabla}\hat{\eta}^h - \underline{\nabla}_h \eta^h, \underline{\nabla}[(I - \pi_U^h)z])] + [(\underline{\nabla}_h \eta^h, \underline{\nabla} z) + (\eta^h, \Delta z)] =: T_1 + T_2. \tag{2.43}$$

We deduce from (2.40), (2.2a) and (2.42) that

$$|T_1| \leqslant C|(I - \pi_U^h)z|_{1,\Omega} \leqslant Ch^{2(1-1/s')}\, |z|_{2,s',\Omega} \leqslant Ch^{2/s}|\theta|_{0,s',\Omega}, \tag{2.44a}$$

and from (2.30) with $\underline{v} = -\underline{\nabla} z$, (2.35) and (2.42) that

$$|T_2| \leqslant Ch|\underline{\nabla}_h \eta^h|_{0,s,\Omega}\, \|z\|_{2,s',\Omega} \leqslant Ch|\theta|_{0,s',\Omega}. \tag{2.44b}$$

Hence, combining (2.43) and (2.44a,2.44b) yields that

$$|(\hat{\eta}^h - \eta^h, \theta)| \leqslant Ch^{2/s}|\theta|_{0,s',\Omega} \qquad \forall \theta \in L^{s'}(\Omega). \tag{2.45}$$

It follows immediately from (2.45) that

$$|\hat{\eta}^h - \eta^h|_{0,s,\Omega} \to 0 \quad \text{as } h \to 0. \tag{2.46}$$

The desired result (2.36b) then follows from (2.41b) and (2.46), as (2.37) implies that $\hat{\eta} = \eta$.

Finally, we need to prove (2.36c). First, we note that

$$(\underline{\nabla}\eta - \underline{\nabla}_h \eta^h, \underline{v}) = -(\eta - \eta^h, \underline{\nabla} \,.\, \underline{v}) - [(\eta^h, \underline{\nabla} \,.\, \underline{v}) + (\underline{\nabla}_h \eta^h, \underline{v})] \quad \forall \underline{v} \in [H^1(\Omega)]^2. \tag{2.47}$$

Hence, it follows from (2.47) and (2.30) that

$$|(\underline{\nabla}\eta - \underline{\nabla}_h \eta^h, \underline{v})| \leqslant C[|\eta - \eta^h|_{0,\Omega} + h\,|\underline{\nabla}_h \eta^h|_{0,\Omega}]|\underline{v}|_{1,\Omega} \quad \forall \underline{v} \in [H^1(\Omega)]^2. \tag{2.48}$$

Therefore, (2.48), (2.36b) and (2.35) yield for the subsequence that

$$\underline{\nabla}_h \eta^h \to \underline{\nabla}\eta \quad \text{strongly in } [[H^1(\Omega)]^\star]^2 \text{ as } h \to 0. \tag{2.49}$$

The desired result (2.36c) follows immediately from (1.21), (2.49) and (2.35). □

3. Convergence

3.1 *Convergence of* $(Q_r^{h,\tau})$ *to* (Q_r^τ)

Similarly to (2.16), we introduce for all $\eta \in C(\bar{\Omega})$

$$\mathfrak{M}^n(\eta)(\underline{x}) := \begin{cases} M_\varepsilon(\eta)(\underline{x}) & \text{in case (i)} \\ k(\underline{x})\hat{M}(\eta(\underline{x}) + b_e(t_n)) & \text{in case (ii)} \\ k(\underline{x}) & \text{in case (iii)} \end{cases} \quad \text{for a.e. } \underline{x} \in \Omega. \tag{3.1}$$

We note from (3.1), (1.10) and Assumption (A1) that there exist $\mathfrak{M}_{\min}$, $\mathfrak{M}_{\max} \in \mathbb{R}$ such that for $n = 1, \ldots, N$

$$0 < \mathfrak{M}_{\min} \leqslant \mathfrak{M}^n(\eta)(\underline{x}) \leqslant \mathfrak{M}_{\max} \quad \forall \eta \in C(\bar{\Omega}) \quad \text{for a.e. } \underline{x} \in \Omega. \tag{3.2}$$

For the purposes of the convergence analysis in this subsection, we introduce for a given $r > 1$: (Q_r^τ) for $n = 1, \ldots, N$, find $w_r^n \in W_0^{1,p}(\Omega)$ and $\underline{q}_r^n \in [L^r(\Omega)]^2$ such that

$$\mathscr{A}\left(\frac{w_r^n - w_r^{n-1}}{\tau_n}, \eta\right) - (\underline{q}_r^n, \underline{\nabla}\eta) = (\mathscr{F}^n, \eta) \quad \forall \eta \in W_0^{1,p}(\Omega), \tag{3.3a}$$

$$(\mathfrak{M}^n(w_r^n)|\underline{q}_r^n|^{r-2}\underline{q}_r^n, \underline{v}) + (\underline{\nabla} w_r^n, \underline{v}) = 0 \quad \forall \underline{v} \in [L^r(\Omega)]^2; \tag{3.3b}$$

where $w_r^0 = w^0$.

THEOREM 3.1 Let Assumptions (A1) and (A2) hold. For any fixed $r \in (1, \frac{4}{3})$ and fixed time partition $\{\tau_n\}_{n=1}^N$, and for all regular partitionings $\mathscr{T}^h$ of Ω, there exists a subsequence of $\{\{W_r^n, \underline{Q}_r^n\}_{n=1}^N\}_{h>0}$ (not indicated), where $\{W_r^n, \underline{Q}_r^n\}_{n=1}^N$ solves $(Q_r^{h,\tau})$, (2.18a,2.18b), such that as $h \to 0$, for any $s \in [1, \infty)$,

$$W_r^n,\ P^h W_r^n \to w_r^n \quad \text{strongly in } L^p(\Omega), \quad n = 1, \ldots, N, \tag{3.4a}$$

$$\mathfrak{M}^{h,n}(P^h W_r^n) \to \mathfrak{M}^n(w_r^n) \quad \text{strongly in } L^s(\Omega), \quad n = 1, \ldots, N, \tag{3.4b}$$

$$\underline{\nabla}_h W_r^n \to \underline{\nabla} w_r^n \quad \text{weakly in } [L^p(\Omega)]^2, \quad n = 1, \ldots, N, \tag{3.4c}$$

$$\underline{\nabla}_h W_r^n \to \underline{\nabla} w_r^n \quad \text{strongly in } [[H^{1/2}(\Omega)]^\star]^2, \quad n = 1, \ldots, N, \tag{3.4d}$$

$$\underline{Q}_r^n \to \underline{q}_r^n \quad \text{weakly in } [L^r(\Omega)]^2, \quad n = 1, \ldots, N; \tag{3.4e}$$

where $\{w_r^n, \underline{q}_r^n\}_{n=1}^N$ is a solution of (Q_r^τ), (3.3a,3.3b).

In addition, we have that

$$\max_{n=0,\ldots,N} \|w_r^n\|_{\mathscr{A}} + \sum_{n=1}^N \|w_r^n - w_r^{n-1}\|_{\mathscr{A}}^2 + \sum_{n=1}^N \tau_n |\underline{q}_r^n|_{0,r,\Omega}^r + \left(\sum_{n=1}^N \tau_n |\underline{\nabla} w_r^n|_{0,p,\Omega}^p\right)^{1/p} \leqslant C. \tag{3.5}$$

Moreover, in case (iii) the solution of (Q_r^τ) is unique, and so the whole sequence converges in (3.4a–3.4e).

Proof. The desired subsequence weak convergence result (3.4e) follows immediately from the bound on $\{\underline{Q}_r^n\}_{n=1}^N$ in (2.20), on noting that the time partition $\{\tau_n\}_{n=1}^N$ is fixed. It follows from (2.20) that

$$|\underline{\nabla}_h W_r^n|_{0,p,\Omega} \leqslant C(\tau_n^{-1}), \quad n = 1, \ldots, N. \tag{3.6}$$

The desired results (3.4a,3.4c,3.4d) then follow immediately from (3.6), Lemma 2.4 and (2.5a,2.5b) on extracting a further subsequence (not indicated). On noting that $\mathfrak{M}^n(\cdot)$ is well-defined on S^h and is continuous with respect to its argument, it follows from (3.4a) for a further subsequence of $\{\{W_r^n\}_{n=1}^N\}_{h>0}$

(not indicated) that as $h \to 0$, for $n = 1, \dots, N$,

$$P^h W_r^n \to w_r^n \quad \text{a.e. in } \Omega \;\Rightarrow\; \mathfrak{M}^n(P^h W_r^n) \to \mathfrak{M}^n(w_r^n) \quad \text{a.e. in } \Omega. \tag{3.7}$$

It follows from (3.7), (3.2) and Lebesgue's general convergence theorem that as $h \to 0$ for any $s \in [1, \infty)$

$$\mathfrak{M}^n(P^h W_r^n) \to \mathfrak{M}^n(w_r^n) \quad \text{strongly in } L^s(\Omega), \quad n = 1, \dots, N. \tag{3.8}$$

Combining (3.1), (2.16), (2.13), (2.5b), (2.2b) and (3.8) yields the desired result (3.4b).

We now need to establish that $\{w_r^n, \underline{q}_r^n\}_{n=1}^N$ solve (Q_r^τ), (3.3a,3.3b). For any $\eta \in C_0^\infty(\Omega)$, we choose $\eta^h = \pi_N^h \eta$ in (2.18a) and now pass to the limit $h \to 0$ for the subsequence, on noting (2.15), (1.28), (1.27a,1.27b), (3.4a,3.4d,3.4e) and (2.2b), to obtain (3.3a) for all $\eta \in C_0^\infty(\Omega)$. Noting that $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, (1.28), (1.27a,1.27b) and that $w_r^n \in W_0^{1,p}(\Omega)$, $\underline{q}_r^n \in [L^r(\Omega)]^d$ and $\mathscr{F}^n \in L^2(\Omega)$, $n = 1, \dots, N$, yields the desired result (3.3a).

For any $\underline{v} \in [C^\infty(\bar{\Omega})]^2$, we choose $\underline{v}^h = \underline{Q}_r^n - \underline{P}^h \underline{v}$ in (2.18b), and then try to pass to the limit for the subsequence as $h \to 0$. First, we note from (2.18a) with $\eta^h = W_r^n$ and (2.8a) that for $n = 1, \dots, N$

$$\begin{aligned}(\underline{\nabla}_h W_r^n, \underline{P}^h \underline{v}) &= (\underline{\nabla}_h W_r^n, \underline{Q}_r^n) + (\mathfrak{M}^{h,n}(P^h W_r^n)|\underline{Q}_r^n|^{r-2}\underline{Q}_r^n, \underline{Q}_r^n - \underline{P}^h \underline{v}) \\ &\geqslant \mathscr{A}^h\left(\frac{W_r^n - W_r^{n-1}}{\tau_n}, W_r^n\right) - (\mathscr{F}^n, W_r^n) + (\mathfrak{M}^{h,n}(P^h W_r^n), |\underline{P}^h \underline{v}|^{r-2}\underline{P}^h \underline{v}, \underline{Q}_r^n - \underline{P}^h \underline{v}).\end{aligned} \tag{3.9}$$

Passing to the limit $h \to 0$ for the subsequence in (3.9) yields, on noting (3.4a–3.4e), (2.5b), (2.15), (1.28) and (1.27a,1.27b), for $n = 1, \dots, N$ that

$$(\underline{\nabla} w_r^n, \underline{v}) \geqslant \mathscr{A}\left(\frac{w^n - w_r^{n-1}}{\tau_n}, w_r^n\right) - (\mathscr{F}^n, w_r^n) + (\mathfrak{M}^n(w_r^n), |\underline{v}|^{r-2}\underline{v}, \underline{q}_r^n - \underline{v}). \tag{3.10}$$

It follows from (3.10) and (3.3a) with $\eta = w_r^n$ that for $n = 1, \dots, N$

$$(\underline{\nabla} w_r^n, \underline{v} - \underline{q}_r^n) \geqslant (\mathfrak{M}^n(w_r^n)\, |\underline{v}|^{r-2}\underline{v}, \underline{q}_r^n - \underline{v}) \quad \forall \underline{v} \in [C^\infty(\bar{\Omega})]^2. \tag{3.11}$$

As $w_r^n \in C(\bar{\Omega})$, $\mathfrak{M}^n(w_r^n) \in L^\infty(\Omega)$ and $\underline{q}_r^n \in [L^r(\Omega)]^2$, it follows that (3.11) holds true for all $\underline{v} \in [L^r(\Omega)]^2$. For any fixed $\underline{z} \in [L^r(\Omega)]^2$, choosing $\underline{v} = \underline{q}_r^n \pm \alpha \underline{z}$ with $\alpha \in \mathbb{R}_{>0}$ in (3.11) and letting $\alpha \to 0$ yields the desired result (3.3b) on repeating the above for any $\underline{z} \in [L^r(\Omega)]^2$.

In addition, it follows from $W_r^0 = \pi_N^h w^0$ and (2.2b) that $w_r^0 = w^0$. Therefore, $\{w_r^n, \underline{q}_r^n\}_{n=1}^N$ is a solution of (Q_r^τ), (3.3a,3.3b). It follows from (2.20), (2.15), (1.27a,1.27b), (1.28), (3.4a,3.4c,3.4d,3.4e) and (2.8a) that (3.5) holds.

Finally, it is a simple matter to establish the uniqueness of the solution of (Q_r^τ) in case (iii). □

COROLLARY 3.2 Let the Assumptions of Theorem 3.1 hold. For $n = 1, \dots, N$ let $\hat{W}_r^n \in U_0^h$ be such that

$$(\underline{\nabla} \hat{W}_r^n, \underline{\nabla} \chi^h) = (\underline{\nabla}_h W_r^n, \underline{\nabla} \chi^h) \quad \forall \chi^h \in U_0^h. \tag{3.12}$$

Then there exists a further subsequence of $\{\{W_r^n, \underline{Q}_r^n\}_{n=1}^N\}_{h>0}$ (not indicated), where $\{W_r^n, \underline{Q}_r^n\}_{n=1}^N$ solves $(Q_r^{h,\tau})$, (2.18a,2.18b), such that as $h \to 0$

$$\hat{W}_r^n \to w_r^n \quad \text{strongly in } L^s(\Omega), \quad n = 1, \ldots, N, \ \ \forall s \in [1, \infty), \tag{3.13a}$$

$$\underline{\nabla} \hat{W}_r^n \to \underline{\nabla} w_r^n \quad \text{weakly in } [L^2(\Omega)]^2, \quad n = 1, \ldots, N; \tag{3.13b}$$

where $\{w_r^n, \underline{q}_r^n\}_{n=1}^N$ is a solution of (Q_r^τ), (3.3a,3.3b). In case (iii), the whole sequence converges in (3.13a,3.13b) as the solution of (Q_r^τ) is unique.

Proof. The proof follows immediately from (3.6), (3.4a,3.4c), (3.12), (2.35), (2.36a,2.36b), (2.39) and (2.41a,2.41b) on noting that $\hat{\eta} = \eta$. □

3.2 *Convergence of (Q_r^τ) to (Q) in case (i)*

It follows from (3.3a), (3.5), (1.28) and (1.30b) in the growing sandpile case that for $n = 1, \ldots, N$

$$\tau_n |(\underline{q}_r^n, \underline{\nabla} \eta)| \leqslant C |\eta|_{0,\Omega} \quad \forall \eta \in C_0^\infty(\Omega). \tag{3.14}$$

Hence, for a fixed time partition $\{\tau_n\}_{n=1}^N$, the distributional divergence of $\underline{q}_r^n$ belongs to $L^2(\Omega)$, $n = 1, \ldots, N$. Therefore, on recalling (1.25a), (Q_r^τ), (3.3a,3.3b), can be reformulated for a given $r \in (1, \frac{4}{3})$ as:

(Q_r^τ) For $n = 1, \ldots, N$, find $w_r^n \in W_0^{1,p}(\Omega)$ and $\underline{q}_r^n \in \underline{V}^r(\Omega)$ such that

$$\left(\frac{w_r^n - w_r^{n-1}}{\tau_n}, \eta \right) + (\underline{\nabla} \,.\, \underline{q}_r^n, \eta) = (f^n, \eta) \quad \forall \eta \in L^2(\Omega), \tag{3.15a}$$

$$(M_\varepsilon(w_r^n) |\underline{q}_r^n|^{r-2} \underline{q}_r^n, \underline{v}) - (w_r^n, \underline{\nabla} \,.\, \underline{v}) = 0 \quad \forall \underline{v} \in \underline{V}^r(\Omega), \tag{3.15b}$$

where $w_r^0 = w_0^\varepsilon$.

The above is the formulation of (Q_r^τ) in Barrett & Prigozhin (2013a, (3.24a,3.24b)). On recalling (1.25b), we state the discrete time approximation of the mixed formulation of the growing sandpile problem; that is, the $r \to 1$ limit of (Q_r^τ):

(Q^τ) For $n = 1, \ldots, N$, find $w^n \in W_0^{1,\infty}(\Omega)$ and $\underline{q}^n \in \underline{V}^{\mathscr{M}}(\Omega)$ such that

$$\left(\frac{w^n - w^{n-1}}{\tau_n}, \eta \right) + (\underline{\nabla} \,.\, \underline{q}^n, \eta) = (f^n, \eta) \quad \forall \eta \in L^2(\Omega), \tag{3.16a}$$

$$\langle |\underline{v}| - |\underline{q}^n|, M_\varepsilon(w^n) \rangle_{C(\bar{\Omega})} - (\underline{\nabla} \,.\, (\underline{v} - \underline{q}^n), w^n) \geqslant 0 \quad \forall \underline{v} \in \underline{V}^{\mathscr{M}}(\Omega); \tag{3.16b}$$

where $w^0 = w_0^\varepsilon$.

Similarly to (1.4), we introduce for $\chi \in W_0^{1,\infty}(\Omega)$ the closed convex nonempty set

$$K_\varepsilon(\chi) := \{\eta \in W_0^{1,\infty}(\Omega) : |\underline{\nabla} \eta| \leqslant M_\varepsilon(\chi) \text{ a.e. on } \Omega\}. \tag{3.17}$$

Then, associated with (Q^τ) is the corresponding approximation of the primal quasi-variational inequality:

(P^τ) For $n = 1, \ldots, N$, find $w^n \in K_\varepsilon(w^n)$ such that

$$\left(\frac{w^n - w^{n-1}}{\tau_n}, \eta - w^n\right) \geqslant (f^n, \eta - w^n) \quad \forall \eta \in K_\varepsilon(w^n), \tag{3.18}$$

where $w^0 = w_0^\varepsilon$.

Similarly to Barrett & Prigozhin (2013a), for our convergence results we require extra assumptions.

(A3) Ω is a strictly star-shaped domain.

(A4) $w_0^\varepsilon \geqslant 0$ and $f \in L^\infty(0,T;L^2(\Omega))$.

THEOREM 3.3 Let the Assumptions (A1)–(A3) hold. For any fixed time partition $\{\tau_n\}_{n=1}^N$, there exists a subsequence of $\{\{w_r^n, \underline{q}_r^n\}_{n=1}^N\}_{r\in(1,4/3)}$ (not indicated), where $\{w_r^n, \underline{q}_r^n\}_{n=1}^N$ solves (Q_r^τ), (3.15a,3.15b), such that as $r \to 1$

$$w_r^n \to w^n \quad \text{strongly in } C(\bar{\Omega}), \quad n = 0, \ldots, N, \tag{3.19a}$$

$$M_\varepsilon(w_r^n) \to M_\varepsilon(w^n) \quad \text{strongly in } C(\bar{\Omega}), \quad n = 0, \ldots, N, \tag{3.19b}$$

$$\underline{q}_r^n \to \underline{q}^n \quad \text{weakly in } [\mathscr{M}(\bar{\Omega})]^2, \quad n = 1, \ldots, N, \tag{3.19c}$$

$$\underline{\nabla} \,.\, \underline{q}_r^n \to \underline{\nabla} \,.\, \underline{q}^n \quad \text{weakly in } L^2(\Omega), \quad n = 1, \ldots, N; \tag{3.19d}$$

where $\{w^n, \underline{q}^n\}_{n=1}^N$ is a solution of (Q^τ), (3.16a,3.16b).

Proof. See the proof of Theorem 3.4 in Barrett & Prigozhin (2013a). We note that the convexity of Ω and the restriction of $\tau \in (0, \frac{1}{2}]$ were also assumed there, as these were required solely to establish the existence of a solution $\{w_r^n, \underline{q}_r^n\}_{n=1}^N$ to (Q_r^τ), see Barrett & Prigozhin (2013a, Theorem 3.3). These constraints on Ω and τ are not required here, see Theorem 3.1. In addition, as the time partition $\{\tau_n\}_{n=1}^N$ is fixed, the bound $|\underline{\nabla} \,.\, \underline{q}^r|_{0,\Omega} \leqslant C(\tau_n^{-1})$, $n = 1, \ldots, N$, which immediately follows from (3.14) is adequate to establish (3.19d). Therefore, the bound on $\underline{\nabla} \,.\, \underline{q}_r^n$ in Barrett & Prigozhin (2013a, (3.47)) is not necessary. □

Next, we note the following monotonicity result on recalling that $f^n \geqslant 0$, $n = 1, \ldots, N$.

THEOREM 3.4 Let Assumptions (A1)–(A3) hold. If $\{w^n, \underline{q}^n\}_{n=1}^N$ is a solution of (Q^τ), (3.16a,3.16b), then $\{w^n\}_{n=1}^N$ solves (P^τ), (3.18), and

$$w^n \geqslant w^{n-1}, \quad n = 1, \ldots, N. \tag{3.20}$$

Proof. See the proof of Theorem 3.6 in Barrett & Prigozhin (2013a). □

We introduce the following notation for $t \in (t_{n-1}, t_n]$, $n = 1, \ldots, N$,

$$\begin{aligned} &f^{\tau,+}(\cdot,t) := f^n(\cdot) \quad w^\tau(\cdot,t) := \frac{(t - t_{n-1})}{\tau_n} w^n(\cdot) + \frac{(t_n - t)}{\tau_n} w^{n-1}(\cdot), \\ &w^{\tau,+}(\cdot,t) := w^n(\cdot), \quad w^{\tau,-}(\cdot,t) := w^{n-1}(\cdot), \quad \underline{q}^{\tau,+}(\cdot,t) := \underline{q}^n(\cdot). \end{aligned} \tag{3.21}$$

We now introduce the weak mixed formulation of the growing sandpile problem:
(Q) Find $w \in L^\infty(0,T;W_0^{1,\infty}(\Omega)) \cap W^{1,\infty}(0,T;[C_0^1(\bar{\Omega})]^\star)$ and $\underline{q} \in L^\infty(0,T;[\mathscr{M}(\bar{\Omega})]^2)$ such that

$$\int_0^T \left[\left\langle \frac{\partial w}{\partial t}, \eta \right\rangle_{C_0^1(\bar{\Omega})} - \langle \underline{q}, \underline{\nabla} \eta \rangle_{C(\bar{\Omega})} - (f, \eta) \right] \mathrm{d}t = 0 \quad \forall \eta \in L^1(0,T;C_0^1(\bar{\Omega})), \tag{3.22a}$$

$$\int_0^T [\langle |\underline{v}| - |\underline{q}|, M_\varepsilon(w) \rangle_{C(\bar{\Omega})} - (\underline{\nabla} \,.\, \underline{v} - f, w)] \,\mathrm{d}t \geqslant \frac{1}{2}[|w(\cdot,T)|_{0,\Omega}^2 - |w_0^\varepsilon(\cdot)|_{0,\Omega}^2] \quad \forall \underline{v} \in L^1(0,T;\underline{V}^{\mathscr{M}}(\Omega)), \tag{3.22b}$$

where $w(\cdot,0) = w_0^\varepsilon(\cdot)$.

Associated with (Q) is the corresponding primal quasi-variational inequality:
(P) Find $w \in L^\infty(0,T;K_\varepsilon(w)) \cap W^{1,\infty}(0,T;[C_0^1(\bar{\Omega})]^\star)$ such that

$$\int_0^T \left[\left\langle \frac{\partial w}{\partial t}, \eta \right\rangle_{C_0^1(\bar{\Omega})} - (f, \eta - w) \right] \mathrm{d}t \geqslant \frac{1}{2}[|w(\cdot,T)|_{0,\Omega}^2 - |w_0^\varepsilon(\cdot)|_{0,\Omega}^2] \quad \forall \eta \in L^1(0,T;K_\varepsilon(w) \cap C_0^1(\bar{\Omega})), \tag{3.23}$$

where $w(\cdot,0) = w_0^\varepsilon(\cdot)$.

For the reasoning behind the formulations (Q) and (P), and Assumption (A4); see Barrett & Prigozhin (2013a, Remarks 3.1 and 3.9).

THEOREM 3.5 Let Assumptions (A1)–(A4) hold. For all time partitions $\{\tau_n\}_{n=1}^N$, there exists a subsequence of $\{\{w^n, \underline{q}^n\}_{n=1}^N\}_{\tau>0}$ (not indicated), where $\{w^n, \underline{q}^n\}_{n=1}^N$ solves (Q^τ), (3.16a,3.16b), such that as $\tau \to 0$

$$w^\tau, w^{\tau,\pm} \to w \quad \text{weak}^\star \text{ in } L^\infty(0,T;W^{1,\infty}(\Omega)), \tag{3.24a}$$

$$\frac{\partial w^\tau}{\partial t} \to \frac{\partial w}{\partial t} \quad \text{weak}^\star \text{ in } L^\infty(0,T;[C_0^1(\bar{\Omega})]^\star), \tag{3.24b}$$

$$w^\tau \to w \quad \text{strongly in } C([0,T];C(\bar{\Omega})), \tag{3.24c}$$

$$w^{\tau,\pm} \to w \quad \text{strongly in } L^2(0,T;C(\bar{\Omega})), \tag{3.24d}$$

$$M_\varepsilon(w^\tau) \to M_\varepsilon(w) \quad \text{strongly in } C([0,T];C(\bar{\Omega})), \tag{3.24e}$$

$$M_\varepsilon(w^{\tau,\pm}) \to M_\varepsilon(w) \quad \text{strongly in } L^2(0,T;C(\bar{\Omega})), \tag{3.24f}$$

$$\underline{q}^{\tau,+} \to \underline{q} \quad \text{weak}^\star \text{ in } L^\infty(0,T;[\mathscr{M}(\bar{\Omega})]^d); \tag{3.24g}$$

where $\{w, \underline{q}\}$ is a solution of (Q), (3.22a,3.22b). Moreover, w solves (P), (3.23).

Proof. See the proof of Theorem 3.8 in Barrett & Prigozhin (2013a). □

3.3 *Convergence of* (Q_r^τ) *to* (Q) *in case (ii)*

In the cylindrical superconductor case, on noting (3.1), (3.3b) becomes

$$(k\hat{M}(w_r^n + b_{\rm e}(t_n))|\underline{q}_r^n|^{r-2}\underline{q}_r^n, \underline{v}) + (\underline{\nabla} w_r^n, \underline{v}) = 0 \quad \forall \underline{v} \in [L^r(\Omega)]^2, \tag{3.25}$$

which can be rewritten, on noting (1.14), as

$$(k|\underline{q}_r^n|^{r-2}\underline{q}_r^n, \underline{v}) + ([\hat{M}(w_r^n + b_{\rm e}(t_n))]^{-1}\underline{\nabla} w_r^n, \underline{v}) = 0 \quad \forall \underline{v} \in [L^r(\Omega)]^2. \tag{3.26}$$

With $\hat{F} \in C(\mathbb{R};\mathbb{R})$ such that

$$(\hat{M}_0)^{-1} \geqslant \hat{F}'(s) = [\hat{M}(s)]^{-1} \geqslant (\hat{M}_1)^{-1} > 0 \quad \text{and} \quad \hat{F}(0) = 0, \tag{3.27}$$

where we have noted (1.14), (3.26) can be rewritten as

$$(k|\underline{q}_r^n|^{r-2}\underline{q}_r^n, \underline{v}) + (\underline{\nabla}[\hat{F}(w_r^n + b_{\rm e}(t_n)) - \hat{F}(b_{\rm e}(t_n))], \underline{v}) = 0 \quad \forall \underline{v} \in [L^r(\Omega)]^2. \tag{3.28}$$

Then similarly to (3.15a,3.15b), on noting the analogue of (3.14), (2.9), (1.30b) and (3.28), (Q_r^τ), (3.3a,3.3b), in the cylindrical superconductor case can be reformulated for a given $r \in (1, \frac{4}{3})$ as

(Q_r^τ) For $n = 1, \ldots, N$, find $w_r^n \in W_0^{1,p}(\Omega)$ and $\underline{q}_r^n \in \underline{V}^r(\Omega)$ such that

$$\left(\frac{(w_r^n + b_{\rm e}(t_n)) - (w_r^{n-1} + b_{\rm e}(t_{n-1}))}{\tau_n}, \eta\right) + (\underline{\nabla}\,.\,\underline{q}_r^n, \eta) = 0 \quad \forall \eta \in L^2(\Omega), \tag{3.29a}$$

$$(k|\underline{q}_r^n|^{r-2}\underline{q}_r^n, \underline{v}) - (\hat{F}(w_r^n + b_{\rm e}(t_n)) - \hat{F}(b_{\rm e}(t_n)), \underline{\nabla}\,.\,\underline{v}) = 0 \quad \forall \underline{v} \in \underline{V}^r(\Omega). \tag{3.29b}$$

It is now a simple matter to establish the uniqueness of $\{w_r^n, \underline{q}_r^n\}_{n=1}^N$ solving (Q_r^τ), (3.29a,3.29b), by exploiting (2.8a) and the monotonicity of $\widehat{F}$, recall (3.27). In addition, we have the following stability result.

LEMMA 3.6 Let Assumptions (A1) and (A2) hold. For any fixed $r \in (1, \frac{4}{3})$ and time partition $\{\tau_n\}_{n=1}^N$, the unique solution $\{w_r^n, \underline{q}_r^n\}_{n=1}^N$ of (Q_r^τ), (3.29a,3.29b), in addition to satisfying (3.5) with $\|\cdot\|_{\mathscr{A}} \equiv |\cdot|_{0,\Omega}$ satisfies

$$\begin{aligned}\frac{r-1}{r}\max_{n=1,\ldots,N}(k, |\underline{q}_r^n|^r) + \sum_{n=1}^N \tau_n \left|\frac{w_r^n - w_r^{n-1}}{\tau_n}\right|_{0,\Omega}^2 + \sum_{n=1}^N \tau_n |\underline{\nabla}\,.\,\underline{q}_r^n|_{0,\Omega}^2 \\ + \max_{n=1,\ldots,N}|\underline{\nabla} w_r^n|_{0,p,\Omega} + \sum_{n=1}^N \tau_n |\underline{q}_r^n|_{0,r,\Omega}^{2r} \leqslant C.\end{aligned} \tag{3.30}$$

Proof. Choosing $\eta \equiv [\hat{F}(w_r^n + b_e(t_n)) - \hat{F}(w_r^{n-1} + b_e(t_{n-1}))] - [\hat{F}(b_e(t_n)) - \hat{F}(b_e(t_{n-1}))]$ in (3.29a), and noting (3.29b) and (2.8b), yields for $n = 2, \ldots, N$ that

$$\left(w_r^n + b_e(t_n) - (w_r^{n-1} + b_e(t_{n-1})), \frac{\hat{F}(w_r^n + b_e(t_n)) - \hat{F}(w_r^{n-1} + b_e(t_{n-1})) - [\hat{F}(b_e(t_n)) - \hat{F}(b_e(t_{n-1}))]}{\tau_n} \right)$$
$$= -(\underline{\nabla} \cdot \underline{q}_r^n, [\hat{F}(w_r^n + b_e(t_n)) - \hat{F}(w_r^{n-1} + b_e(t_{n-1}))] - [\hat{F}(b_e(t_n)) - \hat{F}(b_e(t_{n-1}))])$$
$$= -(k[|\underline{q}_r^n|^{r-2}\underline{q}_r^n - |\underline{q}_r^{n-1}|^{r-2}\underline{q}_r^{n-1}], \underline{q}_r^n)$$
$$\leqslant -\frac{r-1}{r}(k, |\underline{q}_r^n|^r - |\underline{q}_r^{n-1}|^r), \tag{3.31a}$$

and, on noting (1.30a),

$$\left(w_r^1 + b_e(t_1) - (w_r^0 + b_e(t_0)), \frac{[\hat{F}(w_r^1 + b_e(t_1)) - \hat{F}(w_r^0 + b_e(t_0))] - [\hat{F}(b_e(t_1)) - \hat{F}(b_e(t_0))]}{\tau_1} \right)$$
$$= -(\underline{\nabla} \cdot \underline{q}_r^1, [\hat{F}(w_r^1 + b_e(t_1)) - \hat{F}(w_r^0 + b_e(t_0))] - [\hat{F}(b_e(t_1)) - \hat{F}(b_e(t_0))])$$
$$= -(k, |\underline{q}_r^1|^r) - (\underline{q}_r^1, [\hat{M}(w_0 + b_e(t_0))]^{-1}\underline{\nabla} w_0)$$
$$\leqslant -(k, |\underline{q}_r^1|^r) + (k, |\underline{q}_r^1|)$$
$$\leqslant -\frac{r-1}{r}(k, |\underline{q}_r^1|^r) + \frac{1}{p}|k|_{0,1,\Omega}. \tag{3.31b}$$

Summing (3.31a) and including (3.31b) yields for $n = 1, \ldots, N$ that

$$\frac{r-1}{r}(k|\underline{q}_r^n|^r, 1) + \sum_{\ell=1}^{n} \tau_\ell \left(\frac{(w_r^\ell + b_e(t_\ell)) - (w_r^{\ell-1} + b_e(t_{\ell-1}))}{\tau_\ell}, \frac{\hat{F}(w_r^\ell + b_e(t_\ell)) - \hat{F}(w_r^{\ell-1} + b_e(t_{\ell-1}))}{\tau_\ell} \right)$$
$$\leqslant C + \sum_{\ell=1}^{n} \tau_\ell \left(\frac{(w_r^\ell + b_e(t_\ell)) - (w_r^{\ell-1} + b_e(t_{\ell-1}))}{\tau_\ell}, \frac{\hat{F}(b_e(t_\ell)) - \hat{F}(b_e(t_{\ell-1}))}{\tau_\ell} \right). \tag{3.32}$$

The first two bounds in the desired result (3.30) then follow from (3.32), (3.27), (2.9), (2.10) and (1.30b), on using Young's inequality. The third bound in (3.30) then follows from the second bound in (3.30), (3.29a) with $\eta = \underline{\nabla} \cdot \underline{q}_r^n$, (2.9), (2.10) and (1.30b).

Next, we prove the fourth bound in (3.30). First, we note from (3.29b), (A1) and the first bound in (3.30) that for $n = 1, \ldots, N$

$$|(\hat{F}(w_r^n + b_e(t_n)) - \hat{F}(b_e(t_n)), \underline{\nabla} \cdot \underline{v})| \leqslant |k|_{0,\infty,\Omega} |\underline{q}_r^n|_{0,r,\Omega}^{r-1} |\underline{v}|_{0,r,\Omega}$$
$$\leqslant \left[\frac{r-1}{r} |\underline{q}_r^n|_{0,r,\Omega}^r + C \right] |\underline{v}|_{0,r,\Omega}$$
$$\leqslant C|\underline{v}|_{0,r,\Omega} \quad \forall \underline{v} \in \underline{V}^r(\Omega). \tag{3.33}$$

It follows from (3.33), as $C_0^\infty(\Omega)$ is dense in $L^r(\Omega)$, that the distributional gradient of $\hat{F}(w_r^n + b_e(t_n))$ belongs to the dual of $[L^r(\Omega)]^2$. Hence, we deduce from (3.33) that

$$|\underline{\nabla}[\hat{F}(w_r^n + b_e(t_n))]|_{0,p,\Omega} \leqslant C, \quad n = 1, \ldots, N. \tag{3.34}$$

As $\hat{F}$ is globally Lipschitz, recall (3.27), we obtain the fourth bound in (3.30).

Finally, it follows from (3.29b) with $\underline{v} = \underline{q}_r^n$, (1.14), (3.27), (3.5) and the third bound in (3.30) that

$$\begin{aligned} k_{\min}^2 \sum_{n=1}^N \tau_n |\underline{q}_r^n|_{0,r,\Omega}^{2r} \leqslant \sum_{n=1}^N \tau_n [(k, |\underline{q}_r^n|^r)]^2 = \sum_{n=1}^N \tau_n [\hat{F}(w_r^n + b_e(t_n) - \hat{F}(b_e(t_n)), \underline{\nabla} \,.\, \underline{q}_r^n)]^2 \\ \leqslant (\hat{M}_0)^{-2} \sum_{n=1}^N \tau_n |w_r^n|_{0,\Omega}^2 |\underline{\nabla} \,.\, \underline{q}_r^n|_{0,\Omega}^2 \\ \leqslant C \sum_{n=1}^N \tau_n |\underline{\nabla} \,.\, \underline{q}_r^n|_{0,\Omega}^2 \leqslant C. \end{aligned} \tag{3.35}$$

Hence, the final bound in (3.30) holds. □

3.3.1 *Convergence of (Q_r^τ) to (Q_r).* In addition to the notation (3.21), we introduce for $t \in (t_{n-1}, t_n]$, $n = 1, \ldots, N$,

$$b_e^\tau(t) := \frac{(t - t_{n-1})}{\tau_n} b_e(t_n) + \frac{(t_n - t)}{\tau_n} b_e(t_{n-1}), \quad b_e^{\tau,+}(t) := b_e(t_n). \tag{3.36}$$

We also write $w_r^{\tau(,+)}$ to mean with or without the superscript $+$. We note from (3.36), (1.30b), (2.9) and (2.10) that

$$b_e^\tau,\, b_e^{\tau,+} \to b_e \quad \text{strongly in } L^2(0,T), \qquad \frac{\mathrm{d}b_e^\tau}{\mathrm{d}t} \to \frac{\mathrm{d}b_e}{\mathrm{d}t} \quad \text{strongly in } L^2(0,T) \text{ as } \tau \to 0. \tag{3.37}$$

We set also $\Omega_T := \Omega \times (0,T)$.

Adopting the notation (3.21) and (3.36), (Q_r^τ), (3.29a,3.29b), can be restated as: Find $w_r^\tau \in H^1(0,T;L^2(\Omega))$ and $\underline{q}_r^{\tau,+} \in L^2(0,T;\underline{V}^r(\Omega))$ such that

$$\int_0^T \left(\frac{\partial w_r^\tau}{\partial t} + \underline{\nabla} \,.\, \underline{q}_r^+ + \frac{\mathrm{d}b_e^\tau}{\mathrm{d}t}, \eta \right) \mathrm{d}t = 0 \quad \forall \eta \in L^2(\Omega_T), \tag{3.38a}$$

$$\int_0^T [(k|\underline{q}_r^{\tau,+}|^{r-2} \underline{q}_r^{\tau,+}, \underline{v}) - (\hat{F}(w_r^{\tau,+} + b_e^{\tau,+}) - \hat{F}(b_e^{\tau,+}), \underline{\nabla} \,.\, \underline{v})]\, \mathrm{d}t = 0 \quad \forall \underline{v} \in L^2(0,T;\underline{V}^r(\Omega)); \tag{3.38b}$$

where $w_r^\tau(\cdot, 0) = w_0(\cdot)$.

In Theorem 3.7, we show the convergence of (Q_r^τ), (3.38a,3.38b), as $\tau \to 0$ to

(Q_r) Find $w_r \in H^1(0,T;L^2(\Omega))$ and $\underline{q}_r \in L^2(0,T;\underline{V}^r(\Omega))$ such that

$$\int_0^T \left(\frac{\partial w_r}{\partial t} + \underline{\nabla}\,.\,\underline{q}_r + \frac{\mathrm{d}b_e}{\mathrm{d}t}, \eta\right)\mathrm{d}t = 0 \quad \forall \eta \in L^2(\Omega_T), \tag{3.39a}$$

$$\int_0^T [(k|\underline{q}_r|^{r-2}\underline{q}_r, \underline{v}) - (\hat{F}(w_r + b_\mathrm{e}) - \hat{F}(b_\mathrm{e}), \underline{\nabla}\,.\,\underline{v})]\,\mathrm{d}t = 0 \quad \forall \underline{v} \in L^2(0,T;\underline{V}^r(\Omega)), \tag{3.39b}$$

where $w_r(\cdot,0) = w_0(\cdot)$.

THEOREM 3.7 Let the Assumptions (A1) and (A2) hold. For any fixed $r \in (1, \frac{4}{3})$ and for all time partitions $\{\tau_n\}_{n=1}^N$, there exists a subsequence of $\{w_r^\tau, \underline{q}_r^{\tau,+}\}_{\tau>0}$ (not indicated), where $\{w_r^\tau, \underline{q}_r^{\tau,+}\}$ is the unique solution of (Q_r^τ), (3.38a,3.38b), such that as $\tau \to 0$

$$w_r^{\tau(,+)} \to w_r \quad \text{weak-}\star \text{ in } L^\infty(0,T;W^{1,p}(\Omega)), \tag{3.40a}$$

$$\frac{\partial w_r^\tau}{\partial t} \to \frac{\partial w_r}{\partial t} \quad \text{weakly in } L^2(\Omega_T), \tag{3.40b}$$

$$w_r^{\tau(,+)} \to w_r \quad \text{strongly in } L^2(\Omega_T), \tag{3.40c}$$

$$\underline{q}_r^{\tau,+} \to \underline{q}_r \quad \text{weakly in } L^{2r}(0,T;[L^r(\Omega)]^2), \tag{3.40d}$$

$$\underline{\nabla}\,.\,\underline{q}_r^{\tau,+} \to \underline{\nabla}\,.\,\underline{q}_r \quad \text{weakly in } L^2(\Omega_T). \tag{3.40e}$$

Moreover, $\{w_r, \underline{q}_r\}$ solves (Q_r), (3.39a,3.39b).

Proof. The bounds (3.5) and (3.30) yield immediately that

$$\|w_r^{\tau(,+)}\|_{L^\infty(0,T;W^{1,p}(\Omega))} + \left\|\frac{\partial w_r^\tau}{\partial t}\right\|^2_{L^2(\Omega_T)} + \|\underline{q}_r^{\tau,+}\|^{2r}_{L^{2r}(0,T;L^r(\Omega))} + \|\underline{\nabla}\,.\,\underline{q}_r^{\tau,+}\|^2_{L^2(\Omega_T)} \leqslant C, \tag{3.41a}$$

$$\|w_r^\tau - w_r^{\tau,+}\|_{L^2(\Omega_T)} \leqslant \tau^2 \left\|\frac{\partial w_r^\tau}{\partial t}\right\|^2_{L^2(\Omega_T)} \leqslant C\tau^2. \tag{3.41b}$$

The subsequence convergence results (3.40a–3.40e) follow immediately from (3.41a,3.41b). The strong convergence result (3.40c) follows from (3.40a,3.40b), the compactness result (1.26) and (3.41b). As $w_r^\tau(\cdot,0) = w_0(\cdot)$, it follows from the above that $w_r(\cdot,0) = w_0(\cdot)$.

It follows immediately from passing to the limit $\tau \to 0$ in (3.38a) for the subsequence, on noting (3.40b,3.40e) and (3.37), that $\{w_r, \underline{q}_r\}$ satisfy (3.39a).

Given any $\underline{z} \in L^2(0,T;[\underline{V}^r(\Omega)]^2)$, we choose $\underline{v} \equiv \underline{q}_r^{\tau,+} - \underline{z}$ in (3.38b) to yield, on noting (2.8a), that

$$\begin{aligned}\int_0^T (\hat{F}(w_r^{\tau,+} + b_e^{\tau,+}) - \hat{F}(b_e^{\tau,+}), \underline{\nabla}\,.\,(\underline{q}_r^{\tau,+} - \underline{z}))\,\mathrm{d}t &= \int_0^T (k|\underline{q}_r^{\tau,+}|^{r-2}\underline{q}_r^{\tau,+}, \underline{q}_r^{\tau,+} - \underline{z})\,\mathrm{d}t \\ &\geqslant \int_0^T (k\,|\underline{z}|^{r-2}\underline{z}, \underline{q}_r^{\tau,+} - \underline{z})\,\mathrm{d}t.\end{aligned} \tag{3.42}$$

Passing to the limit $\tau \to 0$ in (3.42) for the subsequence yields, on noting (3.40c–3.40e), (3.27) and (3.37), that

$$\int_0^T (\hat{F}(w_r + b_e) - \hat{F}(b_e), \underline{\nabla} \,.\, (\underline{q}_r - \underline{z}))\,dt \geqslant \int_0^T (k\,|\underline{z}|^{r-2}\underline{z}, \underline{q}_r - \underline{z})\,dt. \tag{3.43}$$

For any fixed $\underline{v} \in \underline{V}^r(\Omega)$, choosing $\underline{z} = \underline{q}_r \pm \alpha \underline{v}$ with $\alpha \in \mathbb{R}_{>0}$ in (3.43), and letting $\alpha \to 0$ yields the desired result (3.39b). Hence, $\{w_r, \underline{q}_r\}$ solves (Q_r), (3.39a,3.39b). □

3.3.2 *Convergence of (Q_r) to (Q).* We need an extra assumption.

(A5) $k \in C(\bar{\Omega})$.

Then the weak mixed formulation of the cylindrical superconductor problem is

(Q) Find $w \in H^1(0, T; L^2(\Omega))$ and $\underline{q} \in L^2(0, T; \underline{V}^{\mathscr{M}}(\Omega))$ such that

$$\int_0^T \left(\frac{\partial w}{\partial t} + \underline{\nabla} \,.\, \underline{q} + \frac{db_e}{dt}, \eta \right) dt = 0 \quad \forall \eta \in L^2(0, T; L^2(\Omega)), \tag{3.44a}$$

$$\int_0^T [\langle |\underline{v}| - |\underline{q}|, k \rangle_{C(\bar{\Omega})} - (\underline{\nabla} \,.\, (\underline{v} - \underline{q}), \hat{F}(w + b_e) - \hat{F}(b_e))]\,dt \geqslant 0 \quad \forall \underline{v} \in L^2(0, T; \underline{V}^{\mathscr{M}}(\Omega)), \tag{3.44b}$$

where $w(\cdot, 0) = w_0(\cdot)$.

Recalling (1.13) and (1.14), it follows that

$$\hat{K}(\psi) := \{\eta \in W_0^{1,\infty}(\Omega) : |\underline{\nabla}\eta| \leqslant k\hat{M}(\psi) \text{ a.e. in } \Omega\}. \tag{3.45}$$

Associated with the mixed formulation (Q) is the primal variational inequality:

(P) Find $w \in L^\infty(0, T; \hat{K}(w + b_e)) \cap H^1(0, T; L^2(\Omega))$ such that

$$\int_0^T \left(\frac{\partial w}{\partial t} + \frac{db_e}{dt}, \eta - w \right) dt \geqslant 0 \quad \forall \eta \in L^2(0, T; \hat{K}(w + b_e)), \tag{3.46}$$

where $w(\cdot, 0) = w_0(\cdot)$.

THEOREM 3.8 Let Assumptions (A1)–(A3) and (A5) hold. Then there exists a subsequence of $\{w_r, \underline{q}_r\}_{r \in (1,4/3)}$ (not indicated), where $\{w_r, \underline{q}_r\}$ solves (Q_r), (3.39a,3.39b), such that as $r \to 1$

$$w_r \to w \quad \text{weak-}\star \text{ in } L^\infty(0, T; W^{1,4}(\Omega)), \tag{3.47a}$$

$$\frac{\partial w_r}{\partial t} \to \frac{\partial w}{\partial t} \quad \text{weakly in } L^2(\Omega_T), \tag{3.47b}$$

$$w_r \to w \quad \text{strongly in } C([0, T]; C(\bar{\Omega})), \tag{3.47c}$$

$$\underline{q}_r \to \underline{q} \quad \text{weakly in } L^2(0,T;[\mathscr{M}(\bar{\Omega})]^2), \tag{3.47d}$$

$$\underline{\nabla}\,.\,\underline{q}_r \to \underline{\nabla}\,.\,\underline{q} \quad \text{weakly in } L^2(\Omega_T). \tag{3.47e}$$

Moreover, $\{w, \underline{q}\}$ solves (Q), (3.44a,3.44b).

Proof. On noting that $r \in (1, \frac{4}{3}) \Rightarrow p > 4$, the results (3.41a), (3.40a–3.40e) and (2.8a) yield immediately that

$$\|w_r\|_{L^\infty(0,T;W^{1,4}(\Omega))} + \left\|\frac{\partial w_r}{\partial t}\right\|^2_{L^2(\Omega_T)} + \|\underline{q}_r\|_{L^2(0,T;L^1(\Omega))} + \|\underline{\nabla}\,.\,\underline{q}_r\|^2_{L^2(\Omega_T)} \leqslant C(T). \tag{3.48}$$

The subsequence convergence results (3.47a,3.47b,3.47d,3.47e) follow immediately from (3.48). The strong convergence result (3.47c) follows from (3.47a,3.47b) and the compactness result (1.26). As $w_r(\cdot,0) = w_0(\cdot)$, it follows from the above that $w(\cdot,0) = w_0(\cdot)$. It follows immediately from passing to the limit $r \to 1$ in (3.39a) for the subsequence, on noting (3.47b,3.47e), that $\{w, \underline{q}\}$ satisfy (3.44a).

Given any $\underline{z} \in L^2(0,T;[C^\infty(\bar{\Omega})]^2)$, we choose $\underline{v} \equiv \underline{q}_r - \underline{z}$ in (3.39b) to yield, on noting (2.8a), that

$$\begin{aligned}\int_0^T (\hat{F}(w_r + b_{\mathrm{e}}) - \hat{F}(b_{\mathrm{e}}), \underline{\nabla}\,.\,(\underline{q}_r - \underline{z}))\,\mathrm{d}t &= \int_0^T (k|\underline{q}_r|^{r-2}\underline{q}_r, \underline{q}_r - \underline{z})\,\mathrm{d}t \\ &\geqslant r^{-1}\int_0^T (k, |\underline{q}_r|^r - |\underline{z}|^r)\,\mathrm{d}t.\end{aligned} \tag{3.49}$$

It follows immediately from (3.40c,3.40e) and (3.27) that for any $\underline{z} \in L^2(0,T;[C^\infty(\bar{\Omega})]^2)$

$$\begin{aligned}&\int_0^T (\hat{F}(w_r + b_{\mathrm{e}}) - \hat{F}(b_{\mathrm{e}}), \underline{\nabla}\,.\,(\underline{q}_r^+ - \underline{z}))\,\mathrm{d}t \\ &\quad\to \int_0^T (\hat{F}(w + b_{\mathrm{e}}) - \hat{F}(b_{\mathrm{e}}), \underline{\nabla}\,.\,(\underline{q} - \underline{z}))\,\mathrm{d}t \quad \text{as } r \to 1.\end{aligned} \tag{3.50}$$

Next, we note that for any $\underline{z} \in L^2(0,T;[C^\infty(\bar{\Omega})]^2)$

$$r^{-1}\int_0^T (k|\underline{z}|^r, 1)\,\mathrm{d}t \to \int_0^T (k|\underline{z}|, 1)\,\mathrm{d}t \quad \text{as } r \to 1. \tag{3.51}$$

Finally, it follows from (3.47d), and similarly to (1.24), that

$$\liminf_{r\to 1} r^{-1}\int_0^T (k|\underline{q}_r|^r, 1)\,\mathrm{d}t \geqslant \liminf_{r\to 1}\int_0^T (k|\underline{q}_r|, 1)\,\mathrm{d}t \geqslant \int_0^T \langle|\underline{q}|, k\rangle_{C(\bar{\Omega})}\,\mathrm{d}t. \tag{3.52}$$

Combining (3.49–3.52), it follows that $\{w, \underline{q}\}$ satisfies (3.44b) for any $\underline{v} \in L^2(0,T;[C^\infty(\bar{\Omega})]^2)$. The desired result, $\{w, \underline{q}\}$ satisfies (3.44b) for any $\underline{v} \in L^2(0,T;\underline{V}^{\mathscr{M}}(\Omega))$, and hence $\{w, \underline{q}\}$ solves (Q), (3.44a,3.44b), then follows from the density results (1.22b, 1.22c) in Barrett & Prigozhin (2010) with '=' replaced by '$\leqslant$' in the latter. □

THEOREM 3.9 Let the assumptions of Theorem 3.8 hold. We then have that any solution $\{w, \underline{q}\}$ of (Q), (3.44a,3.44b), satisfies

$$\int_0^T [\langle |\underline{q}|, k\hat{M}(w + b_{\mathrm{e}})\rangle_{C(\bar{\Omega})} - (w, \underline{\nabla}\,.\,\underline{q})]\,\mathrm{d}t = 0. \tag{3.53}$$

Moreover, w solves the quasi-variational inequality (P), (3.46).

Proof. See the proof of Theorem 3.3 in Barrett & Prigozhin (2010). However, we note that one can establish (3.53) by requiring only the density results (1.22b,1.22c) in Barrett & Prigozhin (2010), with '=' replaced by '$\leqslant$' in the latter, as opposed to (1.22a–1.22c) there. To see this, we note that it follows immediately from (3.39b), (3.48) and (3.27) that

$$\int_0^T [(k\hat{M}(w_r + b_{\mathrm{e}})|\underline{q}_r|^{r-2}\underline{q}_r, \underline{v}) - (w_r, \underline{\nabla}\,.\,\underline{v})]\,\mathrm{d}t = 0 \quad \forall \underline{v} \in L^2(0,T;\underline{V}^r(\Omega)). \tag{3.54}$$

For any fixed $\underline{z} \in L^2(0,T;[C(\bar{\Omega})]^2)$, we choose $\underline{v} = \underline{q}_r - \underline{z}$ in (3.54) and deduce from (3.47c–3.47e), similarly to (3.49–3.52), on passing to the limit $r \to 1$ that

$$\int_0^T [\langle |\underline{z}| - |\underline{q}|, k\hat{M}(w + b_{\mathrm{e}})\rangle_{C(\bar{\Omega})} - (\underline{\nabla}\,.\,(\underline{z} - \underline{q}), w)]\,\mathrm{d}t \geqslant 0 \quad \forall \underline{z} \in L^2(0,T;[C(\bar{\Omega})]^2). \tag{3.55}$$

Applying the stated density results from Barrett & Prigozhin (2010), we obtain (3.55) holds for all $\underline{z} \in L^2(0,T;\underline{V}^{\mathscr{M}}(\Omega))$. Then choosing $\underline{z} = \underline{0}$ and $\underline{z} = 2\underline{q}$ in (3.55) yields the desired result (3.53). □

3.4 *Convergence of (Q_r^τ) to (Q) in case (iii)*

It follows from (3.3a), (3.5), (1.28), (1.27b) and (1.30b) in the thin film superconductor case that for $n = 1, \ldots, N$

$$\tau_n |(\underline{q}_r^n, \underline{\nabla}\eta)| \leqslant C\|\eta\|_{H_{00}^{1/2}(\Omega)} \qquad \forall \eta \in C_0^\infty(\Omega). \tag{3.56}$$

Hence, for a fixed time partition $\{\tau_n\}_{n=1}^N$, the distributional divergence of $\underline{q}_r^n$ belongs to $[H_{00}^{1/2}(\Omega)]^\star$, $n = 1, \ldots, N$. On recalling (1.25c), (Q_r^τ), (3.3a,3.3b), can then be reformulated for a given $r \in (1, \frac{4}{3})$ as

(Q_r^τ) For $n = 1, \ldots, N$, find $w_r^n \in W_0^{1,p}(\Omega)$ and $\underline{q}_r^n \in \underline{Z}^r(\Omega)$ such that

$$a\left(\frac{w_r^n - w_r^{n-1}}{\tau_n}, \eta\right) + \langle \underline{\nabla}\,.\,\underline{q}_r^n, \eta\rangle_{H_{00}^{1/2}(\Omega)} = -\left(\frac{b_{\mathrm{e}}(t_n) - b_{\mathrm{e}}(t_{n-1})}{\tau_n}, \eta\right) \quad \forall \eta \in H_{00}^{1/2}(\Omega), \tag{3.57a}$$

$$(k|\underline{q}_r^n|^{r-2}\underline{q}_r^n, \underline{v}) - \langle \underline{\nabla}\,.\,\underline{v}, w_r^n\rangle_{H_{00}^{1/2}(\Omega)} = 0 \quad \forall \underline{v} \in \underline{Z}^r(\Omega), \tag{3.57b}$$

where $w_r^0 = w_0$.

We have the following stability result.

LEMMA 3.10 Let Assumptions (A1) and (A2) hold. For any fixed $r \in (1, \frac{4}{3})$ and time partition $\{\tau_n\}_{n=1}^N$, the unique solution $\{w_r^n, \underline{q}_r^n\}_{n=1}^N$ of (Q_r^τ), (3.57a,3.57b), in addition to satisfying (3.5) with

$\|\cdot\|_{\mathscr{A}} \equiv \|\cdot\|_{H^{1/2}_{00}(\Omega)}$ satisfies

$$\frac{r-1}{r}\max_{n=1,\dots,N}(k,|\underline{q}^n_n|^r) + \sum_{n=1}^{N}\tau_n\left\|\frac{w^n_r - w^{n-1}_r}{\tau_n}\right\|^2_{H^{1/2}_{00}(\Omega)} + \sum_{n=1}^{N}\tau_n\|\underline{\nabla}\,.\,\underline{q}^n_r\|^2_{[H^{1/2}_{00}(\Omega)]^\star} + \sum_{n=1}^{N}\tau_n|\underline{q}^n_r|^{2r}_{0,r,\Omega} \leqslant C. \tag{3.58}$$

Proof. Similarly to (3.31a,3.31b), choosing $\eta \equiv w^n_r - w^{n-1}_r$ in (3.57a), and noting (3.57b) and (2.8b), yields for $n = 2,\dots,N$ that

$$\begin{aligned}
&\tau_n a\left(\frac{w^n_r - w^{n-1}_r}{\tau_n}, \frac{w^n_r - w^{n-1}_r}{\tau_n}\right) + \tau_n\left(\frac{b_{\mathrm{e}}(t_n) - b_{\mathrm{e}}(t_{n-1})}{\tau_n}, \frac{w^n_r - w^{n-1}_r}{\tau_n}\right)\\
&\quad = -\langle\underline{\nabla}\,.\,\underline{q}^n_r, w^n_r - w^{n-1}_r\rangle_{H^{1/2}_{00}(\Omega)}\\
&\quad = -(k[|\underline{q}^n_r|^{r-2}\underline{q}^n_r - |\underline{q}^{n-1}_r|^{r-2}\underline{q}^{n-1}_r], \underline{q}^n_r)\\
&\quad \leqslant -\frac{r-1}{r}(k,|\underline{q}^n_r|^r - |\underline{q}^{n-1}_r|^r),
\end{aligned} \tag{3.59a}$$

and, on noting (1.30a),

$$\begin{aligned}
&\tau_1 a\left(\frac{w^1_r - w^0_r}{\tau_1}, \frac{w^1_r - w^0_r}{\tau_1}\right) + \tau_1\left(\frac{b_{\mathrm{e}}(t_1) - b_{\mathrm{e}}(t_0)}{\tau_1}, \frac{w^1_r - w^0_r}{\tau_1}\right)\\
&\quad = -\langle\underline{\nabla}\,.\,\underline{q}^1_r, w^1_r - w^0_r\rangle_{H^{1/2}_{00}(\Omega)}\\
&\quad = -(k,|\underline{q}^1_r|^r) - (\underline{q}^1_r, \underline{\nabla} w_0)\\
&\quad \leqslant -(k,|\underline{q}^1_r|^r) + (k,|\underline{q}^1_r|)\\
&\quad \leqslant -\frac{r-1}{r}(k,|\underline{q}^1_r|^r) + \frac{1}{p}|k|_{0,1,\Omega}.
\end{aligned} \tag{3.59b}$$

Summing (3.59a) and including (3.59b) yields for $n = 1,\dots,N$ that

$$\begin{aligned}
&\frac{r-1}{r}(k|\underline{q}^n_r|^r, 1) + \sum_{\ell=1}^{n}\tau_\ell a\left(\frac{w^\ell_r - w^{\ell-1}_r}{\tau_\ell}, \frac{w^\ell_r - w^{\ell-1}_r}{\tau_\ell}\right)\\
&\quad \leqslant C + \sum_{\ell=1}^{n}\tau_\ell\left(\frac{b_{\mathrm{e}}(t_\ell) - b_{\mathrm{e}}(t_{\ell-1})}{\tau_\ell}, \frac{w^\ell_r - w^{\ell-1}_r}{\tau_\ell}\right).
\end{aligned} \tag{3.60}$$

The first two bounds in the desired result (3.58) then follow from (3.60), (2.9), (2.10) and (1.30b), on using Young's inequality. The third bound in (3.58) then follows from the second bound in (3.58) and (3.57a).

Finally, similarly to (3.35), it follows from (3.57b) with $\underline{v} = \underline{q}_r^n$, (1.14), (3.27), (3.5) and the third bound in (3.58) that

$$
\begin{aligned}
k_{\min}^2 \sum_{n=1}^{N} \tau_n |\underline{q}_r^n|_{0,r,\Omega}^{2r} &\leqslant \sum_{n=1}^{N} \tau_n [(k, |\underline{q}_r^n|^r)]^2 \\
&= \sum_{n=1}^{N} \tau_n [\langle \underline{\nabla} \cdot \underline{q}_r^n, w_r^n \rangle_{H_{00}^{1/2}(\Omega)}]^2 \\
&\leqslant \sum_{n=1}^{N} \tau_n \|w_r^n\|_{H_{00}^{1/2}(\Omega)}^2 \|\underline{\nabla} \cdot \underline{q}_r^n\|_{[H_{00}^{1/2}(\Omega)]^\star}^2 \\
&\leqslant C \sum_{n=1}^{N} \tau_n \|\underline{\nabla} \cdot \underline{q}_r^n\|_{[H_{00}^{1/2}(\Omega)]^\star}^2 \leqslant C. \qquad (3.61)
\end{aligned}
$$

Hence, the final bound in (3.58) holds. □

3.4.1 *Convergence of (Q_r^τ) to (Q_r).* Adopting the notation (3.21) and (3.36), (Q_r^τ), (3.57a,3.57b), can be rewritten as: Find $w_r^\tau \in H^1(0,T;H_{00}^{1/2}(\Omega))$ and $\underline{q}_r^{\tau,+} \in L^2(0,T;\underline{Z}^r(\Omega))$ such that

$$
\int_0^T \left[a\left(\frac{\partial w_r^\tau}{\partial t}, \eta\right) + \langle \underline{\nabla} \cdot \underline{q}_r^{\tau,+}, \eta \rangle_{H_{00}^{1/2}(\Omega)} + \left(\frac{\mathrm{d}b_{\mathrm{e}}^\tau}{\mathrm{d}t}, \eta\right) \right] \mathrm{d}t = 0 \quad \forall \eta \in L^2(0,T;H_{00}^{1/2}(\Omega)), \qquad (3.62a)
$$

$$
\int_0^T [(k|\underline{q}_r^{\tau,+}|^{r-2}\underline{q}_r^{\tau,+}, \underline{v}) - \langle \underline{\nabla} \cdot \underline{v}, w_r^{\tau,+} \rangle_{H_{00}^{1/2}(\Omega)}] \,\mathrm{d}t = 0 \quad \forall \underline{v} \in L^2(0,T;\underline{Z}^r(\Omega)); \qquad (3.62b)
$$

where $w_r^\tau(\cdot, 0) = w_0(\cdot)$.

In Theorem 3.11, we show the convergence of (Q_r^τ), (3.62a,3.62b), as $\tau \to 0$ to

(Q_r) Find $w_r \in H^1(0,T;H_{00}^{1/2}(\Omega))$ and $\underline{q}_r \in L^2(0,T;\underline{Z}^r(\Omega))$ such that

$$
\int_0^T \left[a\left(\frac{\partial w_r}{\partial t}, \eta\right) + \langle \underline{\nabla} \cdot \underline{q}_r, \eta \rangle_{H_{00}^{1/2}(\Omega)} + \left(\frac{\mathrm{d}b_e}{\mathrm{d}t}, \eta\right) \right] \mathrm{d}t = 0 \quad \forall \eta \in L^2(0,T;H_{00}^{1/2}(\Omega)), \qquad (3.63a)
$$

$$
\int_0^T [(k|\underline{q}_r|^{r-2}\underline{q}_r, \underline{v}) - \langle \underline{\nabla} \cdot \underline{v}, w_r \rangle_{H_{00}^{1/2}(\Omega)}] \,\mathrm{d}t = 0 \quad \forall \underline{v} \in L^2(0,T;\underline{Z}^r(\Omega)); \qquad (3.63b)
$$

where $w_r(\cdot, 0) = w_0(\cdot)$.

Associated with (Q_r) is the corresponding generalized p-Laplacian problem for $p \in (4, \infty)$:

(P_p) Find $w_r \in L^p(0,T;W_0^{1,p}(\Omega)) \cap H^1(0,T;H_{00}^{1/2}(\Omega))$ such that

$$
\int_0^T \left[a\left(\frac{\partial w_r}{\partial t}, \eta\right) + \left(\frac{|\underline{\nabla} w_r|^{p-2}}{k^{p-1}} \underline{\nabla} w_r, \underline{\nabla} \eta\right) + \left(\frac{\mathrm{d}b_e}{\mathrm{d}t}, \eta\right) \right] \mathrm{d}t = 0 \quad \forall \eta \in L^p(0,T;W_0^{1,p}(\Omega)), \qquad (3.64)
$$

where $w_r(\cdot, 0) = w_0(\cdot)$.

THEOREM 3.11 Let Assumptions (A1) and (A2) hold. For any fixed $r \in (1, \frac{4}{3})$ the sequence $\{w_r^\tau, \underline{q}_r^{\tau,+}\}_{\tau>0}$, where $\{w_r^\tau, \underline{q}_r^{\tau,+}\}$ is the unique solution of (Q_r^τ), is such that as $\tau \to 0$

$$w_r^{\tau(,+)} \to w_r \quad \text{weak-}\star \text{ in } L^\infty(0,T;H_{00}^{1/2}(\Omega)), \tag{3.65a}$$

$$\frac{\partial w_r^\tau}{\partial t} \to \frac{\partial w_r}{\partial t} \quad \text{weakly in } L^2(0,T;H_{00}^{1/2}(\Omega)), \tag{3.65b}$$

$$\underline{q}_r^{\tau,+} \to \underline{q}_r \quad \text{weakly in } L^{2r}(0,T;[L^r(\Omega)]^2), \tag{3.65c}$$

$$\underline{\nabla} \,.\, \underline{q}_r^{\tau,+} \to \underline{\nabla} \,.\, \underline{q}_r \quad \text{weakly in } L^2(0,T;[H_{00}^{1/2}(\Omega)]^\star), \tag{3.65d}$$

where $\{w_r, \underline{q}_r\}$ is the unique solution of (Q_r), (3.63a,3.63b). In addition, w_r is the unique solution of (P_p), (3.64).

Proof. It follows immediately from (3.5), (3.58) and (3.21) that

$$\|w_r^{\tau(,+)}\|_{L^\infty(0,T;H_{00}^{1/2}(\Omega))} + \left\|\frac{\partial w_r^\tau}{\partial t}\right\|^2_{L^2(0,T;H_{00}^{1/2}(\Omega))} + \|\underline{q}_r^{\tau,+}\|^{2r}_{L^{2r}(0,T;L^r(\Omega))}$$
$$+ \|\underline{\nabla}\,.\,\underline{q}_r^{\tau,+}\|^2_{L^2(0,T;[H_{00}^{1/2}(\Omega)]^\star)} \leqslant C, \tag{3.66a}$$

$$\|w_r^\tau - w_r^{\tau(,+)}\|^2_{L^2(0,T;H_{00}^{1/2}(\Omega))} \leqslant \tau^2 \left\|\frac{\partial w_r^\tau}{\partial t}\right\|^2_{L^2(0,T;H_{00}^{1/2}(\Omega))} \leqslant C\tau^2. \tag{3.66b}$$

It follows immediately from (3.66a,3.66b) that the results (3.65a–3.65d) hold for a subsequence of $\{w_r^\tau, \underline{q}_r^{\tau,+}\}_{\tau>0}$. We then pass to the limit $\tau \to 0$ in (3.62a) for the above subsequence and obtain (3.63a) for any fixed $\eta \in L^2(0,T;H_{00}^{1/2}(\Omega))$, on noting (3.65b,3.65d) and (3.37).

For any fixed $\underline{\eta} \in L^2(0,T;\underline{Z}^r(\Omega))$, we choose $\underline{v} = \underline{q}_r^{\tau,+} - \underline{z}$ in (3.62b). On noting (3.62a), (2.8a) and (3.66b), we deduce that

$$- \int_0^T \langle \underline{\nabla}\,.\,\underline{z}, w_r^{\tau,+}\rangle_{H_{00}^{1/2}(\Omega)}\,\mathrm{d}t$$
$$= - \int_0^T \langle \underline{\nabla}\,.\,\underline{q}_r^{\tau,+}, w_r^{\tau,+}\rangle_{H_{00}^{1/2}(\Omega)}\,\mathrm{d}t + \int_0^T (k|\underline{q}_r^{\tau,+}|^{r-2}\underline{q}_r^{\tau,+}, \underline{q}_r^{\tau,+} - \underline{z})\,\mathrm{d}t$$
$$\geqslant \int_0^T \left[a\left(\frac{\partial w_r^\tau}{\partial t}, w_r^{\tau,+}\right) + \left(\frac{db_{\mathrm{e}}^\tau}{\mathrm{d}t}, w_r^{\tau,+}\right)\right]\mathrm{d}t + \int_0^T (k|\underline{z}|^{r-2}\underline{z}, \underline{q}_r^{\tau,+} - \underline{z})\,\mathrm{d}t$$

$$\geqslant \frac{1}{2}(\|w_r^\tau(\cdot,T)\|_a^2 - \|w_0\|_a^2) + \int_0^T a\left(\frac{\partial w_r^\tau}{\partial t}, w_r^{\tau,+} - w_r^\tau\right) \mathrm{d}t$$
$$+ \int_0^T \left(\frac{\mathrm{d}b_\mathrm{e}^\tau}{\mathrm{d}t}, w_r^{\tau,+}\right) \mathrm{d}t + \int_0^T (k|\underline{z}|^{r-2}\,\underline{z}, \underline{q}_r^{\tau,+} - \underline{z})\,\mathrm{d}t$$
$$\geqslant \frac{1}{2}(\|w_r^\tau(\cdot,T)\|_a^2 - \|w_0\|_a^2) + \int_0^T \left(\frac{\mathrm{d}b_\mathrm{e}^\tau}{\mathrm{d}t}, w_r^{\tau,+}\right) \mathrm{d}t$$
$$+ \int_0^T (k|\underline{z}|^{r-2}\underline{z}, \underline{q}_r^{\tau,+} - \underline{z})\,\mathrm{d}t - C\tau. \tag{3.67}$$

It follows from (3.66a) that

$$\|w_r^\tau\|_{C([0,T];H_{00}^{1/2}(\Omega))} \leqslant C. \tag{3.68}$$

Hence, we deduce from (3.68), on extraction of a possible further subsequence, that

$$\liminf_{\tau\to 0} \|w_r^\tau(\cdot,T)\|_a^2 \geqslant \|w_r(\cdot,T)\|_a^2. \tag{3.69}$$

On noting (3.65a,3.65c), (3.69) and (3.37), we can pass to the limit $\tau \to 0$ in (3.67) for the above subsequence to obtain

$$-\int_0^T \langle \underline{\nabla} \,.\, \underline{z}, w_r\rangle_{H_{00}^{1/2}(\Omega)}\,\mathrm{d}t \geqslant \frac{1}{2}(\|w_r(\cdot,T)\|_a^2 - \|w_0\|_a^2) + \int_0^T \left(\frac{\mathrm{d}b_\mathrm{e}}{\mathrm{d}t}, w_r\right) \mathrm{d}t$$
$$+ \int_0^T (k|\underline{z}|^{r-2}\underline{z}, \underline{q}_r - \underline{z})\,\mathrm{d}t \quad \forall \underline{z} \in L^2(0,T;\underline{Z}^r(\Omega)). \tag{3.70}$$

It follows from (3.70) and (3.63a) that

$$\int_0^T \langle \underline{\nabla} \,.\, (\underline{q}_r - \underline{z}), w_r\rangle_{H_{00}^{1/2}(\Omega)}\,\mathrm{d}t \geqslant \int_0^T (k|\underline{z}|^{r-2}\underline{z}, \underline{q}_r - \underline{z})\,\mathrm{d}t \quad \forall \underline{z} \in L^2(0,T;\underline{Z}^r(\Omega)). \tag{3.71}$$

For any fixed $\underline{v} \in L^2(0,T;\underline{Z}^r(\Omega))$, choosing $\underline{z} = \underline{q}_r \pm \alpha\,\underline{v}$ with $\alpha \in \mathbb{R}_{>0}$ in (3.71), and letting $\alpha \to 0$ yields the desired result (3.63b). Hence, $\{w_r, \underline{q}_r\}$ solves (Q_r), (3.63a,3.63b).

In addition, we obtain from (3.62b) for any fixed $\underline{v} \in C([0,T];[C^\infty(\bar{\Omega})]^2)$, on noting the third bound in (3.66a), that

$$\int_0^T (w_r^{\tau,+}, \underline{\nabla}\,.\,\underline{v})\,\mathrm{d}t = \int_0^T (k|\underline{q}_r^{\tau,+}|^{r-2}\underline{q}_r^{\tau,+}, \underline{v})\,\mathrm{d}t \leqslant C\int_0^T |\underline{q}_r^{\tau,+}|_{0,r,\Omega}^{r-1} |\underline{v}|_{0,r,\Omega}\,\mathrm{d}t$$
$$\leqslant C(T)\|\underline{v}\|_{L^r(0,T;L^r(\Omega))}. \tag{3.72}$$

Passing to the limit $h,\ \tau \to 0$ in (3.72), on noting (3.65a), yields that

$$\int_0^T (w_r, \underline{\nabla}\,.\,\underline{v})\,\mathrm{d}t \leqslant C\|\underline{v}\|_{L^r(0,T;L^r(\Omega))} \quad \forall \underline{v} \in C([0,T];[C^\infty(\bar{\Omega})]^2). \tag{3.73}$$

Hence, it follows that

$$w_r \in L^p(0,T;W_0^{1,p}(\Omega)) \quad \text{and} \quad \|w_r\|_{L^p(0,T;W^{1,p}(\Omega))} \leqslant C. \tag{3.74}$$

To show the uniqueness of this solution $\{w_r, \underline{q}_r\}$ of (Q_r), (3.63a,3.63b), and that w_r is the unique solution of (P_p), (3.64), see the proof of Theorem 3.1 in Barrett & Prigozhin (2013b). □

3.4.2 *Convergence of (Q_r) to (Q).* On recalling (1.25d) and assuming (A5), the weak mixed formulation of the thin film superconductor problem is

(Q) Find $w \in H^1(0,T;H^{1/2}_{00}(\Omega))$ and $\underline{q} \in L^2(0,T;\underline{Z}^{\mathscr{M}}(\Omega))$ such that

$$\int_0^T \left[a\left(\frac{\partial w}{\partial t}, \eta\right) + \langle \underline{\nabla}\,.\,\underline{q}, \eta\rangle_{H^{1/2}_{00}(\Omega)} + \left(\frac{\mathrm{d}b_e}{\mathrm{d}t}, \eta\right)\right] \mathrm{d}t = 0 \quad \forall \eta \in L^2(0,T;H^{1/2}_{00}(\Omega)), \tag{3.75a}$$

$$\int_0^T [\langle |\underline{v}| - |\underline{q}|, k\rangle_{C(\bar{\Omega})} - \langle \underline{\nabla}\,.\,(\underline{v} - \underline{q}), w\rangle_{H^{1/2}_{00}(\Omega)}]\, \mathrm{d}t \geqslant 0 \quad \forall \underline{v} \in L^2(0,T;\underline{Z}^{\mathscr{M}}(\Omega)); \tag{3.75b}$$

where $w(\cdot,0) = w_0(\cdot)$.

Let

$$K_k := \{\eta \in W^{1,\infty}_0(\Omega) : |\underline{\nabla}\eta| \leqslant k \text{ a.e. in } \Omega\}. \tag{3.76}$$

Associated with the mixed formulation (Q) is the primal variational inequality:

(P) Find $w \in L^\infty(0,T;K_k) \cap H^1(0,T;H^{1/2}_{00}(\Omega))$ such that

$$\int_0^T \left[a\left(\frac{\partial w}{\partial t}, \eta - w\right) + \left(\frac{\mathrm{d}b_e}{\mathrm{d}t}, \eta - w\right)\right] \mathrm{d}t \geqslant 0 \quad \forall \eta \in L^2(0,T;K_k), \tag{3.77}$$

where $w(\cdot,0) = w_0(\cdot)$.

THEOREM 3.12 Let Assumptions (A1), (A2), (A3) and (A5) hold. Then there exists a subsequence of $\{w_r, \underline{q}_r\}_{r\in(1,4/3)}$ (not indicated), where $\{w_r, \underline{q}_r\}$ is the unique solution of (Q_r), such that as $r \to 1$

$$w_r \to w \quad \text{weak-}\star \text{ in } L^\infty(0,T;H^{1/2}_{00}(\Omega)),\ \text{weakly in } L^2(0,T;H^1_0(\Omega)), \tag{3.78a}$$

$$\frac{\partial w_r}{\partial t} \to \frac{\partial w}{\partial t} \quad \text{weakly in } L^2(0,T;H^{1/2}_{00}(\Omega)), \tag{3.78b}$$

$$\underline{q}_r \to \underline{q} \quad \text{weakly in } L^2(0,T;[\mathscr{M}(\bar{\Omega})]^2), \tag{3.78c}$$

$$\underline{\nabla}\,.\,\underline{q}_r \to \underline{\nabla}\,.\,\underline{q} \quad \text{weakly in } L^2(0,T;[H^{1/2}_{00}(\Omega)]^\star), \tag{3.78d}$$

where $\{w, \underline{q}\}$ solves (Q), (3.75a,3.75b). In addition, w is unique; and the possible nonuniqueness in $\underline{q}$ is restricted to the following: If there were two solutions $\underline{q}^i$, $i = 1,2$, then

$$\underline{\nabla}\,.\,(\underline{q}^2 - \underline{q}^1) = 0 \quad \text{a.e. in } \Omega_T \quad \text{and} \quad \int_0^T \langle |\underline{q}^2|, k\rangle_{C(\bar{\Omega})}\, \mathrm{d}t = \int_0^T \langle |\underline{q}^1|, k\rangle_{C(\bar{\Omega})}\, \mathrm{d}t. \tag{3.79}$$

Finally, w is the unique solution of (P), (3.77).

Proof. See the proof of Theorem 3.2 in Barrett & Prigozhin (2013b). □

4. Numerical algorithm and simulation results

Our iterative procedure for solving the nth step of $(\mathrm{Q}_r^{h,\tau})$, (2.18a,2.18b), for W_r^n and $\underline{Q}_r^n$ is as follows.

Set $W_r^{n,0} = W_r^{n-1} \in N_0^h$ and $\underline{Q}_r^{n,0} = \underline{Q}_r^{n-1} \in \underline{S}^h$. For $m \geqslant 1$, given iterates $W_r^{n,m-1} \in N_0^h$ and $\underline{Q}_r^{n,m-1} \in \underline{S}^h$, we use the following linearized version of (2.18b):

$$\mathfrak{M}^{h,n}(P^h W_r^{n,m-1})[|\underline{Q}_r^{n,m-1}|_\delta^{r-2}\underline{Q}_r^{n,m} - (|\underline{Q}_r^{n,m-1}|_\delta^{r-2} - |\underline{Q}_r^{n,m-1}|^{r-2})\underline{Q}_r^{n,m-1}] + \underline{\nabla}_h W_r^{n,m} = \underline{0}, \tag{4.1}$$

where $|\underline{v}|_\delta := (|\underline{v}|^2 + \delta^2)^{1/2}$ with $\delta^2 \ll 1$, to obtain

$$\underline{Q}_r^{n,m} = \underline{Q}_r^{n,m-1} - \frac{|\underline{Q}_r^{n,m-1}|^{r-2}\underline{Q}_r^{n,m-1} + [\mathfrak{M}^{h,n}(P^h W_r^{n,m-1})]^{-1}\underline{\nabla}_h W_r^{n,m}}{|\underline{Q}_r^{n,m-1}|_\delta^{r-2}}. \tag{4.2}$$

Substituting (4.2) into an iterative version of (2.18a), yields the following linear system for $W_r^{n,m} \in N_0^h$

$$\mathscr{A}_h\left(\frac{W_r^{n,m} - W_r^{n-1}}{\tau_n}, \eta^h\right) + ([\mathfrak{M}^{h,n}(P^h W_r^{n,m-1})]^{-1}|\underline{Q}_r^{n,m-1}|_\delta^{2-r}\underline{\nabla}_h W_r^{n,m}, \underline{\nabla}_h \eta^h)$$
$$= (\mathscr{F}^n, \eta^h) + (|\underline{Q}_r^{n,m-1}|_\delta^{2-r}[|\underline{Q}_r^{n,m-1}|_\delta^{r-2} - |\underline{Q}_r^{n,m-1}|^{r-2}]\underline{Q}_r^{n,m-1}, \underline{\nabla}_h \eta^h) \quad \forall \eta^h \in N_0^h. \tag{4.3}$$

Clearly, the linear system (4.3) is well-posed. Solving it for $W_r^{n,m} \in N_0^h$, we then obtain $\underline{Q}_r^{n,m} \in \underline{S}^h$ from (4.2).

Prior to the next iteration, $\underline{Q}_r^{n,m}$ is then replaced by $\alpha\underline{Q}_r^{n,m} + (1-\alpha)\underline{Q}_r^{n,m-1}$, where $\alpha \in (0,1)$ (under-relaxation) was sometimes needed for convergence in case (i) and $\alpha > 1$ (over-relaxation) led to acceleration of convergence in cases (ii) and (iii). Although we have no convergence proof of this procedure, in practice it worked well. In particular, the number of iterations was almost independent of the value of r, and only depended slightly on the mesh size. We note that similar algorithms have been used in Barrett & Prigozhin (2010, 2012, 2013a,b), but there a linear system of similar size to (4.3) was solved on each iteration for the dual variable $\underline{Q}_r^{n,m}$ and then $W_r^{n,m}$ was updated explicitly.

Being a solution to the quasi-variational inequality (P), the primal variable w is rate-independent. It can be shown, similarly to Barrett & Prigozhin (2013b, Section 4), that if the direction of the dual variable $\underline{q}$ does not change with time a.e. in the incident set $|\nabla w| = \mathfrak{M}(w)$ and this set increases monotonically in time, then the primal variable w at time t depends solely on w^0 and $\int_0^t \mathscr{F}(\cdot,s)\,\mathrm{d}s$. These conditions are satisfied in our examples below. However, the dual variable $\underline{q}$ is not rate-independent. Hence, our time-step strategy for approximating both w and $\underline{q}$ at time T, on assuming that they are changing gradually with time, was to choose a large time step τ_1 followed by a small time step $\tau_2 = T - \tau_1$. Then $W_r^2(\cdot)$ is regarded as an approximation to $w(\cdot,T)$, whereas $\underline{Q}_r^2(\cdot)$ can be regarded as an approximation to either the mean of $\underline{q}(\cdot,t)$ over the time interval $(T-\tau_2, T)$ or, as we did in this work, to $\underline{q}(\cdot, T - 0.5\tau_2)$.

As in Barrett & Prigozhin (2013a), for ease of implementation in case (i) we replaced w_0^ε by w_0 in (1.30a) and (2.11). Throughout, we set $\delta = 10^{-10}$, chose $r = 1 + 10^{-9}$, and adopted the stopping criterion

$$\frac{|\pi_N^h[|W_r^{n,m} - W_r^{n,m-1}|]|_{0,1,\Omega}}{|\pi_N^h[|W_r^{n,m}|]|_{0,1,\Omega}} < 10^{-6} \quad \text{and} \quad \frac{|\underline{Q}_r^{n,m} - \underline{Q}_r^{n,m-1}|_{0,1,\Omega}}{|\underline{Q}_r^{n,m}|_{0,1,\Omega}} < 2 \times 10^{-5}. \tag{4.4}$$

The simulations have been performed in Matlab R2012b (64 bit) on a PC with an Intel Core i5-2400 3.1 GHz processor and 16Gb RAM. The Matlab PDE Toolbox was used for the triangulation of Ω,

TABLE 1 *Growing sandpile. Approximation by the Raviart–Thomas (RT) and the nonconforming linear (NC) element*

					Number of iterations	
h	Finite element	$\delta(w)$ (%)	$\delta(\underline{q})$ (%)	CPU time (min)	Step 1	Step 2
0.04	RT	0.38	5.0	1.0	187	204
	NC	0.26	4.3	1.1	314	847
0.02	RT	0.14	2.5	5.3	255	185
	NC	0.08	2.3	6.8	400	1032

which was quasi-uniform. Although for the convergence analysis in the previous sections, we assumed, for ease of exposition, that Ω was polygonal and that the bilinear form $c(\underline{\nabla}_h\cdot, \underline{\nabla}_h\cdot)$ on $N_0^h \times N_0^h$ was calculated exactly; in practice curved domain boundaries were approximated by polygonal ones and $c(\underline{\nabla}_h\cdot, \underline{\nabla}_h\cdot)$ was approximated, see the appendix in Barrett & Prigozhin (2012) for details.

To compare our nonconforming approximations $(Q_r^{h,\tau})$, (2.18a,2.18b), with those in Barrett & Prigozhin (2010, 2013a,b) based on the Raviart–Thomas element, we considered three problems with known analytical solutions.

Our first example is a sandpile growing upon the initial support surface $w_0 = \max(0.4 - |\underline{x}|, 0)$ below the source f, which was uniform in its support $|\underline{x}| \leqslant 0.2$ with $\int_\Omega f \, \mathrm{d}\underline{x} = 1$. Owing to the radial symmetry, the analytical solution to the unregularized problem is easily found, see Barrett & Prigozhin (2013a). We approximated the regularized (with $\varepsilon = 0.01$ in (1.10)) quasi-variational inequality problem in the square $\Omega = (-1, 1)^2$, with the internal friction of sand $k_0 = 0.4$, both by $(Q_r^{h,\tau})$, (2.18a,2.18b), proposed in this work and by the Raviart–Thomas approximation in Barrett & Prigozhin (2013a). In both cases two time steps, $\tau_1 = 0.19$ and $\tau_2 = 0.01$, were made to obtain the approximation at $T = 0.2$. On recalling (2.4), we estimated the relative errors by

$$\delta(w) := \frac{|P^h W_r^2 - w^*|_{0,1,\Omega}}{|w^*|_{0,1,\Omega}} \quad \text{and} \quad \delta(\underline{q}) := \frac{|\underline{Q}_r^2 - \underline{q}^*|_{0,1,\Omega}}{|\underline{q}^*|_{0,1,\Omega}},$$

for two meshes with $h = 0.04$ and $h = 0.02$. Here, $w^* \in S^h$ and $\underline{q}^* \in \underline{S}^h$ with $w^*|_\sigma = w(\underline{x}^\sigma, 0.2)$ and $\underline{q}^*|_\sigma = \underline{q}(\underline{x}^\sigma, 0.195)$, where $\underline{x}^\sigma$ is the centroid of triangle $\sigma \in \mathcal{T}^h$. For the Raviart–Thomas approximation, Barrett & Prigozhin (2013a), the best convergence was achieved with $\alpha = 0.7$ (under-relaxation), while for $(Q_r^{h,\tau})$ no relaxation was needed with the fastest convergence being for $\alpha = 1$. Although more iterations at each time step were needed, the latter method produced a more accurate approximation, see Table 1, and was much simpler to realize.

As our second example, let us consider a cylindrical superconductor and assume the Kim critical state model with $j_c(b) = (1 + |b|/B_0)^{-1}$ with $B_0 \in \mathbb{R}_{>0}$, where we recall (1.14). Let $w_0 = 0$ and the external field grow monotonically, $\mathrm{d}b_e/\mathrm{d}t \geqslant 0$, with $b_e(0) = 0$. Then the magnetic field in the superconductor, $b(\underline{x}, t) = w(\underline{x}, t) + b_e(t)$, can be found analytically, see Barrett & Prigozhin (2010). At any point in time, this field is a function of the distance $d(\underline{x}) := \mathrm{dist}(\underline{x}, \partial\Omega)$ to the domain boundary: $b(\underline{x}, t) = [u(d(\underline{x}), t)]_+$, where $s_+ := \max\{s, 0\}$ and $u(d, t)$ satisfies $\partial u/\partial d = -j_c(u)$ with $u(0, t) = b_e(t)$. Solving this equation, we obtain that

$$b(\underline{x}, t) = -B_0 + \sqrt{B_0^2 + 2B_0[d_0(t) - d(\underline{x})]_+}, \tag{4.5}$$

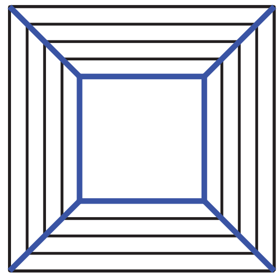

FIG. 1. Current streamlines (thin black) and current density discontinuity lines (thick blue).

TABLE 2 *Cylindrical superconductor*

h	Finite element	$\delta(w)$ (%)	$\delta(\underline{q})$ (%)	CPU time (min)	Number of iterations Step 1	Step 2
0.02	RT	0.25	3.5	4.5	44	78
	NC	0.15	3.5	0.4	168	268
0.01	RT	0.07	2.0	95	61	89
	NC	0.05	1.9	2.8	219	355

where $d_0(t) = b_e(t)(1 + 0.5b_e(t)/B_0)$ is the depth of the field penetration zone at time t. The current density $\underline{j}$ is critical in the penetration zone, $|\underline{j}(\underline{x}, t)| = j_c(b(\underline{x}, t))$ for $d(\underline{x}) \leqslant d_0(t)$, and zero outside of it. As $\underline{j} = \underline{\nabla} \times w = \underline{\nabla} \times b$, the current streamlines are the level contours of b. It is more difficult to find the electric field for a general domain Ω but, if Ω is a rectangle, the analytical solution for $\underline{e}$ can be found in Brandt (1995) for the Bean model, and can be easily extended to the Kim model with a field-dependent critical current density.

If $\Omega = (0, 1)^2$, the field penetration zone consists of four regions of unidirectional current density, see Fig. 1. Current discontinuity lines separate these regions from each other and the central zero current region. Noting that the direction of the electric field should coincide with that of the current density and, as $\nabla \times \underline{e} = -\mathrm{d}b_e/\mathrm{d}t$, the tangential component of $\underline{e}$ must be continuous along these discontinuity lines, it follows that the electric field should vanish on these lines. Let $R_1(t) := \{\underline{x} \in \Omega : x_1 \in (0, 1),\ x_2 \in (0, s(x_1, t))\}$, where $x_2 = s(x_1, t) := \min(x_1, d_0(t), 1 - x_1)$ for $x_1 \in [0, 1]$ is part of the discontinuity lines. We have that $\underline{e} = [e_1(\underline{x}, t), 0]^{\mathrm{T}}$ in $R_1(t)$ and $e_1(x_1, s(x_1, t), t) = 0$ for $x_1 \in [0, 1]$. Faraday's law, which in $R_1(t)$ reduces to $\partial e_1/\partial x_2 = \partial b/\partial t$, and (4.5) yield that

$$e_1(\underline{x}, t) = \left[\sqrt{B_0^2 + 2B_0(d_0(t) - s(x_1, t))} - \sqrt{B_0^2 + 2B_0(d_0(t) - x_2)}\right]\frac{\mathrm{d}}{\mathrm{d}t} d_0(t) \quad \text{in } R_1(t).$$

Similarly, one can find the electric field in the three other regions of the penetration zone. Solving the problem numerically, we chose $B_0 = 0.05$, $b_e(t) = t$ and used two time steps, $\tau_1 = 0.09$ and $\tau_2 = 0.01$ to find the numerical solution at $T = 0.1$; see Table 2 for a comparison of $(Q_r^{h,\tau})$, (2.18a,2.18b), to the method in Barrett & Prigozhin (2010) based on the Raviart–Thomas element.

For both methods, over-relaxation with $\alpha = 1.8$ led to the fastest convergence. The finite element scheme in Barrett & Prigozhin (2010) was based on the modified formulation (1.32) of (1.31b) which,

TABLE 3 *Thin film magnetization*

h	Finite element	$\delta(\underline{j})$ (%)	$\delta(\underline{q})$ (%)	CPU time (min)	Number of iterations Step 1	Step 2
0.06	RT	0.89	3.3	4.1	69	176
	NC	0.15	0.3	2.6	76	82
0.03	RT	0.46	1.3	131	100	202
	NC	0.06	0.2	127	76	80

probably, was less efficiently realized in our program. This could be the reason for the vast difference in computation times of the two methods in this case, even though less iterations were needed for the method in Barrett & Prigozhin (2010). However, the programming of this scheme is more involved, and the computed primal variable is less accurate.

Our last example is the magnetization of a thin superconducting disc. For the Bean model, $j_c \equiv 1$, the sheet current density and the magnetic field are known, see Mikheenko & Kuzovlev (1993) and Clem & Sanchez (1994). Using this analytical solution, the electric field can also be calculated, see Barrett & Prigozhin (2012). The primal variable, the magnetization function w, in thin film magnetization problems is an auxiliary variable. Of main interest in such problems are the sheet current density $\underline{j} = \underline{\nabla} \times w$ and the electric field $\underline{e}$. In addition, the magnetic field can be determined from $\underline{j}$ by means of the Biot–Savart law, (1.16). To compare the nonconforming approximation $(Q_r^{h,\tau})$, (2.18a,2.18b), with the Raviart–Thomas approximation in Barrett & Prigozhin (2012, 2013b), we present the numerical errors for the two main variables, $\delta(\underline{j})$ and $\delta(\underline{e}) = \delta(\underline{q})$ in Table 3, where $\delta(\underline{j})$ is defined similarly to $\delta(\underline{q})$. Since the bilinear form $c(\underline{\nabla}_h\cdot, \underline{\nabla}_h\cdot)$ on $N_0^h \times N_0^h$ leads to a dense matrix, the numerical solution of (4.3) is both memory and time consuming for fine meshes. We note that the computation times in Table 3 do not include the time for assembling the entries of $c(\underline{\nabla}_h\cdot, \underline{\nabla}_h\cdot)$ on $N_0^h \times N_0^h$. Here we recall that these entries were approximated, see the appendix in Barrett & Prigozhin (2012) for details. In this example, we chose Ω to be the unit disc, $b_e(t) = t$, and found the numerical solution at $T = 0.65$ using two time steps, $\tau_1 = 0.6$ and $\tau_2 = 0.05$. Over-relaxation with $\alpha = 1.8$ was employed in both iterative procedures.

For the approximation in Barrett & Prigozhin (2012, 2013b), employing the lowest order Raviart–Thomas element for $\underline{q}$ and the continuous piecewise linear element for w, the approximate current density was calculated directly as $\underline{J}_r^n = \underline{\nabla} \times W_r^n \in \underline{S}^h$. The same approach was used here for the nonconforming approximation on each element $\sigma \in \mathscr{T}^h$. However, we note that such a simple procedure may lead to an inaccurate approximation of $\underline{j}$ in thin film problems involving transport currents, which lead to nonhomogenous time-dependent boundary data for w_r and singular time-dependent forcing data $\mathscr{F}$ in (1.31a,1.31b). Problems of this type have been approximated using the appropriately modified nonconforming approximation $(Q_r^{h,\tau})$, (2.18a,2.18b), in Barrett *et al.* (2013). There, on recalling (3.12) and (3.13a,3.13b), instead of setting $\underline{J}_r^n = \underline{\nabla} \times W_r^n$ on each $\sigma \in \mathscr{T}^h$, we set $\underline{J}_r^n = \underline{\nabla} \times \hat{W}_r^n \in \underline{S}^h$ and this led to a more accurate approximation of $\underline{j}$. We note that the cost of the postprocessing step (3.12) is negligible compared to solving (4.3).

REFERENCES

ARONSSON, G., EVANS, L. C. & WU, Y. (1996) Fast/slow diffusion and growing sandpiles. *J. Differential Equations*, **131**, 304–335.

BARRETT, J. W. & PRIGOZHIN, L. (2000) Bean's critical-state model as the $p \to \infty$ limit of an evolutionary p-Laplacian equation. *Nonlinear Anal.*, **42**, 977–993.

Barrett, J. W. & Prigozhin, L. (2006) Dual formulations in critical state problems. *Interfaces Free Bound.*, **8**, 349–370.

Barrett, J. W. & Prigozhin, L. (2010) A quasi-variational inequality problem in superconductivity. *Math. Models Methods Appl. Sci.*, **20**, 679–706.

Barrett, J. W. & Prigozhin, L. (2012) Electric field formulation for thin film magnetization problems. *Superconductor Sci. Technol.*, **25**, 104022.

Barrett, J. W. & Prigozhin, L. (2013a) A quasi-variational inequality problem arising in the modeling of growing sandpiles. *ESAIM: Math. Model. Numer. Anal.*, **47**, 1133–1165.

Barrett, J. W. & Prigozhin, L. (2013b) Existence and approximation of a mixed formulation for thin film magnetization problems in superconductivity. *Math. Models Methods Appl. Sci.*, to appear. Preprint available at http://www.worldscientific.com/doi/pdf/10.1142/S0218202513500747.

Barrett, J. W., Prigozhin, L. & Sokolovsky, V. (2013) Transport current and magnetization problems for thin type-II superconducting films. *Superconductor Sci. Technol.*, **26**, 105009.

Bean, C. P. (1964) Magnetization of high-field superconductors. *Rev. Mod. Phys.*, **36**, 31–39.

Brandt, E. H. (1995) Electric field in superconductors with rectangular cross section. *Phys. Rev. B*, **52**, 15442–15457.

Clem, J. R. & Sanchez, A. (1994) Hysteretic AC losses and susceptibility of thin superconducting disks. *Phys. Rev. B*, **50**, 9355–9362.

Dumont, S. & Igbida, N. (2009) On a dual formulation for the growing sandpile problem. *Eur. J. Appl. Math.*, **20**, 169–185.

Folland, G. B. (1984) *Real Analysis: Modern Techniques and their Applications*. Pure and Applied Mathematics. New York: Wiley.

Gilbarg, D. & Trudinger, N. S. (1983) *Elliptic Partial Differential Equations of Second Order*, 2nd edn. Grundlehren der Mathematischen Wissenschaften, vol. 224. Berlin: Springer.

Grisvard, P. (1985) *Elliptic Problems in Nonsmooth Domains*. Monographs and Studies in Mathematics, vol. 24. Boston, MA: Pitman.

Kim, Y. B., Hempstead, C. F. & Strnad, A. R. (1962) Critical persistent currents in hard superconductors. *Phys. Rev. Lett.*, **9**, 306–309.

Marini, L. D. (1985) An inexpensive method for the evaluation of the solution of the lowest order Raviart–Thomas mixed method. *SIAM J. Numer. Anal.*, **22**, 493–496.

Mikheenko, P. N. & Kuzovlev, Y. E. (1993) Inductance measurements of HTSC films with high critical currents. *Physica C*, **204**, 229–236.

Prigozhin, L. (1986) A quasivariational inequality in the problem of filling a shape. *USSR Comput. Math. Math. Phys.*, **26**, 74–79.

Prigozhin, L. (1993) A variational problem of bulk solids mechanics and free-surface segregation. *Chem. Eng. Sci.*, **48**, 3647–3656.

Prigozhin, L. (1994) Sandpiles and river networks: extended systems with nonlocal interactions. *Phys. Rev. E*, **49**, 1161–1167.

Prigozhin, L. (1996) Variational model of sandpile growth. *Eur. J. Appl. Math.*, **7**, 225–235.

Rodrigues, J. F. & Santos, L. (2012) Quasivariational solutions for first order quasilinear equations with gradient constraint. *Arch. Rat. Mech. Anal.*, **205**, 493–514.

Simon, J. (1987) Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl.*, **146**, 65–96.

Temam, R. (1984) *Navier–Stokes Equations. Theory and Numerical Analysis*. Studies in Mathematics and its Applications, vol. 2. Amsterdam: North-Holland.