Partial L^1 Monge-Kantorovich Problem: Variational Formulation and Numerical Approximation

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Abstract

We consider the Monge-Kantorovich problem with transportation cost equal to distance and a relaxed mass balance condition: instead of optimally transporting one given distribution of mass onto another with the same total mass, only a given amount of mass, m, has to be optimally transported. In this partial problem the given distributions are allowed to have different total masses and m should not exceed the least of them. We derive and analyze a variational formulation of the arising free boundary problem in optimal transportation. Furthermore, we introduce and analyse the finite element approximation of this formulation using the lowest order Raviart-Thomas element. Finally, we present some numerical experiments where both approximations to the optimal transportation domains and the optimal transport between them are computed.

1 Introduction

The classical Monge-Kantorovich (MK) problem in optimal transportation consists in finding an optimal mass preserving map from one distribution of mass, f^+, onto another one, f^-. In the relaxed formulation of this problem due to Kantorovich, the map is replaced by a transport plan, a measure minimizing the transportation cost

$$\mathcal{C} := \int_{\overline{\Omega}\times\overline{\Omega}} c(x,y)\,\gamma(x,y) \tag{1.1}$$

among all measures $\gamma \in \mathcal{M}^+(\overline{\Omega}\times\overline{\Omega})$ satisfying $\int_{A\times\overline{\Omega}}\gamma = \int_A f^+$ and $\int_{\overline{\Omega}\times A}\gamma = \int_A f^-$ for any Borel set $A \subseteq \overline{\Omega}$. Here Ω is a connected bounded open set in $\mathbb{R}^n$ such that the supports of f^+ and f^- are in $\overline{\Omega}$; and $c(\underline{x},\underline{y})$ is the cost function, usually determined by the distance $d_\Omega(\underline{x},\underline{y})$ measured inside $\overline{\Omega}$. In addition, $\mathcal{M}^+(\cdot)$ is the set of non-negative Radon measures, whilst the set of all Radon measures will be denoted by $\mathcal{M}(\cdot)$. Clearly, if a solution exists, the two distributions should have the same mass, $m = \int_{\overline{\Omega}} f^+ = \int_{\overline{\Omega}} f^- < \infty$.

The problem has a variety of applications and has been studied recently with a renewed interest, see [1, 13, 14, 21]. Numerical methods have been derived for the quadratic cost function (cost equals square of the distance), see [7, 3] and the references therein; and, more recently, for the so called L^1 MK problems where the cost function is linear (cost equals distance), see [5].

In the latter work the cost $c(\underline{x},\underline{y})$ of transporting one unit of mass from a point $\underline{x} \in \overline{\Omega}$ to a point $\underline{y} \in \overline{\Omega}$ was assumed equal to the generalized distance

$$d_{\Omega,k}(\underline{x},\underline{y}) = \inf\left\{\int_0^1 k(s(t))|s'(t)|\,dt : s \in C^{0,1}([0,1];\overline{\Omega}),\ s(0)=\underline{x},\ s(1)=\underline{y}\right\}, \tag{1.2}$$

where $k : \overline{\Omega} \to \mathbb{R}_{>0}$ is a given function. In this case an equivalent dual formulation of the MK problem, see [8] and also [1, 19], can be written for a vectorial measure $\underline{q}$, representing the transport flux,

$$\mathcal{C} = \min\left\{\int_{\overline{\Omega}} k\,|\underline{q}| \ :\ \underline{\nabla}\,.\,\underline{q} = f^+ - f^- \ \text{ in } \Omega \quad \text{and} \quad \underline{q}\,.\,\underline{\nu} = 0 \text{ on } \partial\Omega\right\}. \tag{1.3}$$

Here $\partial\Omega$ is the boundary of Ω with normal $\underline{\nu}$; and the constraints are understood in the sense:

$$-\langle \underline{q}, \underline{\nabla}\varphi\rangle_{C(\overline{\Omega})} = \langle f^+ - f^-, \varphi\rangle_{C(\overline{\Omega})} \qquad \forall\ \varphi \in C^1(\overline{\Omega})\,, \tag{1.4}$$

where $\langle\cdot,\cdot\rangle_{C(\overline{\Omega})}$ is the duality pairing on $[C(\overline{\Omega})]^\star \times C(\overline{\Omega})$ with $\mathcal{M}(\overline{\Omega}) \equiv [C(\overline{\Omega})]^\star$ being the dual of $C(\overline{\Omega})$, and is naturally extended to vector arguments. The dual formulation (1.3) was the basis of the present authors numerical approximation in [5]. We note that the flux contains all the information on the direction and density of the optimal transportation and its cost; but there as well as in this paper, we do not find the optimal plan γ.

If the total masses do not agree, the maximal mass that can be transferred from f^+ to f^- is $m = \min\{\int_{\overline{\Omega}} f^+, \int_{\overline{\Omega}} f^-\}$. Determining an optimal transportation plan for this amount m, we will call an *unbalanced* MK problem. One can also seek an optimal transportation plan for a given amount of mass, $m < \min\{\int_{\overline{\Omega}} f^+, \int_{\overline{\Omega}} f^-\}$, which we will call a *partial* MK problem. The unbalanced and partial MK problems are free boundary problems, as the optimal transportation domains, as well as the optimal plan, have to be found. These problems have been recently introduced and studied theoretically, mainly for the quadratic cost function, by Caffarelli and McCann [10]. In this work we consider the unbalanced and partial MK problems with the linear transportation cost (L^1 MK problems).

First, we reformulate the problem as a balanced one over an extended region $\overline{\Omega_E} \supset \overline{\Omega}$ by introducing fictitious sources and an auxiliary zone of free transportation Σ (a free Dirichlet region, see [9]) in such a way that if the balanced transportation plan is optimal, only the required mass m is directly transported in $\overline{\Omega}$ from f^+ to f^-, and this part of the transport plan is a solution to the partial MK problem.

Secondly, we make use of the dual formulation (1.3), which is equivalent to the balanced L^1 MK problem also if there is a closed free Dirichlet set. In this case $d_{\Omega_E,k}(x, y)$ becomes a semidistance, since $k|_\Sigma = 0$, and the equivalence was shown by Pratelli [19] under the assumption that $k \geq k_0 > 0$ in $\overline{\Omega_E} \setminus \Sigma$ and is a lower semicontinuous function. We will assume for simplicity that $k \equiv 1$ in $\overline{\Omega}$ and is extended to 1 in $\overline{\Omega_E} \setminus \Sigma$, and make use of the special structure of the auxiliary balanced problem to simplify the flux formulation (1.3), and arrive at a new variational formulation of the partial L^1 MK problem, see (2.7) below. A further simplification is possible for the unbalanced problem, where some constraints can be accounted for directly, see (2.10); and this is our reason to distinguish between these two cases. We note that a different approach to the resolution of unbalanced L^2 MK problems has been suggested in [6].

Recently, Ekeland [11] has studied theoretically, for the quadratic cost function, the following *optimal matching* problem. Several kinds of goods, whose spatial distributions are given measures $f^{(i)} \in \mathcal{M}^+(\overline{\Omega})$ with the same total $\int_{\overline{\Omega}} f^{(i)} = m$, have to be brought together to produce m units of a final product (one unit of each of these constitutive parts is necessary to manufacture one unit of the final product). It is required to determine the measure $g \in \mathcal{M}^+(\overline{\Omega})$ such that $\int_{\overline{\Omega}} g = m$ and the total cost of transporting all the measures $f^{(i)}$ onto g is minimal. Here we show that our variational formulation of the unbalanced MK problem can be easily adapted, see (2.11) below, and used for the numerical approximation of the partial L^1 optimal matching problem for a given $m = \int_{\overline{\Omega}} g \leq \min\{\int_{\overline{\Omega}} f^{(i)}\}$ with the measures $f^{(i)}$ not necessarily balanced.

To approximate the problems numerically, we first regularize the non-differentiable function $|\underline{v}|$ in (2.7), (2.10) and (2.11) by $\frac{1}{r}|\underline{v}|^r$. In Section 3, we prove existence and uniqueness to the Euler-Lagrange equations of these regularized problems; and, in addition, show subsequence convergence, as $r \to 1$, to the Euler-Lagrange inequality systems associated with (2.7), (2.10) and (2.11). This proves existence of a solution to these inequality systems and hence to the associated minimization problems (2.7), (2.10) and (2.11).

In Section 4 we discretize the resulting regularized Euler-Lagrange systems using the Raviart-Thomas finite element of the lowest order with vertex sampling on nonlinear terms. We prove well-posedness of these approximations; and, moreover, we prove convergence as the mesh parameter, h, goes to zero. In Section 5 we introduce, and prove convergence of, augmented Lagrangian methods to handle the constraints in the resulting problems. In Section 6 we discuss algorithms to solve the resulting nonlinear algebraic systems. Finally, in Section 7 we present numerical experiments to show the effectiveness of our approach. In these experiments, we use adaptive mesh refinement to improve the accuracy of the flux near its singularities and to enhance the resolution of the free boundaries surrounding the domains in $\overline{\Omega}$ from/to which the mass is directly transported.

Finally, we note that our formulation for the unbalanced MK problem collapses in the classical balanced case to an interesting modification of the standard variational formulation in terms of the transport flux. In addition, the derived numerical method for the unbalanced MK problem solves the balanced problem at least as efficiently as the mixed scheme in [5].

2 Unbalanced and Partial L^1 MK Problems

Let $\Omega \subset \mathbb{R}^n$ be a connected bounded open Lipschitz set, f^+, $f^- \in \mathcal{M}^+(\overline{\Omega})$ be the two non-negative mass distributions and $m \in (0, \min\{\int_{\overline{\Omega}} f^+, \int_{\overline{\Omega}} f^-\}]$ be the amount of mass that should be transported from f^+ to f^-. Following [10] we define the set $\Gamma_{\leq}(f^+, f^-)$ as the subset of $\mathcal{M}^+(\overline{\Omega}\times\overline{\Omega})$ whose left and right marginals are dominated by f^+ and f^-, respectively: i.e. $\gamma \in \Gamma_{\leq}(f^+, f^-)$ if $\gamma \in \mathcal{M}^+(\overline{\Omega}\times\overline{\Omega})$ and

$$\int_{A\times\overline{\Omega}} \gamma \leq \int_A f^+ \qquad \text{and} \qquad \int_{\overline{\Omega}\times A} \gamma \leq \int_A f^-$$

for all Borel sets $A \subseteq \overline{\Omega}$.

Our aim is to solve the partial (unbalanced) L^1 MK problem, i.e. to minimize the cost functional

$$\int_{\overline{\Omega}\times\overline{\Omega}} d_{\Omega,1}(\underline{x}, \underline{y})\, \gamma(\underline{x}, \underline{y})$$

over all measures in $\Gamma^m_{\leq}(f^+, f^-) := \{\gamma \in \Gamma_{\leq}(f^+, f^-) \ : \ \int_{\overline{\Omega}\times\overline{\Omega}} \gamma = m\}$. For ease of exposition, we have assumed here that $k \equiv 1$ in $\overline{\Omega}$.

Let us imbed the space $\mathbb{R}^n \equiv \{\underline{x}\} \equiv \{x_1, x_2, \cdots, x_n\}$ into $\mathbb{R}^{n+1} \equiv \{\underline{x}, x_{n+1}\}$ as the subspace $x_{n+1} = 0$. We then define two auxiliary sources, $\widetilde{f}^-$, $\widetilde{f}^+ \in \mathcal{M}^+(\overline{\Omega})$, supported in two parallel hyperplanes, $x_{n+1} = \ell$ and $x_{n+1} = -\ell$, respectively, where

$$\ell > \tfrac{1}{2} \max\{d_{\Omega,1}(\underline{x}, \underline{y}) \ : \ \underline{x} \in \operatorname{supp}(f^+),\ \underline{y} \in \operatorname{supp}(f^-)\} \tag{2.1}$$

and such that

$$\int_{\overline{\Omega}} \widetilde{f}^+ = \int_{\overline{\Omega}} f^- - m \qquad \text{and} \qquad \int_{\overline{\Omega}} \widetilde{f}^- = \int_{\overline{\Omega}} f^+ - m\,; \tag{2.2}$$

see Figure 1. We assume that the transport in these two auxiliary hyperplanes, $x_{n+1} = \pm\ell$, is free, i.e. $k = 0$ on them, so that the distribution of the sources in each hyperplane is

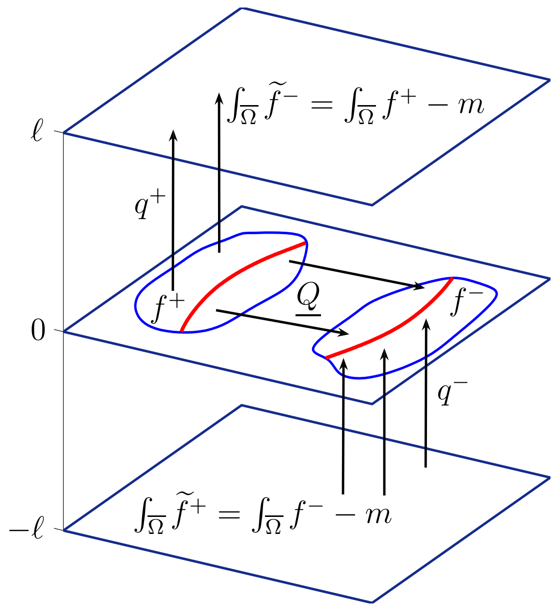

Figure 1: The planes, sources and fluxes.

unimportant. We assume that $k = 1$ everywhere else in $\mathbb{R}^{n+1}$. On noting (2.2), the resulting MK problem is then balanced in $\overline{\Omega_E} := \overline{\Omega} \times [-\ell, \ell]$ with

$$\int_{\overline{\Omega_E}} \left[f^+(\underline{x})\, \delta(x_{n+1}) + \widetilde{f}^+(\underline{x})\, \delta(x_{n+1} + \ell) \right] = \int_{\overline{\Omega}} \left[f^+ + \widetilde{f}^+ \right] = \int_{\overline{\Omega}} \left[f^+ + f^- \right] - m$$
$$= \int_{\overline{\Omega}} \left[f^- + \widetilde{f}^- \right] = \int_{\overline{\Omega_E}} \left[f^-(\underline{x})\, \delta(x_{n+1}) + \widetilde{f}^-(\underline{x})\, \delta(x_{n+1} - \ell) \right],$$

where $\delta \in \mathcal{M}(\mathbb{R})$ with $\int_{\mathbb{R}} \delta = 1$ and $\operatorname{supp}(\delta) = \{0\}$. Since the auxiliary hyperplanes, $x_{n+1} = \pm\ell$, are a distance ℓ from the main one, $x_{n+1} = 0$; it follows from (2.1) that the transportation cannot be optimal if any material is transported to the main hyperplane from an auxiliary one (or vice versa) and then back. Let $\Omega^\pm$ be open Lipschitz sets such that $\operatorname{supp}(f^\pm) \subseteq \overline{\Omega^\pm} \subseteq \overline{\Omega}$. Then as the transport in the auxiliary hyperplanes is free, optimal interplane transport brings, via straight transport rays orthogonal to the main hyperplane and all having the same length ℓ, the mass $\int_{\overline{\Omega}} \widetilde{f}^+$ from the lower auxiliary hyperplane to $\overline{\Omega^-}$ in the main hyperplane and transfers the mass $\int_{\overline{\Omega}} \widetilde{f}^-$ from $\overline{\Omega^+}$ in the main hyperplane to the upper hyperplane. We denote the corresponding fluxes as $q^\pm(\underline{x})\, \underline{e}_{n+1}$, $\underline{x} \in \overline{\Omega^\pm}$, respectively, where $\underline{e}_{n+1} \in \mathbb{R}^{n+1}$ is the unit vector in the direction of increasing x_{n+1}. We introduce also the characteristic function χ_A such that $\chi_A(y) = 1$ if $y \in A$, and otherwise $\chi_A(y) = 0$. The optimal flux, $\underline{q} \in \mathbb{R}^{n+1}$, in $\overline{\Omega_E}$ should then have the following form:

$$\underline{q}(\underline{x}, x_{n+1}) = \begin{bmatrix} \underline{Q}(\underline{x}) \\ 0 \end{bmatrix} \delta(x_{n+1}) + \left[q^{+}(\underline{x})\,\chi_{(0,\ell)}(x_{n+1}) + q^{-}(\underline{x})\,\chi_{(-\ell,0)}(x_{n+1})\right]\,\underline{e}_{n+1}$$
$$+ \begin{bmatrix} \underline{q}_F^{+}(\underline{x}) \\ 0 \end{bmatrix} \delta(x_{n+1} - \ell) + \begin{bmatrix} \underline{q}_F^{-}(\underline{x}) \\ 0 \end{bmatrix} \delta(x_{n+1} + \ell)$$
$$\forall\ \underline{x} \in \overline{\Omega},\ x_{n+1} \in [-\ell, \ell]\,. \qquad (2.3)$$

Here $\underline{Q}(\underline{x}) \in \mathbb{R}^n$ is the flux confined to the main hyperplane, $x_{n+1} = 0$, and $\underline{q}_F^{\pm}(\underline{x}) \in \mathbb{R}^n$ is the free transport flux in the auxiliary hyperplanes $x_{n+1} = \pm\ell$. In the above and below, we have extended $q^{\pm}$ from $\overline{\Omega^{\pm}}$ by zero.

Substituting (2.3) into corresponding dual formulation, (1.3), for the auxiliary balanced problem over $\overline{\Omega_E}$, we have that

$$\int_{\overline{\Omega}\times[-\ell,\ell]} k\,|\underline{q}| = \int_{\overline{\Omega}} \left[\,|\underline{Q}| + \ell\,(|q^{+}| + |q^{-}|)\,\right]\,. \qquad (2.4)$$

The corresponding mass balance equation, (1.4), over $\overline{\Omega_E}$ with the auxiliary sources yields for (2.3) that

$$-\langle \underline{Q}, \underline{\nabla}\varphi\rangle_{C(\overline{\Omega})} + \langle q^{+} - q^{-}, \varphi\rangle_{C(\overline{\Omega})} = \langle f^{+} - f^{-}, \varphi\rangle_{C(\overline{\Omega})} \qquad \forall\ \varphi \in C^1(\overline{\Omega})\,, \qquad (2.5a)$$
$$\langle \underline{q}_F^{+}, \underline{\nabla}\varphi\rangle_{C(\overline{\Omega})} + \langle q^{+}, \varphi\rangle_{C(\overline{\Omega})} = \langle \widetilde{f}^{-}, \varphi\rangle_{C(\overline{\Omega})} \qquad \forall\ \varphi \in C^1(\overline{\Omega})\,, \qquad (2.5b)$$
$$-\langle \underline{q}_F^{-}, \underline{\nabla}\varphi\rangle_{C(\overline{\Omega})} + \langle q^{-}, \varphi\rangle_{C(\overline{\Omega})} = \langle \widetilde{f}^{+}, \varphi\rangle_{C(\overline{\Omega})} \qquad \forall\ \varphi \in C^1(\overline{\Omega})\,. \qquad (2.5c)$$

As we are not directly interested in the free flux $\underline{q}_F^{\pm}$, since it does not appear in (2.4), and the distribution of the auxiliary sources $\widetilde{f}^{\pm}$ in $\overline{\Omega}$ is not important; we need only consider constant test functions φ in the equations (2.5b,c). In which case, they collapse, on noting (2.2), to the mass balance equations

$$\int_{\overline{\Omega}} q^{+} = \int_{\overline{\Omega}} f^{+} - m \qquad \text{and} \qquad \int_{\overline{\Omega}} q^{-} = \int_{\overline{\Omega}} f^{-} - m\,. \qquad (2.6)$$

Minimizing (2.4), on recalling from (2.1) that moving mass to/from an auxiliary plane and back cannot be optimal, yields that $q^{\pm} \geq 0$. On the other hand, the cost of transport between planes is the same for any admissible flux (2.3) with $q^{\pm} \geq 0$. Hence the transportation inside the main plane, $\underline{Q}$, should itself be optimal for any optimal plan (2.3) subject to the constraints (2.5a) and (2.6).

Defining $K_{\mathcal{P}} \subset [\mathcal{M}(\overline{\Omega})]^n \times \mathcal{M}(\overline{\Omega^{+}}) \times \mathcal{M}(\overline{\Omega^{-}})$ as the set of triples $(\underline{Q}, q^{+}, q^{-})$ satisfying (2.5a) and (2.6), we arrive at the following variational formulation of the partial L^1 MK problem for given ℓ satisfying (2.1):

($\mathcal{P}$) Minimize

$$\int_{\overline{\Omega}} \left[\,|\underline{V}| + \ell\,(|v^{+}| + |v^{-}|)\,\right] \qquad (2.7)$$

over all $(\underline{V}, v^+, v^-) \in K_{\mathcal{P}}$.

We note that the minimum value is $\mathcal{C} + \ell\,[\int_{\overline{\Omega}}(f^+ + f^-) - 2\,m]$. In addition, the problem can be simplified in the unbalanced case. If $m = \int_{\overline{\Omega}} f^+ \leq \int_{\overline{\Omega}} f^-$, then the minimizer $(\underline{Q}, q^+, q^-)$ of (2.7) is such that $q^+ \equiv 0$. If this were not the case, as $\int_{\overline{\Omega}} q^+ = 0$ on recalling (2.6), one can solve the balanced L^1 MK problem:

$$\min\left\{\int_{\overline{\Omega}} |\underline{V}| \;:\; \underline{\nabla}\,.\,\underline{V} = -q^+ \text{ in } \Omega \quad \text{and} \quad \underline{V}\,.\,\underline{\nu} = 0 \text{ on } \partial\Omega\right\}. \tag{2.8}$$

As a flux $\underline{Q}^\star$ solving (2.8) yields that $\mathcal{C}^\star = \int_{\overline{\Omega}} |\underline{Q}^\star| \leq \text{diam}(\Omega) \int_{\overline{\Omega}} |[q^+]_+| < \ell \int_{\overline{\Omega}} |q^+|$ if $\ell > \frac{1}{2}\,\text{diam}(\Omega)$ with respect to $d_{\Omega,1}$, recall (1.2); then $(\underline{Q} - \underline{Q}^\star, 0, q^-) \in K_{\mathcal{P}}$ and contradicts $(\underline{Q}, q^+, q^-)$ being a minimizer of (2.7). Hence $q^+ \equiv 0$ and (2.5a) yields that $q^- = \underline{\nabla}\,.\,\underline{Q} - f^+ + f^-$ in Ω and $\underline{Q}\,.\,\underline{\nu} = 0$ on $\partial\Omega$ in a weak sense. Hence both conditions in (2.6) are satisfied automatically; similarly, if $m = \int_{\overline{\Omega}} f^- \leq \int_{\overline{\Omega}} f^+$. Setting

$$\underline{V}_0^{\mathcal{M}}(\Omega) := \{\underline{v} \in [\mathcal{M}(\overline{\Omega})]^n : \exists\, w := \underline{\nabla}\,.\,\underline{v} \in \mathcal{M}(\overline{\Omega}) \text{ such that} \\ \langle w, \varphi\rangle_{C(\overline{\Omega})} = -\langle \underline{v}, \underline{\nabla}\varphi\rangle_{C(\overline{\Omega})} \quad \forall\, \varphi \in C^1(\overline{\Omega})\}\,; \tag{2.9}$$

we have the following variational formulation of the unbalanced L^1 MK problem for given ℓ satisfying (2.1):

$(\mathcal{U})$ Minimize

$$\int_{\overline{\Omega}} \left[\, |\underline{V}| + \ell\, |\underline{\nabla}\,.\,\underline{V} - f^+ + f^-|\,\right] \tag{2.10}$$

over all $\underline{V} \in \underline{V}_0^{\mathcal{M}}(\Omega)$.

For the balanced L^1 MK problem, $q^+ = q^- = 0$ and we deduce from (2.7) the known variational formulation (1.3) with $k \equiv 1$. However, here we obtain also (2.10) as an alternative formulation, which is equivalent provided ℓ satisfies the inequality (2.1).

Let us now assume we are given measures, $f^{(i)} \in \mathcal{M}^+(\overline{\Omega})$, $i = 1 \to I$, and we want to transport the same amount of mass, $m \leq \min\{\int_{\overline{\Omega}} f^{(i)}\}$, from each of these distributions onto another measure, $g \in \mathcal{M}^+(\overline{\Omega})$ with $\int_{\overline{\Omega}} g = m$. We need then to solve I unbalanced MK problems (2.10) with $f^+ = f^{(i)}$, $f^- = g$. Since these problems are independent, the optimal transportation fluxes $\underline{Q}^{(i)}$ can also be found, if g was given, by minimizing over $\{\underline{V}^{(i)}\}_{i=1}^I$, where $\underline{V}^{(i)} \in \underline{V}_0^{\mathcal{M}}(\Omega)$, the functional $\sum_{i=1}^I \int_{\overline{\Omega}} \left[|\underline{V}^{(i)}| + \ell\,|\underline{\nabla}\,.\,\underline{V}^{(i)} - f^{(i)} + g|\right]$ with ℓ sufficiently large; e.g. $\ell > \frac{1}{2}\text{diam}(\Omega)$. The partial optimal matching problem consists in determining the measure $g \in \mathcal{M}^+(\overline{\Omega})$ for which the total transportation cost $\sum_{i=1}^I \int_{\overline{\Omega}} |\underline{V}^{(i)}|$ is minimal. This leads to the following variational formulation of the partial L^1 optimal matching problem for given $\ell > \frac{1}{2}\,\text{diam}(\Omega)$:

$(\mathcal{PM})$ Minimize

$$\sum_{i=1}^{I} \int_{\overline{\Omega}} \left[\, |\underline{V}^{(i)}| + \ell\,|\underline{\nabla}\,.\,\underline{V}^{(i)} - f^{(i)} + g|\,\right] \tag{2.11}$$

over all $\{\underline{V}^{(i)}\}_{i=1}^{I}$, $\underline{V}^{(i)} \in \underline{V}_0^{\mathcal{M}}(\Omega)$, and $g \in \mathcal{M}^+(\overline{\Omega})$ satisfying $\int_{\overline{\Omega}} g = m$.

We note that in some applications one may want to restrict the support of g to a given set $\overline{\Omega_g} \subset \overline{\Omega}$, and hence seek $g \in \mathcal{M}^+(\overline{\Omega_g})$ satisfying $\int_{\overline{\Omega_g}} g = m$. In addition, one can simplify the problem by assuming that $g \in L^\infty(\Omega_g)$ with $g \leq G$ for a given constant $G \geq \frac{m}{|\Omega_g|}$; which we shall do in this paper.

We end this section with a few remarks about the notation employed in this paper. Throughout we adopt the standard notation for Sobolev spaces on a bounded Lipschitz open set D, denoting the norm of $W^{j,p}(D)$ ($j \in \mathbb{N}$, $p \in [1,\infty]$) by $\|\cdot\|_{j,p,D}$ and the semi-norm by $|\cdot|_{j,p,D}$. Of course, we have that $|\cdot|_{0,p,D} \equiv \|\cdot\|_{0,p,D}$. We extend these norms and semi-norms in the natural way to the corresponding spaces of vector valued functions. For $p = 2$, $W^{j,2}(D)$ will be denoted by $H^j(D)$ with the associated norm and semi-norm written as, respectively, $\|\cdot\|_{j,D}$ and $|\cdot|_{j,D}$. We introduce $L^p_M(D) := \{\eta \in L^p(D) : \int_D \eta \, d\underline{x} = 0\}$, and recall the Poincaré inequality for any $p \in [1,\infty]$:

$$|\eta|_{0,p,D} \leq C_D^\star \, |\underline{\nabla}\eta|_{0,p,D} \qquad \forall\ \eta \in W^{1,p}_M(D) := W^{1,p}(D) \cap L^p_M(D)\,, \tag{2.12}$$

where $C_D^\star$ depends on D, but is independent of p. The measure of D will be denoted by $|D|$. Throughout for any Banach space V, its dual will be denoted by $V^\star$.

For our regularized problems in the next section we require, for $r \in (1,\infty)$, the reflexive Banach space

$$\underline{V}_0^r(\Omega) := \{\underline{v} \in \underline{V}^r(\Omega)\ :\ \underline{v}\,.\,\underline{\nu} = 0 \text{ on } \partial\Omega\}, \tag{2.13a}$$

$$\text{where} \qquad \underline{V}^r(\Omega) := \{\underline{v} \in [L^r(\Omega)]^n\ :\ \underline{\nabla}\,.\,\underline{v} \in L^r(\Omega)\}, \tag{2.13b}$$

$$\text{with norm} \qquad \|\underline{v}\|_{\underline{V}^r(\Omega)} := \left[\, |\underline{v}|^r_{0,r,\Omega} + |\underline{\nabla}\cdot\underline{v}|^r_{0,r,\Omega}\,\right]^{\frac{1}{r}}\,. \tag{2.13c}$$

We note that $\underline{V}_0^r(\Omega)$ is the strong closure of $[C_0^\infty(\Omega)]^n$ in the norm $\|\cdot\|_{\underline{V}^r(\Omega)}$; e.g. this can be shown by a simple extension of the argument for the case $r = 2$ in [17, p26–29].

Let $C(\overline{D})$ denote the space of continuous functions on $\overline{D}$, and $C_M(\overline{D}) := \{\eta \in C(\overline{D}) : \int_D \eta = 0\}$. As one can identify $L^1(D)$ as a closed subspace of $\mathcal{M}(\overline{D})$, it is convenient to adopt the notation

$$\int_{\overline{D}} |\mu| \equiv \|\mu\|_{\mathcal{M}(\overline{D})} := \sup_{\eta \in C(\overline{D})} \frac{\langle \mu, \eta \rangle_{C(\overline{D})}}{|\eta|_{0,\infty,D}} < \infty\,. \tag{2.14}$$

We note that if $\{\mu_j\}_{j\geq 0}$ is a bounded sequence in $\mathcal{M}(\overline{D})$, then there exist a subsequence $\{\mu_{j_\ell}\}_{j_\ell \geq 0}$ and a $\mu \in \mathcal{M}(\overline{D})$ such that as $j_\ell \to \infty$

$$\mu_{j_\ell} \to \mu \quad \text{vaguely in } \mathcal{M}(\overline{D}); \quad \text{i.e.} \quad \langle \mu_{j_\ell} - \mu, \eta \rangle_{C(\overline{D})} \to 0 \quad \forall\ \eta \in C(\overline{D})\,. \tag{2.15}$$

In addition, we have that

$$\liminf_{j_\ell \to \infty} \int_{\overline{D}} |\mu_{j_\ell}| \geq \int_{\overline{D}} |\mu|\,; \tag{2.16}$$

see e.g. [12, p5] and [16, p223]. Moreover we note that for all $\underline{v} \in \underline{V}_0^{\mathcal{M}}(\Omega)$, there exists $\underline{v}_j \in [C_0^\infty(\Omega)]^n$ such that

$$\underline{v}_j \to \underline{v} \quad \text{vaguely in } [\mathcal{M}(\overline{\Omega})]^n\,, \quad \underline{\nabla}\,.\,\underline{v}_j \to \underline{\nabla}\,.\,\underline{v} \quad \text{vaguely in } \mathcal{M}(\overline{\Omega}) \quad \text{as } j \to \infty\,, \tag{2.17a}$$

$$\text{and} \qquad \limsup_{j\to\infty} \int_{\overline{\Omega}} |\underline{v}_j| \leq \int_{\overline{\Omega}} |\underline{v}|\,; \tag{2.17b}$$

e.g. see the proof of Lemma 2.4 in [5], which is based on the standard techniques of partition of unity, local change of variable and mollification. Similarly to (2.17a,b), for all $v \in \mathcal{M}(\overline{\Omega})$ there exist $v_j \in C_0^\infty(\Omega)$ such that

$$v_j \to v \quad \text{vaguely in } \mathcal{M}(\overline{\Omega}) \quad \text{as } j \to \infty\,, \qquad \text{and} \qquad \limsup_{j\to\infty} \int_{\overline{\Omega}} |v_j| \leq \int_{\overline{\Omega}} |v|\,. \tag{2.18}$$

Finally, throughout C denotes a generic positive constant independent of the regularization parameter, $r \in (1,\infty)$, and the mesh parameter h. Whereas, $C_{s_1,s_2,\cdots s_I}$ denotes a generic positive constant dependent on $\{s_i\}_{i=1}^I$.

3 Existence Theory for $(\mathcal{P})$, $(\mathcal{U})$ and $(\mathcal{PM})$ via Regularization

Firstly, we gather together our assumptions on the data.

(A1) Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded connected open set with a Lipschitz boundary $\partial\Omega$, if $n \geq 2$. For $(\mathcal{P})$ we assume that $f^\pm \in \mathcal{M}^+(\overline{\Omega^\pm})$ with $\Omega^\pm$ an open Lipschitz subset of Ω, and $m \in (0, \min\{\int_{\overline{\Omega^\pm}} f^\pm\}]$. In all cases for simplicity, we choose the parameter $\ell > \frac{1}{2}\,\mathrm{diam}(\Omega)$ with respect to $d_{\Omega,1}$, recall (1.2). Of course, one could assume the relaxed version (2.1) in the case of $(\mathcal{P})$ and $(\mathcal{U})$.

Throughout, we extend functions defined on subsets D of Ω by zero to Ω; and in addition, we define $(\eta_1,\eta_2)_D := \displaystyle\int_D \eta_1\,\eta_2\,d\underline{x}$, and this is naturally extended to vector functions.

For any $r > 1$, we regularize the non-differentiable nonlinearity $|\cdot|$ by the strictly convex function $\frac{1}{r}\,|\cdot|^r$. We note for all $\underline{b}$, $\underline{c} \in \mathbb{R}^n$ that

$$\frac{1}{r}\,\frac{\partial|\underline{b}|^r}{\partial b_i} = |\underline{b}|^{r-2}\,b_i \quad \Rightarrow \quad |\underline{b}|^{r-2}\,\underline{b}\,.\,(\underline{b}-\underline{c}) \geq \tfrac{1}{r}\,[\,|\underline{b}|^r - |\underline{c}|^r\,]\,. \tag{3.1}$$

In addition, for $r \in (1,2]$ we have for all $\underline{b}$, $\underline{c} \in \mathbb{R}^n$ that

$$\begin{aligned}\left(|\underline{b}|^{r-2}\,\underline{b} - |\underline{c}|^{r-2}\,\underline{c}\right)\,.\,(\underline{b}-\underline{c}) &= \left(|\underline{b}|^{r-1} - |\underline{c}|^{r-1}\right)\,(|\underline{b}|-|\underline{c}|) + \left(|\underline{b}|^{r-2} + |\underline{c}|^{r-2}\right)\,(|\underline{b}|\,|\underline{c}| - \underline{b}\,.\,\underline{c})\\ &\geq (r-1)\,(|\underline{b}|+|\underline{c}|)^{r-2}\,\left[\,(|\underline{b}|-|\underline{c}|)^2 + 2\,(|\underline{b}|\,|\underline{c}| - \underline{b}\,.\,\underline{c})\,\right]\\ &\geq (r-1)\,\left(|\underline{b}|+|\underline{c}|\right)^{r-2}\,|\underline{b}-\underline{c}|^2\,.\end{aligned} \tag{3.2}$$

3.1 Existence for $(\mathcal{P})$

For a given $r > 1$, on setting $p = \frac{r}{r-1}$ such that $\frac{1}{r} + \frac{1}{p} = 1$, we then consider the following problem for given $f_r^\pm \in L^r(\Omega_r^\pm)$ with $\overline{\Omega^\pm} \subseteq \overline{\Omega_r^\pm} \subseteq \overline{\Omega}$ and $(f_r^\pm, 1)_{\Omega_r^\pm} = \int_{\overline{\Omega^\pm}} f^\pm$:

$(\mathcal{P}_r)$ Find $\underline{Q}_r \in \underline{V}_0^r(\Omega)$, $q_r^\pm \in L^r(\Omega_r^\pm)$, $u_r \in L_M^p(\Omega)$ and $\lambda_r^\pm \in \mathbb{R}$ such that

$$(|\underline{Q}_r|^{r-2}\,\underline{Q}_r, \underline{v})_\Omega = (u_r, \underline{\nabla}\,.\,\underline{v})_\Omega \qquad \forall\ \underline{v} \in \underline{V}_0^r(\Omega)\,, \tag{3.3a}$$
$$\ell\,(|q_r^+|^{r-2}\,q_r^+, v^+)_{\Omega_r^+} = (u_r - \lambda_r^+, v^+)_{\Omega_r^+} \qquad \forall\ v^+ \in L^r(\Omega_r^+)\,, \tag{3.3b}$$
$$\ell\,(|q_r^-|^{r-2}\,q_r^-, v^-)_{\Omega_r^-} = -(u_r + \lambda_r^-, v^-)_{\Omega_r^-} \qquad \forall\ v^- \in L^r(\Omega_r^-)\,, \tag{3.3c}$$
$$(\underline{\nabla}\,.\,\underline{Q}_r, \eta)_\Omega = (f_r^+ - q_r^+, \eta)_{\Omega_r^+} - (f_r^- - q_r^-, \eta)_{\Omega_r^-} \qquad \forall\ \eta \in L^p(\Omega)\,, \tag{3.3d}$$
$$(f_r^+ - q_r^+, 1)_{\Omega_r^+} = (f_r^- - q_r^-, 1)_{\Omega_r^-} = m\,. \tag{3.3e}$$

For $\rho \in L_M^r(\Omega)$, let

$$\underline{X}^r(\rho) := \{\underline{v} \in \underline{V}_0^r(\Omega) : (\underline{\nabla}\,.\,\underline{v}, \eta)_\Omega = (\rho, \eta)_\Omega \quad \forall\ \eta \in L^p(\Omega)\}\,. \tag{3.4}$$

In addition, we introduce

$$\underline{Y}^r(f_r^+, f_r^-, m) := \{\ (\underline{v}, v^+, v^-) \in \underline{X}^r([f_r^+ - v^+] - [f_r^- - v^-]) \times L^r(\Omega_r^+) \times L^r(\Omega_r^-) : \\ (f_r^+ - v^+, 1)_{\Omega_r^+} = (f_r^- - v^-, 1)_{\Omega_r^-} = m\,\}\,. \tag{3.5}$$

It follows from (3.1), (3.4) and (3.5) that a solution of $(\mathcal{P}_r)$ is such that $(\underline{Q}_r, q_r^+, q_r^-) \in \underline{Y}^r(f_r^+, f_r^-, m)$ and

$$E_\ell^r(\underline{Q}_r, q_r^+, q_r^-) \leq E_\ell^r(\underline{v}, v^+, v^-) := \frac{1}{r}\left[\int_\Omega |\underline{v}|^r\,d\underline{x} + \ell \int_{\Omega_r^+} |v^+|^r\ d\underline{x} + \ell \int_{\Omega_r^-} |v^-|^r\ d\underline{x}\right] \\ \forall\ (\underline{v}, v^+, v^-) \in \underline{Y}^r(f_r^+, f_r^-, m)\,; \tag{3.6}$$

that is, a regularized version of $(\mathcal{P})$.

Lemma 3.1 *Let the Assumptions (A1) hold. Then for all $r > 1$ with $p = \frac{r}{r-1}$, given $\rho \in L_M^r(\Omega)$ there exists $\underline{Q}_r^\rho \in \underline{X}^r(\rho)$ and*

$$\|\underline{Q}_r^\rho\|_{\underline{V}^r(\Omega)} \leq (C_\Omega^\star + 1)\,|\rho|_{0,r,\Omega}\,, \tag{3.7}$$

where $C_\Omega^\star$ is the constant appearing in (2.12) with $D \equiv \Omega$. Hence it follows that

$$\inf_{\eta \in L_M^p(\Omega)} \sup_{\underline{v} \in \underline{V}_0^r(\Omega)} \frac{(\underline{\nabla}\,.\,\underline{v}, \eta)_\Omega}{\|\underline{v}\|_{\underline{V}^r(\Omega)}\,|\eta|_{0,p,\Omega}} \geq \beta^{-1}\,, \tag{3.8}$$

where $\beta = 2\,(C_\Omega^\star + 1)$ and so is independent of r and p.

Proof. See the proof of Lemma 2.1 in [5]. □

Theorem 3.1 *Let the Assumptions (A1) hold. Then for any given* $r \in (1,2]$ *there exists a unique solution,* $(\underline{Q}_r, q_r^+, q_r^-, u_r, \lambda_r^+, \lambda_r^-) \in \underline{V}_0^r(\Omega) \times L^r(\Omega_r^+) \times L^r(\Omega_r^-) \times L_M^p(\Omega) \times \mathbb{R} \times \mathbb{R}$, *to* $(\mathcal{P}_r)$*. In addition, we have that*

$$\|\underline{Q}_r\|^r_{\underline{V}^r(\Omega)} + \ell\, |q_r^+|^r_{0,r,\Omega_r^+} + \ell\, |q_r^-|^r_{0,r,\Omega_r^-} \leq C_{\Omega,\ell} \left[|f_r^+|^r_{0,r,\Omega_r^+} + |f_r^-|^r_{0,r,\Omega_r^-} \right], \tag{3.9a}$$

$$\|u_r\|_{1,p,\Omega} + |\Omega^+|^{\frac{r-1}{r}} |\lambda_r^+| + |\Omega^-|^{\frac{r-1}{r}} |\lambda_r^-| \leq C_{\Omega,\ell} \left[|f_r^+|^{r-1}_{0,r,\Omega_r^+} + |f_r^-|^{r-1}_{0,r,\Omega_r^-} \right]; \tag{3.9b}$$

where $p = \frac{r}{r-1}$*.*

Proof. For proving the existence and uniqueness of a solution to $(\mathcal{P}_r)$ and the bounds (3.9a–c); we adapt the proof of Theorem 2.1 in [5] for the balanced case, when $m = (f_r^\pm, 1)_{\Omega_r^\pm}$, $q_r^\pm \equiv 0$, and $\lambda_r^\pm = 0$.

Firstly, we define $q_r^{f,\pm} \in L^\infty(\Omega)$ such that

$$q_r^{f,\pm} := \begin{cases} \frac{(f_r^\pm,1)_{\Omega_r^\pm} - m}{|\Omega_r^\pm|} \geq 0 & \text{in } \Omega_r^\pm, \\ 0 & \text{in } \Omega \setminus \Omega_r^\pm \end{cases} \qquad \Rightarrow \qquad (f_r^\pm - q_r^{f,\pm}, 1)_{\Omega_r^\pm} = m\,. \tag{3.10}$$

It follows from (3.10) that

$$|q_r^{f,\pm}|_{0,r,\Omega_r^\pm} \leq |f_r^\pm|_{0,r,\Omega_r^\pm}\,. \tag{3.11}$$

Next, we set $\rho := [f_r^+ - q_r^{f,+}] - [f_r^- - q_r^{f,-}] \in L_M^r(\Omega)$. It follows from (3.7) and (3.11) that there exists $\underline{Q}_r^\rho \in \underline{X}^r(\rho)$ and

$$\|\underline{Q}_r^\rho\|_{\underline{V}^r(\Omega)} \leq (C_\Omega^\star + 1)\, |\rho|_{0,r,\Omega} \leq 2\,(C_\Omega^\star + 1)\, \left[|f_r^+|_{0,r,\Omega_r^+} + |f_r^-|_{0,r,\Omega_r^-} \right]. \tag{3.12}$$

Therefore, on setting $\widehat{\underline{Q}}_r := \underline{Q}_r - \underline{Q}_r^\rho$ and $\widehat{q}_r^\pm := q_r^\pm - q_r^{f,\pm}$, $(\mathcal{P}_r)$, (3.3a–e), can be reduced to: Find $(\widehat{\underline{Q}}_r, \widehat{q}_r^+, \widehat{q}_r^-) \in \underline{Y}^r(0,0,0)$ such that

$$\begin{aligned} (|\widehat{\underline{Q}}_r + \underline{Q}_r^\rho|^{r-2}(\widehat{\underline{Q}}_r + \underline{Q}_r^\rho), \underline{v})_\Omega + \ell\, (|\widehat{q}_r^+ + q_r^{f,+}|^{r-2}(\widehat{q}_r^+ + q_r^{f,+}), v^+)_{\Omega_r^+} \\ + \ell\, (|\widehat{q}_r^- + q_r^{f,-}|^{r-2}(\widehat{q}_r^- + q_r^{f,-}), v^-)_{\Omega_r^-} = 0 \qquad \forall\, (\underline{v}, v^+, v^-) \in \underline{Y}^r(0,0,0)\,; \end{aligned} \tag{3.13}$$

which, on recalling (3.6) and (3.1), is the Euler-Lagrange equation for the minimization problem

$$\inf_{(\underline{v},v^+,v^-) \in \underline{Y}^r(0,0,0)} E_\ell^r(\underline{v} + \underline{Q}_r^\rho, v^+ + q_r^{f,+}, v^- + q_r^{f,-})\,. \tag{3.14}$$

As $E_\ell^r(\cdot,\cdot)$ is strictly convex and continuous over the convex set $\underline{Y}^r(0,0,0)$, there exists a unique solution $(\widehat{\underline{Q}}_r, \widehat{q}_r^+, \widehat{q}_r^-) \in \underline{Y}^r(0,0,0)$ to (3.13). On setting $\underline{Q}_r = \widehat{\underline{Q}}_r + \underline{Q}_r^\rho$ and $q_r^\pm = \widehat{q}_r^\pm + q_r^{f,\pm}$, it follows from (3.13) that $(\underline{Q}_r, q_r^+, q_r^-) \in \underline{Y}^r(f_r^+, f_r^-, m)$ is such that

$$(|\underline{Q}_r|^{r-2}\underline{Q}_r, \underline{v})_\Omega + \ell\, (|q_r^+|^{r-2} q_r^+, v^+)_{\Omega_r^+} + \ell\, (|q_r^-|^{r-2} q_r^-, v^-)_{\Omega_r^-} = 0 \qquad \forall\, (\underline{v}, v^+, v^-) \in \underline{Y}^r(0,0,0)\,; \tag{3.15a}$$

and hence, in particular with $v^{\pm} \equiv 0$, that

$$(|\underline{Q}_r|^{r-2}\underline{Q}_r, \underline{v})_\Omega = 0 \qquad \forall\, \underline{v} \in \underline{X}^r(0). \tag{3.15b}$$

It is also easily deduced from (3.3a–e) and (3.5) that any solution $(\underline{Q}_r, q_r^{\pm})$ of $(\mathcal{P}_r)$ solves (3.15a), and from (3.2) that it is unique. In addition, it follows from (3.14) and (3.5) that

$$|\underline{Q}_r|^r_{0,r,\Omega} + \ell\,|q_r^+|^r_{0,r,\Omega_r^+} + \ell\,|q_r^-|^r_{0,r,\Omega_r^-} \leq |\underline{Q}_r^\rho|^r_{0,r,\Omega} + \ell\,|q_r^{f,+}|^r_{0,r,\Omega_r^+} + \ell\,|q_r^{f,-}|^r_{0,r,\Omega_r^-}\,, \tag{3.16a}$$

$$|\underline{\nabla}\,.\,\underline{Q}_r|^r_{0,r,\Omega} \leq \left[|f_r^+ - q_r^+|_{0,r,\Omega_r^+} + |f_r^- - q_r^-|_{0,r,\Omega_r^-}\right]^r. \tag{3.16b}$$

The bound (3.9a) follows immediately from (3.16a,b), (3.12) and (3.11). Obviously, the introduction of (3.14) is not necessary for proving the existence and uniqueness of (3.6); but is for the convenience in obtaining the bounds (3.9a,b).

Let $\mathcal{B} \equiv \underline{\nabla}\,.$, then it follows from (2.13a,b) that $\mathcal{B} : \underline{V}_0^r(\Omega) \to [L^p_M(\Omega)]^*$ and the dual operator $\mathcal{B}^* : L^p_M(\Omega) \to [\underline{V}_0^r(\Omega)]^*$. Moreover, it follows from (3.8), on noting Lemma I.4.1 and Remark I.4.2 in [17], that $\mathcal{B}^*$ is an isomorphism from $L^p_M(\Omega)$ onto $Z := \{\underline{v}^* \in [\underline{V}_0^r(\Omega)]^* : \langle \underline{v}^*, \underline{v}\rangle_{\underline{V}_0^r(\Omega)} = 0 \quad \forall\, \underline{v} \in \ker(\mathcal{B}) \subset \underline{V}_0^r(\Omega)\,\}$, where $\langle \cdot,\cdot\rangle_{\underline{V}_0^r(\Omega)}$ is the duality pairing on $[\underline{V}_0^r(\Omega)]^* \times \underline{V}_0^r(\Omega)$. Hence it follows from (3.15b), as $\ker(\mathcal{B}) \equiv \underline{X}^r(0)$, that there exists a unique $u_r \in L^p_M(\Omega)$ satisfying (3.3a). In addition, we have from (3.8) and (3.3a) that

$$|u_r|_{0,p,\Omega} \leq \beta \sup_{\underline{v} \in \underline{V}_0^r(\Omega)} \frac{(\,|\underline{Q}_r|^{r-2}\underline{Q}_r, \underline{v})_\Omega}{\|\underline{v}\|_{\underline{V}^r(\Omega)}} \leq \beta\,|\underline{Q}_r|^{r-1}_{0,r,\Omega}\,. \tag{3.17}$$

Choosing $v^{\mp} \equiv 0$ in (3.15a), we have that $v^{\pm} = \mp\underline{\nabla}\,.\,\underline{v}$; and hence, on noting (3.3a), that $\ell\,|q_r^{\pm}|^{r-2}q_r^{\pm} \mp u_r$ are constant on $\Omega_r^{\pm}$. On choosing

$$\lambda_r^{\pm} = -\ell\,|q_r^{\pm}|^{r-2}\,q_r^{\pm} \pm u_r\,, \tag{3.18}$$

we obtain that (3.3b,c) hold for general $v^{\pm} \in L^r(\Omega_r^{\pm})$. Therefore, we have proved that there exists a unique solution to $(\mathcal{P}_r)$, (3.3a–e), which is equivalent to the minimization problem (3.6). Furthermore, we have from (3.18) that

$$|\lambda_r^{\pm}| \leq |\Omega_r^{\pm}|^{-\frac{r-1}{r}} \left[\ell\,|q_r^{\pm}|^{r-1}_{0,r,\Omega_r^{\pm}} + |u_r|_{0,p,\Omega}\right]. \tag{3.19}$$

Finally, it follows from (3.3a) that

$$|(u_r, \underline{\nabla}\,.\,\underline{v})_\Omega| = |(|\underline{Q}_r|^{r-2}\underline{Q}_r, \underline{v})_\Omega| \leq |\underline{Q}_r|^{r-1}_{0,r,\Omega}\,|\underline{v}|_{0,r,\Omega} \qquad \forall\, \underline{v} \in \underline{V}_0^r(\Omega)\,. \tag{3.20}$$

Therefore noting that $\Omega^{\pm} \subseteq \Omega_r^{\pm}$, and combining (3.17), (3.9a), (3.19) and (3.20) immediately yield the desired result (3.9b). □

Next we are more precise about our choice of regularized data $f_r^{\pm}$ for $(\mathcal{P}_r)$. With $f^{\pm} \in \mathcal{M}^+(\overline{\Omega^{\pm}})$ being the data for problem $(\mathcal{P})$; we then choose, for any $r > 1$, corresponding regularized data $f_r^{\pm}$ as follows. If $f^{\pm} \in L^r(\Omega^{\pm})$ we set $f_r^{\pm} \equiv f^{\pm}$. If $\overline{\Omega^{\pm}} \subset \Omega$ we set

$$f_r^{\pm}(\underline{x}) := \tfrac{1}{|B(\underline{x},r-1)|}\,\langle f^{\pm}, 1\rangle_{C(B(\underline{x},r-1))} \qquad \forall\, \underline{x} \in \Omega\,, \tag{3.21}$$

where $B(\underline{x},\rho)$ is the closed ball in $\mathbb{R}^n$ centred at $\underline{x}$ with radius $\rho > 0$ and $f^\pm$ is extended from $\overline{\Omega^\pm}$ to $\mathbb{R}^n$ by zero. It follows for $r-1$ sufficiently small that

$$\overline{\Omega^\pm} \subseteq \overline{\Omega^\pm_r} := \overline{\Omega^\pm} \cup \operatorname{supp}(f^\pm_r) \subseteq \overline{\Omega}\,, \qquad f^\pm_r \in L^r(\Omega^\pm_r)$$
$$\text{with} \qquad f^\pm_r \geq 0 \quad \text{a.e. in } \Omega \qquad \text{and} \qquad (f^\pm_r, 1)_{\Omega^\pm_r} = \int_{\overline{\Omega^\pm}} f^\pm\,; \tag{3.22a}$$

and, moreover, that

$$\lim_{r\to 1} |\Omega^\pm_r \setminus \Omega^\pm| = 0\,, \qquad \limsup_{r\to 1} |f^\pm_r|^r_{0,r,\Omega^\pm_r} \leq \int_{\overline{\Omega^\pm}} |f^\pm|$$
$$\text{and} \qquad f^\pm_r \to f^\pm \quad \text{vaguely in } \mathcal{M}(\overline{\Omega}) \quad \text{as } r \to 1. \tag{3.22b}$$

For general $f^\pm \in \mathcal{M}^+(\overline{\Omega})$, the construction (3.21) can be modified, so that (3.22a,b) still hold. For example, one can partition Ω into a finite number of strictly star-shaped sets with "centres" $\underline{x}_\ell$ and employ the local change of variable $\underline{\tau}_t(\underline{x}) = \underline{x}_\ell + t\,(\underline{x} - \underline{x}_\ell)$ for $t \in (0,1)$, inducing a push forward measure, before applying the mollification (3.21); for details see the proof of Lemma 2.4 in [5], where these techniques are used.

In the theorem below, we will show that the unique solution of $(\mathcal{P}_r)$ with the above choice of data $f^\pm_r$ converges, as $r \to 1$, to a solution of:

$(\widehat{\mathcal{P}})$ Find $\underline{Q} \in \underline{V}^{\mathcal{M}}_0(\Omega)$, $q^\pm \in \mathcal{M}(\overline{\Omega^\pm})$, $u \in C_M(\overline{\Omega})$ and $\lambda^\pm \in \mathbb{R}$ such that

$$\int_{\overline{\Omega}} |\underline{v}| - \int_{\overline{\Omega}} |\underline{Q}| \geq \langle \underline{\nabla}\,.\,(\underline{v} - \underline{Q}), u\rangle_{C(\overline{\Omega})} \qquad \forall\, \underline{v} \in \underline{V}^{\mathcal{M}}_0(\Omega)\,, \tag{3.23a}$$
$$\ell\left[\int_{\overline{\Omega^+}} |v^+| - \int_{\overline{\Omega^+}} |q^+|\right] \geq \langle v^+ - q^+, u - \lambda^+\rangle_{C(\overline{\Omega^+})} \qquad \forall\, v^+ \in \mathcal{M}(\overline{\Omega^+})\,, \tag{3.23b}$$
$$\ell\left[\int_{\overline{\Omega^-}} |v^-| - \int_{\overline{\Omega^-}} |q^-|\right] \geq -\langle v^- - q^-, u + \lambda^-\rangle_{C(\overline{\Omega^-})} \qquad \forall\, v^- \in \mathcal{M}(\overline{\Omega^-})\,, \tag{3.23c}$$
$$\langle \underline{\nabla}\,.\,\underline{Q}, \eta\rangle_{C(\overline{\Omega})} = \langle f^+ - q^+, \eta\rangle_{C(\overline{\Omega^+})} - \langle f^- - q^-, \eta\rangle_{C(\overline{\Omega^-})} \qquad \forall\, \eta \in C(\overline{\Omega})\,, \tag{3.23d}$$
$$\int_{\overline{\Omega^+}} (f^+ - q^+) = \int_{\overline{\Omega^-}} (f^- - q^-) = m\,. \tag{3.23e}$$

We now introduce the measure analogues of (3.4) and (3.5). For $\rho \in \mathcal{M}_M(\overline{\Omega}) := \{\mu \in \mathcal{M}(\overline{\Omega}) : \int_{\overline{\Omega}} \mu = 0\}$, let

$$\underline{X}^{\mathcal{M}}(\rho) := \left\{\underline{v} \in \underline{V}^{\mathcal{M}}_0(\Omega) : \langle \underline{\nabla}\,.\,\underline{v}, \eta\rangle_{C(\overline{\Omega})} = \langle \rho, \eta\rangle_{C(\overline{\Omega})} \quad \forall\, \eta \in C(\overline{\Omega})\right\}\,; \tag{3.24}$$

and

$$\underline{Y}^{\mathcal{M}}(f^+, f^-, m) := \Bigg\{ (\underline{v}, v^+, v^-) \in \underline{X}^{\mathcal{M}}([f^+ - v^+] - [f^-_r - v^-]) \times \mathcal{M}(\overline{\Omega^+}) \times \mathcal{M}(\overline{\Omega^-}) :$$
$$\int_{\overline{\Omega^+}} (f^+ - v^+) = \int_{\overline{\Omega^-}} (f^- - v^-) = m \Bigg\}\,. \tag{3.25}$$

It follows from (3.23a–e), (3.24) and (3.25) that a solution of $(\widehat{\mathcal{P}})$ is such that $(\underline{Q}, q^+, q^-) \in \underline{Y}^{\mathcal{M}}(f^+, f^-, m)$ and

$$E_\ell^{\mathcal{M}}(\underline{Q}, q^+, q^-) \le E_\ell^{\mathcal{M}}(\underline{v}, v^+, v^-) := \int_\Omega |\underline{v}| + \ell \int_{\overline{\Omega^+}} |v^+| + \ell \int_{\overline{\Omega^-}} |v^-| \qquad \forall\, (\underline{v}, v^+, v^-) \in \underline{Y}^{\mathcal{M}}(f^+, f^-, m)\,; \quad (3.26)$$

that is; $(\mathcal{P})$, (2.7), but where we have restricted the support of $\underline{q}^\pm$ to $\overline{\Omega^\pm}$. Hence, proving existence for $(\widehat{\mathcal{P}})$ yields existence for $(\mathcal{P})$ too. However, as the solution of $(\mathcal{P})$ is possibly not unique, we do not have the equivalence of the problems $(\widehat{\mathcal{P}})$ and $(\mathcal{P})$.

Theorem 3.2 *Let the Assumptions (A1) hold. Then there exists a subsequence of* $\{(\underline{Q}_{r_j}, q^\pm_{r_j}, u_{r_j}, \lambda^\pm_{r_j})\}_{r_j>1}$ *of* $\{(\underline{Q}_r, q^\pm_r, u_r, \lambda^\pm_r)\}_{r>1}$*, where* $(\underline{Q}_r, q^+_r, q^-_r, u_r, \lambda^+_r, \lambda^-_r) \in \underline{V}^r_0(\Omega) \times L^r(\Omega^+_r) \times L^r(\Omega^-_r) \times L^p_M(\Omega) \times \mathbb{R} \times \mathbb{R}$ *is the unique solution of* $(\mathcal{P}_r)$ *with* $f^\pm_r$ *satisfying (3.22a,b), such that as* $r_j \to 1$

$$\begin{aligned}
\underline{Q}_{r_j} &\to \underline{Q} && \text{vaguely in } [\mathcal{M}(\overline{\Omega})]^n\,, && (3.27a)\\
\underline{\nabla} \cdot \underline{Q}_{r_j} &\to \underline{\nabla} \cdot \underline{Q} && \text{vaguely in } \mathcal{M}(\overline{\Omega}), && (3.27b)\\
q^\pm_{r_j} &\to q^\pm && \text{vaguely in } \mathcal{M}(\overline{\Omega})\,, && (3.27c)\\
u_{r_j} &\to u && \text{strongly in } C(\overline{\Omega})\,; && (3.27d)\\
\lambda^\pm_{r_j} &\to \lambda^\pm\,. && && (3.27e)
\end{aligned}$$

Moreover, $(\underline{Q}, q^+, q^-, u, \lambda^+, \lambda^-) \in \underline{V}^{\mathcal{M}}_0(\Omega) \times \mathcal{M}(\overline{\Omega^+}) \times \mathcal{M}(\overline{\Omega^-}) \times C_M(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ *solves* $(\widehat{\mathcal{P}})$*, (3.23a–e).*

Proof. It follows from (3.9a,b) and (3.22a,b) that for all r sufficiently close to 1

$$|\underline{Q}_r|_{0,1,\Omega} + |\underline{\nabla} \cdot \underline{Q}_r|_{0,1,\Omega} + |q^\pm_r|_{0,1,\Omega^\pm_r} + \|u_r\|_{1,p^\star,\Omega} + |\lambda^\pm_r| \le C_{\Omega,\Omega^\pm,\ell,f^\pm}, \qquad (3.28)$$

where $p^\star > n$. The subsequence convergence results (3.27a–e), where $\{\underline{Q}, q^\pm, u, \lambda^\pm\} \in \underline{V}^{\mathcal{M}}_0(\Omega) \times \mathcal{M}(\overline{\Omega}) \times \mathcal{M}(\overline{\Omega}) \times C(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ then follow immediately from (3.28), (2.9) and noting the compact Sobolev embedding $W^{1,p^\star}(\Omega) \hookrightarrow\hookrightarrow C(\overline{\Omega})$. As $(u_r, 1)_\Omega = 0$ for all $r > 1$, and recalling (3.22b) it follows that the limits $\{q^\pm, u\} \in \mathcal{M}(\overline{\Omega^\pm}) \times C_M(\overline{\Omega})$.

Passing to the $r_j \to 1$ limit in the r_j versions of (3.3d) with $\eta \in C(\overline{\Omega})$ and (3.3e) yields, on noting (3.27b,c) and (3.22b), that (3.23d,e) hold.

For any $\underline{\xi} \in [C^\infty_0(\Omega)]^n$, we choose $\underline{v} = \underline{\xi} - \underline{Q}_{r_j}$ in the r_j version of (3.3a) to obtain, on noting (3.1), that

$$(u_{r_j}, \underline{\nabla} \cdot (\underline{Q}_{r_j} - \underline{\xi}))_\Omega = (|\underline{Q}_{r_j}|^{r_j-2}\underline{Q}_{r_j}, \underline{Q}_{r_j} - \underline{\xi})_\Omega \ge \frac{1}{r_j}\int_\Omega |\underline{Q}_{r_j}|^{r_j}\, d\underline{x} - \frac{1}{r_j}\int_\Omega |\underline{\xi}|^{r_j}\, d\underline{x}\,. \qquad (3.29)$$

For all $\underline{\xi} \in [C_0^\infty(\Omega)]^n$, we have that

$$\frac{1}{r}\int_\Omega |\underline{\xi}|^r \, d\underline{x} \to \int_\Omega |\underline{\xi}| \, d\underline{x} \qquad \text{as } r \to 1\,; \tag{3.30}$$

and it follows from (3.27b,d) that

$$(\underline{\nabla}\,.\,(\underline{Q}_{r_j} - \underline{\xi}), u_{r_j})_\Omega \to \langle \underline{\nabla}\,.\,(\underline{Q} - \underline{\xi}), u\rangle_{C(\overline{\Omega})} \qquad \text{as } r_j \to 1. \tag{3.31}$$

Next we note from (3.27a) and (2.16) that

$$\liminf_{r_j \to 1} \frac{1}{r_j}\int_\Omega |\underline{Q}_{r_j}|^{r_j} \, d\underline{x} \geq \liminf_{r_j \to 1} \int_\Omega |\underline{Q}_{r_j}| \, d\underline{x} \geq \int_{\overline{\Omega}} |\underline{Q}|\,. \tag{3.32}$$

Combining (3.29)–(3.32) yields that

$$\int_{\overline{\Omega}} |\underline{\xi}| - \int_{\overline{\Omega}} |\underline{Q}| \geq \langle \underline{\nabla}\,.\,(\underline{\xi} - \underline{Q}), u\rangle_{C(\overline{\Omega})} \qquad \forall\, \underline{\xi} \in [C_0^\infty(\Omega)]^n. \tag{3.33}$$

Noting (2.17a,b) yields the desired result (3.23a).

Similarly, using (3.27c–e) one can pass to the limit in the r_j versions of (3.3b,c) with $v^\pm = \xi^\pm - q^\pm_{r_j}$, for any $\xi^\pm \in C_0^\infty(\Omega^\pm)$, to obtain (3.23b,c) with $v^\pm = \xi^\pm$. Noting (2.18) then yields the desired results (3.23b,c). Hence we have that $(\underline{Q}, q^\pm, u, \lambda^\pm) \in \underline{V}_0^{\mathcal{M}}(\Omega) \times \mathcal{M}(\overline{\Omega^+}) \times \mathcal{M}(\overline{\Omega^-}) \times C_M(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ solves $(\widehat{\mathcal{P}})$. $\square$

In the next two subsections, we adapt the arguments above for $(\mathcal{P})$ to show the existence of $(\mathcal{U})$, (2.10), and $(\mathcal{PM})$, (2.11), respectively. In both cases, we just briefly state the key differences.

3.2 Existence for $(\mathcal{U})$

As we saw in Section 2, if we solve $(\mathcal{P})$, (2.7), with $\int_{\overline{\Omega^+}} f^+ = m < \int_{\overline{\Omega^-}} f^-$ then $q^+ \equiv 0$, or similarly if $\int_{\overline{\Omega^-}} f^- = m < \int_{\overline{\Omega^+}} f^+$ then $q^- \equiv 0$. Hence existence of a solution to $(\mathcal{U})$, (2.10), follows immediately from our existence proof for $(\mathcal{P})$ above. However, as our numerical approximation of $(\mathcal{U})$ is based on the discretization of a regularized problem $(\mathcal{U}_r)$ for $r > 1$, we now show that $(\mathcal{U}_r)$ converges to $(\mathcal{U})$ as $r \to 1$.

We shall assume here, in addition to the assumptions (A1), that Ω is star-shaped and $f^\pm \in L^{r_0}(\Omega^\pm)$ for some $r_0 > 1$. Let $f := f^+ - f^-$.

For a given $r \in (1, r_0)$, we then consider the following problem :

$(\mathcal{U}_r)$ Find $\underline{Q}_r \in \underline{V}_0^r(\Omega)$ such that

$$(|\underline{Q}_r|^{r-2}\,\underline{Q}_r, \underline{v})_\Omega + \ell\,(|\underline{\nabla}\,.\,\underline{Q}_r - f|^{r-2}\,(\underline{\nabla}\,.\,\underline{Q}_r - f), \underline{\nabla}\,.\,\underline{v})_\Omega = 0 \qquad \forall\, \underline{v} \in \underline{V}_0^r(\Omega)\,. \tag{3.34}$$

$(\mathcal{U}_r)$ is the Euler-Lagrange equation for the strictly convex minimization problem: Find $\underline{Q}_r \in \underline{V}_0^r(\Omega)$ such that

$$E_\ell^r(\underline{Q}_r) \leq E_\ell^r(\underline{v}) := \frac{1}{r} \int_\Omega [\, |\underline{v}|^r + \ell\, |\underline{\nabla} \cdot \underline{v} - f|^r \,] \, d\underline{x} \qquad \forall\, \underline{v} \in \underline{V}_0^r(\Omega)\,. \tag{3.35}$$

Hence, there exists a unique solution $\underline{Q}_r \in \underline{V}_0^r(\Omega)$ to $(\mathcal{U}_r)$, (3.34) and (3.35) are equivalent problems; and moreover,

$$|\underline{Q}_r|_{0,r,\Omega}^r + \ell\, |\underline{\nabla} \cdot \underline{Q}_r - f|_{0,r,\Omega}^r \leq \ell\, |f|_{0,r,\Omega}^r\,. \tag{3.36}$$

It follows from (3.36) that there exists $\underline{Q} \in \underline{V}_0^{\mathcal{M}}(\Omega)$ such that there exists a subsequence $\{\underline{Q}_{r_j}\}_{r_j>1}$, where $\underline{Q}_r$ is the unique solution of $(\mathcal{U}_r)$, such that (3.27a,b) hold as $r_j \to 1$.

For any $\underline{\xi} \in [C_0^\infty(\Omega)]^n$, we have that $E_\ell^{r_j}(\underline{\xi}) \geq E_\ell^{r_j}(\underline{Q}_{r_j})$. Similarly to (3.32), we have on noting (3.27a,b) and (2.16) that

$$\liminf_{r_j \to 1} E_\ell^{r_j}(\underline{Q}_{r_j}) \geq \int_\Omega [\, |\underline{Q}| + \ell\, |\underline{\nabla} \cdot \underline{Q} - f| \,]\,. \tag{3.37}$$

Similarly to (3.30), we have that

$$\limsup_{r_j \to 1} E_\ell^{r_j}(\underline{\xi}) \leq \int_\Omega [\, |\underline{\xi}| + \ell\, |\underline{\nabla} \cdot \underline{\xi} - f| \,]\,. \tag{3.38}$$

Hence, it follows on combining the above that $\underline{Q} \in \underline{V}_0^{\mathcal{M}}(\Omega)$ is such that

$$\int_\Omega [\, |\underline{\xi}| + \ell\, |\underline{\nabla} \cdot \underline{\xi} - f| \,] \geq \int_\Omega [\, |\underline{Q}| + \ell\, |\underline{\nabla} \cdot \underline{Q} - f| \,] \qquad \forall\, \underline{\xi} \in [C_0^\infty(\Omega)]^n\,. \tag{3.39}$$

Without loss of generality, we assume that the "centre" of the star-shaped Ω is the origin, and set $d := \operatorname{dist}(\underline{0}, \partial\Omega)$. Let $j \in C^\infty(\mathbb{R}^n)$, with compact support in $B(\underline{0},1)$, such that

$$\int_{B(\underline{0},1)} j(\underline{x})\, d\underline{x} = 1, \qquad j(\underline{x}) \geq 0 \qquad \text{and} \qquad j(-\underline{x}) = j(\underline{x})\,.$$

For any $\varepsilon > 0$, let $J_\varepsilon : \mathcal{M}(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n)$ be such that

$$(J_\varepsilon\, \eta)(\underline{x}) = \langle \eta(\cdot), j_\varepsilon(\underline{x} - \cdot)\rangle_{C(\mathbb{R}^n)} \qquad \forall\, \underline{x} \in \mathbb{R}^n\,,$$

where $j_\varepsilon(\underline{x}) = \varepsilon^{-n} j(\varepsilon^{-1}\underline{x})$; and let $\underline{\tau}_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ be such that $\underline{\tau}_\varepsilon(\underline{x}) = (1+\varepsilon)^{-1}\, \underline{x}$ inducing the push forward $\underline{\tau}_{\varepsilon\#} : \mathcal{M}(\mathbb{R}^n) \to \mathcal{M}(\mathbb{R}^n)$, see e.g. [2, p32]. We extend J_ε and $\underline{\tau}_{\varepsilon\#}$ in the natural way, so that $\underline{J}_\varepsilon : [\mathcal{M}(\mathbb{R}^n)]^n \to [C_0^\infty(\mathbb{R}^n)]^n$ and $\underline{\tau}_{\varepsilon\#} : [\mathcal{M}(\mathbb{R}^n)]^n \to [\mathcal{M}(\mathbb{R}^n)]^n$. It follows for any $v \in \mathcal{M}(\overline{\Omega})$ that $v_k := J_{\frac{d}{3k}}(\underline{\tau}_{\frac{1}{k}\#} v) \in C_0^\infty(\Omega)$ and $\{v_k\}_{k\geq 1}$ satisfy (2.18) as $k \to \infty$; and for any $\underline{v} \in \underline{V}_0^{\mathcal{M}}(\Omega)$ that $\underline{v}_k := \underline{J}_{\frac{d}{3k}}(\underline{\tau}_{\frac{1}{k}\#}\underline{v}) \in [C_0^\infty(\Omega)]^n$ and $\{\underline{v}_k\}_{k\geq 1}$ satisfy (2.17a,b) as $k \to \infty$. We note that

$$\underline{\nabla} \cdot (\underline{J}_{\frac{d}{3k}}(\underline{\tau}_{\frac{1}{k}\#}\underline{v})) = J_{\frac{d}{3k}}(\underline{\nabla} \cdot (\underline{\tau}_{\frac{1}{k}\#}\underline{v})) = (1 + \tfrac{1}{k})\, J_{\frac{d}{3k}}(\, \underline{\tau}_{\frac{1}{k}\#}\, (\underline{\nabla} \cdot \underline{v})\,) \qquad \forall\, \underline{v} \in \underline{V}_0^{\mathcal{M}}(\Omega)\,, \tag{3.40}$$

and

$$\int_\Omega |J_{\frac{d}{3k}}(\,\tau_{\frac{1}{k}\#}\,v)|\,d\underline{x} \leq \int_{\overline{\Omega}} |\tau_{\frac{1}{k}\#}\,v| \leq \int_{\overline{\Omega}} |v| \qquad \forall\, v \in \mathcal{M}(\overline{\Omega})\,. \tag{3.41}$$

In addition, we note that for any $r \in [1, \infty)$ and any $v \in L^r(\Omega)$ that

$$|v - J_{\frac{d}{3k}}(\tau_{\frac{1}{k}\#}v)|_{0,r,\Omega} \to 0 \quad \text{as} \quad k \to \infty. \tag{3.42}$$

For any $\underline{v} \in \underline{V}_0^{\mathcal{M}}(\Omega)$, we then choose $\underline{\xi} \equiv \underline{J}_{\frac{d}{3k}}(\tau_{\frac{1}{k}\#}\underline{v})$ in (3.39). It follows from (2.17b), (3.40), (3.41) and (3.42) that

$$\begin{aligned}
\limsup_{k\to\infty} \int_{\overline{\Omega}} &\left[|\underline{J}_{\frac{d}{3k}}(\tau_{\frac{1}{k}\#}\underline{v})| + \ell\, |\underline{\nabla}\,.\,(\underline{J}_{\frac{d}{3k}}(\tau_{\frac{1}{k}\#}\underline{v})) - f| \right] \\
&\leq \int_{\overline{\Omega}} |\underline{v}| + \ell \limsup_{k\to\infty} \int_{\overline{\Omega}} \left[|\underline{\nabla}\,.\,(\underline{J}_{\frac{d}{3k}}(\tau_{\frac{1}{k}\#}\underline{v})) - J_{\frac{d}{3k}}(\tau_{\frac{1}{k}\#}f)| + |f - J_{\frac{d}{3k}}(\tau_{\frac{1}{k}\#}f)| \right] \\
&\leq \int_{\overline{\Omega}} |\underline{v}| + \ell \limsup_{k\to\infty} \int_{\overline{\Omega}} |(1 + \tfrac{1}{k})\,(\underline{\nabla}\,.\,\underline{v}) - f| \leq \int_{\overline{\Omega}} [\,|\underline{v}| + \ell\, |\underline{\nabla}\,.\,\underline{v} - f|\,]\,.
\end{aligned} \tag{3.43}$$

Hence, a solution to

$(\mathcal{U})$ Find $\underline{Q} \in \underline{V}_0^{\mathcal{M}}(\Omega)$ such that

$$\int_{\overline{\Omega}} [\,|\underline{v}| + \ell\, |\underline{\nabla}\,.\,\underline{v} - f|\,] \geq \int_{\overline{\Omega}} [\,|\underline{Q}| + \ell\, |\underline{\nabla}\,.\,\underline{Q} - f|\,] \qquad \forall\, \underline{v} \in \underline{V}_0^{\mathcal{M}}(\Omega)\,; \tag{3.44}$$

that is, (2.10) exists (which is already known for more general data via $(\mathcal{P})$) and, moreover, is the limit of the solutions to the regularized problems $(\mathcal{U}_r)$ as $r \to 1$.

3.3 Existence for $(\mathcal{PM})$

Similarly to the previous subsection, we shall assume here, in addition to the assumptions (A1), that Ω is star-shaped and that the non-negative $f^{(i)} \in L^{r_0}(\Omega)$, $i = 1 \to I$, for some $r_0 > 1$. In addition, we assume that $m \in (0, \min_{i=1\to I}\{\int_{\overline{\Omega}} f^{(i)}\}]$, and Ω_g is an open Lipschitz subset of Ω. Moreover, we will assume that for a given constant $G \geq \frac{m}{|\Omega_g|}$

$$\begin{aligned}
&g \in K_G(m) := \{\eta \in K_G : \int_{\Omega_g} \eta\, d\underline{x} = m\}\,, \\
\text{where} \qquad &K_G := \{\eta \in L^\infty(\Omega_g) : \eta \in [0, G] \text{ a.e. in } \Omega_g\}\,;
\end{aligned} \tag{3.45}$$

as opposed to $g \in \mathcal{M}^+(\overline{\Omega_g})$, with $\int_{\overline{\Omega_g}} g = m$.

For a given $r \in (1, r_0)$, we then consider the following problem

$(\mathcal{PM}_r)$ Find $\underline{Q}_r^{(i)} \in \underline{V}_0^r(\Omega)$, $i = 1 \to I$, and $g_r \in K_G(m)$ such that

$$(|\underline{Q}_r^{(i)}|^{r-2}\,\underline{Q}_r^{(i)}, \underline{v})_\Omega + \ell\,(|\underline{\nabla}\,.\,\underline{Q}_r^{(i)} - f^{(i)} + g_r|^{r-2}\,(\underline{\nabla}\,.\,\underline{Q}_r^{(i)} - f^{(i)} + g_r), \underline{\nabla}\,.\,\underline{v})_\Omega = 0 \qquad \forall\, \underline{v} \in \underline{V}_0^r(\Omega)\,,\ i = 1 \to I, \tag{3.46a}$$

$$\sum_{i=1}^{I}(|\underline{\nabla}\,.\,\underline{Q}_r^{(i)} - f^{(i)} + g_r|^{r-2}\,(\underline{\nabla}\,.\,\underline{Q}_r^{(i)} - f^{(i)} + g_r), \eta - g_r)_{\Omega_g} \geq 0 \qquad \forall\, \eta \in K_G(m)\,. \tag{3.46b}$$

$(\mathcal{PM}_r)$ is the Euler-Lagrange system for the strictly convex minimization problem: Find $\underline{Q}_r^{(i)} \in \underline{V}_0^r(\Omega)$, $i = 1 \to I$, and $g_r \in K_G(m)$ such that

$$E_\ell^r(\{\underline{Q}_r^{(i)}\}_{i=1}^I, g_r) \leq E_\ell^r(\{\underline{v}^{(i)}\}_{i=1}^I, \eta) := \frac{1}{r}\sum_{i=1}^{I}\int_\Omega \left[\,|\underline{v}^{(i)}|^r + \ell\,|\underline{\nabla}\,.\,\underline{v}^{(i)} - f^{(i)} + \eta|^r\,\right]\,d\underline{x} \qquad \forall\, \underline{v}^{(i)} \in \underline{V}_0^r(\Omega),\ i = 1 \to I, \quad \forall\, \eta \in K_G(m)\,. \tag{3.47}$$

Hence, there exists a unique solution $(\{\underline{Q}_r^{(i)}\}_{i=1}^I, g_r)$ to $(\mathcal{PM}_r)$, (3.46a,b) and (3.47) are equivalent problems; and moreover,

$$\sum_{i=1}^{I}\left[|\underline{Q}_r^{(i)}|_{0,r,\Omega}^r + \ell\,|\underline{\nabla}\,.\,\underline{Q}_r^{(i)} - f^{(i)} + g_r|_{0,r,\Omega}^r\right] \leq \ell\,\sum_{i=1}^{I}|\frac{m}{|\Omega_g|} - f^{(i)}|_{0,r,\Omega}^r\,. \tag{3.48}$$

It follows from (3.48) that there exist $\underline{Q}^{(i)} \in \underline{V}_0^{\mathcal{M}}(\Omega)$, $i = 1 \to I$, and $g \in K_G(m)$ such that there exists a subsequence $\{\{\underline{Q}_{r_j}^{(i)}\}_{=1}^I, g_{r_j}\}_{r_j>1}$, where $(\{\underline{Q}_r^{(i)}\}_{i=1}^I, g_r)$ is the unique solution of $(\mathcal{PM}_r)$, such that

$$\underline{Q}_{r_j}^{(i)} \to \underline{Q}^{(i)} \qquad \text{vaguely in } [\mathcal{M}(\overline{\Omega})]^n\,, \qquad i = 1 \to I, \tag{3.49a}$$

$$\underline{\nabla}\,.\,\underline{Q}_{r_j}^{(i)} \to \underline{\nabla}\,.\,\underline{Q}^{(i)} \qquad \text{vaguely in } \mathcal{M}(\overline{\Omega}), \qquad i = 1 \to I, \tag{3.49b}$$

$$g_{r_j} \to g \qquad \text{vaguely in } L^\infty(\Omega_g) \tag{3.49c}$$

hold as $r_j \to 1$.

For any $\underline{\xi}^{(i)} \in [C_0^\infty(\Omega)]^n$, $i = 1 \to I$, and $\eta \in K_g(m)$, we have that $E_\ell^{r_j}(\{\underline{\xi}^{(i)}\}_{i=1}^I, \eta) \geq E_\ell^{r_j}(\{\underline{Q}_{r_j}^{(i)}\}_{i=1}^I, g_{r_j})$. Similarly to (3.32), we have on noting (3.49a–c) and (2.16) that

$$\liminf_{r_j \to 1} E_\ell^{r_j}(\{\underline{Q}_{r_j}^{(i)}\}_{i=1}^I, g_{r_j}) \geq \sum_{i=1}^{I}\int_{\overline{\Omega}}\left[\,|\underline{Q}^{(i)}| + \ell\,|\underline{\nabla}\,.\,\underline{Q}^{(i)} - f^{(i)} + g|\,\right]\,. \tag{3.50}$$

Similarly to (3.30), we have that

$$\limsup_{r_j \to 1} E_\ell^{r_j}(\{\underline{\xi}^{(i)}\}_{i=1}^I, \eta) \leq \sum_{i=1}^{I}\int_{\overline{\Omega}}\left[\,|\underline{\xi}^{(i)}| + \ell\,|\underline{\nabla}\,.\,\underline{\xi}^{(i)} - f^{(i)} + \eta|\,\right]\,. \tag{3.51}$$

Hence, it follows on combining the above that $\underline{Q}^{(i)} \in \underline{V}_0^{\mathcal{M}}(\Omega)$, $i = 1 \to I$, and $g \in K_G(m)$ are such that

$$\sum_{i=1}^{I} \int_\Omega \Big[\, |\underline{\xi}^{(i)}| + \ell\, |\underline{\nabla}\,.\,\underline{\xi}^{(i)} - f^{(i)} + \eta| \,\Big] \geq \sum_{i=1}^{I} \int_\Omega \Big[\, |\underline{Q}^{(i)}| + \ell\, |\underline{\nabla}\,.\,\underline{Q}^{(i)} - f^{(i)} + g| \,\Big]$$
$$\forall\, \underline{\xi}^{(i)} \in [C_0^\infty(\Omega)]^n\,,\ i = 1 \to I, \quad \forall\, \eta \in K_G(m)\,. \quad (3.52)$$

For any $\underline{v}^{(i)} \in \underline{V}_0^{\mathcal{M}}(\Omega)$, we then choose $\underline{\xi}^{(i)} \equiv \underline{J}_{\frac{d}{3k}}(\underline{\tau}_{\frac{1}{k}\#}\underline{v}^{(i)})$ in (3.52). It follows from (2.17b), (3.40), (3.41) and (3.42) that for any $\eta \in K_G(m)$

$$\begin{aligned}
\limsup_{k\to\infty} &\int_\Omega \Big[|\underline{J}_{\frac{d}{3k}}(\underline{\tau}_{\frac{1}{k}\#}\underline{v}^{(i)})| + \ell\, |\underline{\nabla}\,.\,(\underline{J}_{\frac{d}{3k}}(\underline{\tau}_{\frac{1}{k}\#}\underline{v}^{(i)})) - f^{(i)} + \eta| \Big] \\
&\leq \int_\Omega |\underline{v}^{(i)}| + \ell\, \limsup_{k\to\infty} \int_\Omega \Big[|\underline{\nabla}\,.\,(\underline{J}_{\frac{d}{3k}}(\underline{\tau}_{\frac{1}{k}\#}\underline{v}^{(i)})) - J_{\frac{d}{3k}}(\tau_{\frac{1}{k}\#}(f^{(i)} - \eta))| \\
&\qquad\qquad + |(f^{(i)} - \eta) - J_{\frac{d}{3k}}(\tau_{\frac{1}{k}\#}(f^{(i)} - \eta))| \Big] \\
&\leq \int_\Omega |\underline{v}^{(i)}| + \ell\, \limsup_{k\to\infty} \int_\Omega |(1 + \tfrac{1}{k})\,(\underline{\nabla}\,.\,\underline{v}^{(i)}) - f^{(i)} + \eta| \\
&\leq \int_\Omega \big[\, |\underline{v}^{(i)}| + \ell\, |\underline{\nabla}\,.\,\underline{v}^{(i)} - f^{(i)} + \eta| \,\big]\,. \qquad (3.53)
\end{aligned}$$

Hence, there exists a solution to

$(\mathcal{PM})$ Find $\underline{Q}^{(i)} \in \underline{V}_0^{\mathcal{M}}(\Omega)$, $i = 1 \to I$, and $g \in K_G(m)$, such that

$$\int_\Omega \big[\, |\underline{v}| + \ell\, |\underline{\nabla}\,.\,\underline{v} - f^{(i)} + g| \,\big] \geq \int_\Omega \Big[\, |\underline{Q}^{(i)}| + \ell\, |\underline{\nabla}\,.\,\underline{Q}^{(i)} - f^{(i)} + g| \,\Big]$$
$$\forall\, \underline{v} \in \underline{V}_0^{\mathcal{M}}(\Omega)\,,\ i = 1 \to I, \quad (3.54a)$$

$$\sum_{i=1}^{I} \int_\Omega |\underline{\nabla}\,.\,\underline{Q}^{(i)} - f^{(i)} + \eta| \geq \sum_{i=1}^{I} \int_\Omega |\underline{\nabla}\,.\,\underline{Q}^{(i)} - f^{(i)} + g| \qquad \forall\, \eta \in K_G(m)\,; \quad (3.54b)$$

that is, (2.11) with the restriction of g to Ω_g and the upper bound constraint G on g.

4 Finite Element Approximation of $(\mathcal{P}_r)$, $(\mathcal{U}_r)$ and $(\mathcal{PM}_r)$

For ease of exposition, we assume the following.

(A2) Let $n \leq 3$ with Ω, $\Omega^\pm$ and Ω_g polytopes, if $n \geq 2$. Let $\{\mathcal{T}^h\}_{h>0}$ be a regular family of partitionings of Ω into disjoint open simplices σ with $h_\sigma := \mathrm{diam}(\sigma)$ and $h := \max_{\sigma\in\mathcal{T}^h} h_\sigma$, so that

$$\overline{\Omega} = \cup_{\sigma\in\mathcal{T}^h}\,\overline{\sigma}, \quad \text{with} \quad \overline{\Omega^\pm} = \cup_{\sigma\in\mathcal{T}^h_\pm}\,\overline{\sigma} \quad \text{and} \quad \overline{\Omega_g} = \cup_{\sigma\in\mathcal{T}^h_g}\,\overline{\sigma}\,. \quad (4.1)$$

We shall also assume that there exists $r_0 \in (1, \frac{4}{3}]$, required for the proof of Lemma 4.1 below, such that for all $r \in (1, r_0)$

$$\Omega^\pm_r \equiv \Omega^\pm \,. \tag{4.2}$$

Let $\underline{\nu}_{\partial\sigma}$ be the outward unit normal to $\partial\sigma$, the boundary of σ. We then introduce

$$\begin{aligned}
\underline{V}^h &:= \{\underline{v}^h \in [L^\infty(\Omega)]^n : \underline{v}^h\,|_\sigma = \underline{a}_\sigma + b_\sigma\,\underline{x},\ \underline{a}_\sigma \in \mathbb{R}^n,\ b_\sigma \in \mathbb{R} \quad \forall\,\sigma \in \mathcal{T}^h \text{ and} \\
&\qquad\qquad \underline{v}^h\,.\,\underline{\nu}_{\partial\sigma} \text{ is continuous across simplex boundaries}\} \\
&\subset \{\underline{v} \in [L^\infty(\Omega)]^n : \underline{\nabla}\,.\,\underline{v} \in L^\infty(\Omega)\}\,, && (4.3a)\\
S^h &:= \{\eta^h \in L^\infty(\Omega) : \eta^h\,|_\sigma = c_\sigma \in \mathbb{R} \quad \forall\,\sigma \in \mathcal{T}^h\,\}\,, && (4.3b)\\
S^h_\pm &:= \{\eta^h \in L^\infty(\Omega^\pm) : \eta^h\,|_\sigma = c_\sigma \in \mathbb{R} \quad \forall\,\sigma \in \mathcal{T}^h_\pm\,\}\,, && (4.3c)\\
S^h_g &:= \{\eta^h \in L^\infty(\Omega_g) : \eta^h\,|_\sigma = c_\sigma \in \mathbb{R} \quad \forall\,\sigma \in \mathcal{T}^h_g\,\}\,, && (4.3d)\\
\underline{V}^h_0 &:= \{\underline{v}^h \in \underline{V}^h : \underline{v}^h\,.\,\underline{\nu} = 0 \text{ on } \partial\Omega\} \quad \text{and} \quad S^h_M := \{\eta^h \in S^h : (\eta^h, 1)_\Omega = 0\,\}\,. && (4.3e)
\end{aligned}$$

In order for our finite element approximation to be practical, we introduce

$$(\underline{v}, \underline{z})^h := \sum_{\sigma\in\mathcal{T}^h} (\underline{v}, \underline{z})^h_\sigma \quad \text{with} \quad (\underline{v}, \underline{z})^h_\sigma := \tfrac{1}{n+1}\,|\sigma| \sum_{j=1}^{n+1} \underline{v}(P^\sigma_j)\,.\,\underline{z}(P^\sigma_j)$$
$$\forall\,\underline{v},\ \underline{z} \in [C(\overline{\sigma})]^n, \quad \forall\,\sigma \in \mathcal{T}^h\,; \tag{4.4}$$

where $\{P^\sigma_j\}_{j=1}^{n+1}$ are the vertices of σ. Therefore $(\underline{v}, \underline{z})^h$ averages the integrand $\underline{v}\,.\,\underline{z}$ over each simplex σ at its vertices and hence is exact if $\underline{v}\,.\,\underline{z}$ is piecewise linear over the partitioning $\mathcal{T}^h$. For any $r \geq 1$ and for any $\underline{v}^h \in \underline{V}^h$, we have from the equivalence of norms for $\underline{v}^h$ and the convexity of $|\cdot|^r$ that

$$C_r\,(\,|\underline{v}^h|^r, 1)^h_\sigma \leq \int_\sigma |\underline{v}^h|^r\,d\underline{x} \leq (\,|\underline{v}^h|^r, 1)^h_\sigma := \tfrac{1}{n+1}\,|\sigma| \sum_{j=1}^{n+1} |\underline{v}^h(P^\sigma_j)|^r \qquad \forall\,\sigma \in \mathcal{T}^h\,. \tag{4.5}$$

4.1 Approximation of $(\mathcal{P}_r)$

Our fully practical approximation of $(\mathcal{P}_r)$ for given $r \in (1, r_0)$, recall (4.2), by $\underline{V}^h_0$ and S^h, on employing (4.4), is then:

$(\mathcal{P}^h_r)$ Find $\underline{Q}^h_r \in \underline{V}^h_0$, $q^{\pm,h}_r \in S^h_\pm$, $u^h_r \in S^h_M$ and $\lambda^{\pm,h} \in \mathbb{R}$ such that

$$\begin{aligned}
(|\underline{Q}^h_r|^{r-2}\,\underline{Q}^h_r, \underline{v}^h)^h &= (u^h_r, \underline{\nabla}\,.\,\underline{v}^h)_\Omega && \forall\,\underline{v}^h \in \underline{V}^h_0\,, && (4.6a)\\
\ell\,(|q^{+,h}_r|^{r-2}\,q^{+,h}_r, v^{+,h})_{\Omega^+} &= (u^h_r - \lambda^{+,h}_r, v^{+,h})_{\Omega^+} && \forall\,v^{+,h} \in S^h_+\,, && (4.6b)\\
\ell\,(|q^{-,h}_r|^{r-2}\,q^{-,h}_r, v^{-,h})_{\Omega^-} &= -(u^h_r + \lambda^{-,h}_r, v^{-,h})_{\Omega^-} && \forall\,v^{-,h} \in S^h_-\,, && (4.6c)\\
(\underline{\nabla}\,.\,\underline{Q}^h_r, \eta^h)_\Omega &= (f^+_r - q^{+,h}_r, \eta^h)_{\Omega^+} - (f^-_r - q^{-,h}_r, \eta^h)_{\Omega^-} && \forall\,\eta^h \in S^h\,, && (4.6d)\\
(f^+_r - q^{+,h}_r, 1)_{\Omega^+} &= (f^-_r - q^{-,h}_r, 1)_{\Omega^-} = m\,. && && (4.6e)
\end{aligned}$$

Let $P^h_{(\pm)} : L^1(\Omega^{(\pm)}) \to S^h_{(\pm)}$ be such that

$$((I - P^h_{(\pm)})z, \eta^h)_{\Omega^{(\pm)}} = 0 \qquad \forall\, \eta^h \in S^h_{(\pm)}\,, \tag{4.7}$$

where $(\pm)$ means with or without these subscripts and superscripts. It follows for all $z \in L^s(\Omega^{(\pm)})$, where $s \in [1, \infty]$, that

$$|P^h_{(\pm)}\, z|_{0,s,\Omega^{(\pm)}} \leq |z|_{0,s,\Omega^{(\pm)}} \qquad \text{and} \qquad \lim_{h\to 0} |(I - P^h_{(\pm)})\, z|_{0,s,\Omega^{(\pm)}} = 0\,. \tag{4.8}$$

Then, similarly to (3.4) and (3.5), we introduce for $\rho \in L^1_M(\Omega)$

$$\underline{X}^h(\rho) := \{\underline{v}^h \in \underline{V}^h_0 : (\underline{\nabla}\,.\,\underline{v}^h, \eta^h)_\Omega = (\rho, \eta^h)_\Omega \quad \forall\, \eta^h \in S^h\}\,, \tag{4.9}$$

and

$$\underline{Y}^h(f^+_r, f^-_r, m) := \{\, (\underline{v}^h, v^{+,h}, v^{-,h}) \in \underline{X}^h(P^h(\,[f^+_r - v^{+,h}] - [f^-_r - v^{-,h}]\,)\,) \times S^h_+ \times S^h_- : \\ (f^+_r - v^{+,h}, 1)_{\Omega^+} = (f^-_r - v^{-,h}, 1)_{\Omega^-} = m\,\}\,. \tag{4.10}$$

It follows from (3.1), (4.9) and (4.10) that a solution of $(\mathcal{P}^h_r)$ is such that $(\underline{Q}^h_r, q^{+,h}_r, q^{-,h}_r) \in \underline{Y}^h(f^+_r, f^-_r, m)$ and

$$E^{r,h}_\ell(\underline{Q}^h_r, q^{+,h}_r, q^{-,h}_r) \\ \leq E^{r,h}_\ell(\underline{v}^h, v^{+,h}, v^{-,h}) := \frac{1}{r}\left[(|\underline{v}^h|^r, 1)^h + \ell\,(|v^{+,h}|^r, 1)_{\Omega^+} + \ell\,(|v^{-,h}|^r, 1)_{\Omega^-}\right] \\ \forall\, (\underline{v}^h, v^{+,h}, v^{-,h}) \in \underline{Y}^h(f^+_r, f^-_r, m)\,. \tag{4.11}$$

For $r > 1$, let $\underline{I}^h : \underline{V}^r_0(\Omega) \cap [W^{1,r}(\Omega)]^n \to \underline{V}^h_0$ be the generalised interpolation operator satisfying

$$\int_{\partial_i\sigma} (\underline{v} - \underline{I}^h\underline{v})\,.\,\underline{\nu}_{\partial_i\sigma}\, ds = 0 \qquad i = 1 \to n+1, \quad \forall\, \sigma \in \mathcal{T}^h\,; \tag{4.12}$$

where $\partial\sigma \equiv \cup_{i=1}^{n+1}\partial_i\sigma$ and $\underline{\nu}_{\partial_i\sigma}$ are the corresponding outward unit normals on $\partial_i\sigma$. It follows that

$$(\underline{\nabla}\,.\,(\underline{v} - \underline{I}^h\underline{v}), \eta^h) = 0 \qquad \forall\, \eta^h \in S^h. \tag{4.13}$$

In addition, we have for all $\sigma \in \mathcal{T}^h$ and any $s \in (1, \infty]$ that

$$|\underline{v} - \underline{I}^h\underline{v}|_{0,s,\sigma} \leq C_s\, h_\sigma\, |\underline{v}|_{1,s,\sigma} \qquad \text{and} \qquad |\underline{I}^h\underline{v}|_{1,s,\sigma} \leq C_s\, |\underline{v}|_{1,s,\sigma}\,, \tag{4.14a}$$

e.g. see [15, Lemma 3.1] and the proof given there for $s \geq 2$ is also valid for any $s \in (1, \infty]$; and, if $\underline{v}$ is sufficiently smooth, that

$$|\underline{\nabla}\,.\,(\underline{v} - \underline{I}^h\underline{v})|_{0,\sigma} \leq C\, h\, |\underline{\nabla}\,.\,\underline{v}|_{1,\sigma}\,, \tag{4.14b}$$

see e.g. [20, p553], Furthermore, we note from (3.1), (4.4) and (4.14a) that for all $\sigma \in \mathcal{T}^h$ and $\underline{v} \in [W^{2,\infty}(\sigma)]^n$

$$
\begin{aligned}
|\int_\sigma |\underline{v}|^r \, d\underline{x} - (|\underline{I}^h \underline{v}|^r, 1)^h_\sigma| &\leq \int_\sigma |\, |\underline{v}|^r - |\underline{I}^h \underline{v}|^r \,| \, d\underline{x} + |\int_\sigma |\underline{I}^h \underline{v}|^r \, d\underline{x} - (|\underline{I}^h \underline{v}|^r, 1)^h_\sigma| \\
&\leq r \, (|\underline{v}|^{r-1}_{0,\infty,\sigma} + |\underline{I}^h \underline{v}|^{r-1}_{0,\infty,\sigma}) \, \left[\, |\underline{v} - \underline{I}^h \underline{v}|_{0,1,\sigma} + h \, |\underline{I}^h \underline{v}|_{1,1,\sigma} \,\right] \\
&\leq C_r \, h \, \|\underline{v}\|^r_{2,\infty,\sigma} \, .
\end{aligned}
\tag{4.15}
$$

We have the following discrete version of Lemma 3.1

Lemma 4.1 *Let the Assumptions (A1) and (A2) hold. Then for all $r \in (1, r_0)$ with $p = \frac{r}{r-1}$, given $\rho^h \in S^h_M$ there exists $\underline{Q}^{\rho^h,h}_r \in \underline{X}^h(\rho^h)$ and*

$$\|\underline{Q}^{\rho^h,h}_r\|_{\underline{V}^r(\Omega)} \leq \mu_r \, |\rho^h|_{0,r,\Omega} \, , \tag{4.16}$$

where $\mu_r \in \mathbb{R}_{>0}$ is independent of h, but possibly dependent on r. Hence it follows that

$$\inf_{\eta^h \in S^h_M} \sup_{\underline{v}^h \in \underline{V}^h_0} \frac{(\underline{\nabla} \, . \, \underline{v}^h, \eta^h)}{\|\underline{v}^h\|_{\underline{V}^r(\Omega)} \, |\eta^h|_{0,p,\Omega}} \geq [2 \, \mu_r]^{-1} \, . \tag{4.17}$$

Proof. See the proof of Lemma 3.1 in [5]. □

Theorem 4.1 *Let the Assumptions (A1) and (A2) hold. Then for any given $r \in (1, r_0)$ there exists a unique solution, $(\underline{Q}^h_r, q^{+,h}_r, q^{-,h}_r, u^h_r, \lambda^{+,h}_r, \lambda^{-,h}_r) \in \underline{V}^h_0 \times S^h_+ \times S^h_- \times S^h_M \times \mathbb{R} \times \mathbb{R}$, to $(\mathcal{P}^h_r)$. In addition, we have that*

$$\|\underline{Q}^h_r\|^r_{\underline{V}^r(\Omega)} + \ell \, |q^{+,h}_r|^r_{0,r,\Omega^+} + \ell \, |q^{-,h}_r|^r_{0,r,\Omega^-} \leq C_{\Omega,\ell,r} \, \left[\, |f^+_r|^r_{0,r,\Omega^+} + |f^-_r|^r_{0,r,\Omega^-} \,\right] \, , \tag{4.18a}$$

$$|u^h_r|_{0,p,\Omega} + |\Omega^+|^{\frac{r-1}{r}} \, |\lambda^{+,h}_r| + |\Omega^-|^{\frac{r-1}{r}} \, |\lambda^{-,h}_r| \leq C_{\Omega,\ell,r} \, \left[\, |f^+_r|^{r-1}_{0,r,\Omega^+} + |f^-_r|^{r-1}_{0,r,\Omega^-} \,\right] \, ; \tag{4.18b}$$

where $p = \frac{r}{r-1}$.

Proof. The proof is a discrete analogue of the proof of Theorem 3.1. First, we set $\rho := [f^+_r - q^{f,+}_r] - [f^-_r - q^{f,-}_r] \in L^r_M(\Omega)$, where $q^{f,\pm}_r$ are defined by (3.10). It follows from (4.16), (4.8) and (3.11) that there exists $\underline{Q}^{\rho,h}_r \in \underline{X}^r(P^h \rho)$ and

$$\|\underline{Q}^{\rho,h}_r\|_{\underline{V}^r(\Omega)} \leq C_{\Omega,r} \, |P^h \rho|_{0,r,\Omega} \leq C_{\Omega,r} \, |\rho|_{0,r,\Omega} \leq C_{\Omega,r} \, \left[\, |f^+_r|_{0,r,\Omega^+} + |f^-_r|_{0,r,\Omega^-} \right] \, . \tag{4.19}$$

Therefore, on setting $\widehat{\underline{Q}}^h_r := \underline{Q}^h_r - \underline{Q}^{\rho,h}_r$ and $\widehat{q}^{\pm,h}_r := q^{\pm,h}_r - P^h_\pm \, q^{f,\pm}_r$, $(\mathcal{P}^h_r)$, (4.6a–e), can be reduced to: Find $(\widehat{\underline{Q}}^h_r, \widehat{q}^{+,h}_r, \widehat{q}^{-,h}_r) \in \underline{Y}^h(0,0,0)$ such that

$$
\begin{aligned}
(|\widehat{\underline{Q}}^h_r + \underline{Q}^{\rho,h}_r|^{r-2}(\widehat{\underline{Q}}^h_r + \underline{Q}^{\rho,h}_r), \underline{v}^h)^h &+ \ell \, (|\widehat{q}^{+,h}_r + P^h_+ \, q^{f,+}_r|^{r-2}(\widehat{q}^{+,h}_r + P^h_+ \, q^{f,+}_r), v^{+,h})_{\Omega^+} \\
&+ \ell \, (|\widehat{q}^{-,h}_r + P^h_- \, q^{f,-}_r|^{r-2}(\widehat{q}^{-,h}_r + P^h_- \, q^{f,-}_r), v^{-,h})_{\Omega^-} = 0 \\
&\qquad \forall \, (\underline{v}^h, v^{+,h}, v^{-,h}) \in \underline{Y}^h(0,0,0) \, ;
\end{aligned}
\tag{4.20}
$$

which is the Euler-Lagrange equation for the strictly convex minimization problem

$$\inf_{(\underline{v}^h, v^{+,h}, v^{-,h}) \in \underline{Y}^h(0,0,0)} E_\ell^{r,h}(\underline{v}^h + \underline{Q}_r^{\rho,h}, v^{+,h} + P_+^h\, q_r^{f,+}, v^{-,h} + P_-^h\, q_r^{f,-})\,. \tag{4.21}$$

Hence there exists a unique solution $(\widehat{\underline{Q}}_r^h, \widehat{q}_r^{+,h}, \widehat{q}_r^{-,h}) \in \underline{Y}^h(0,0,0)$ to (4.20). On setting $\underline{Q}_r^h = \widehat{\underline{Q}}_r^h + \underline{Q}_r^{\rho,h}$ and $q_r^{\pm,h} = \widehat{q}_r^{\pm,h} + P_\pm^h\, q_r^{f,\pm}$, it follows from (4.20) that $(\underline{Q}_r^h, q_r^{+,h}, q_r^{-,h}) \in \underline{Y}^h(f_r^+, f_r^-, m)$ is such that

$$(|\underline{Q}_r^h|^{r-2}\underline{Q}_r^h, \underline{v}^h)^h + \ell\,(|q_r^{+,h}|^{r-2}q_r^{+,h}, v^{+,h})_{\Omega^+} + \ell\,(|q_r^{-,h}|^{r-2}q_r^{-,h}, v^{-,h})_{\Omega^-} = 0 \qquad \forall\, (\underline{v}^h, v^{+,h}, v^{-,h}) \in \underline{Y}^h(0,0,0)\,; \tag{4.22a}$$

and hence, in particular with $v^{\pm,h} \equiv 0$, that

$$(|\underline{Q}_r^h|^{r-2}\underline{Q}_r^h, \underline{v}^h)^h = 0 \qquad \forall\, \underline{v}^h \in \underline{X}^{r,h}(0). \tag{4.22b}$$

It is also easily deduced from (4.6a–e) and (4.10) that any solution $(\underline{Q}_r^h, q_r^{\pm,h})$ of $(\mathcal{P}_r^h)$ solves (4.22a), and from (3.2) that it is unique. In addition, it follows from (4.21), (4.5), (4.8) and (4.10) that

$$|\underline{Q}_r^h|_{0,r,\Omega}^r + |q_r^{+,h}|_{0,r,\Omega^+}^r + |q_r^{-,h}|_{0,r,\Omega^-}^r \leq C_{\Omega,\ell,r}\left[|\underline{Q}_r^{\rho,h}|_{0,r,\Omega}^r + |q_r^{f,+}|_{0,r,\Omega^+}^r + |q_r^{f,-}|_{0,r,\Omega^-}^r\right], \tag{4.23a}$$

$$|\underline{\nabla}\,.\,\underline{Q}_r^h|_{0,r,\Omega}^r \leq \left[|f_r^+ - q_r^{+,h}|_{0,r,\Omega^+} + |f_r^- - q_r^{-,h}|_{0,r,\Omega^-}\right]^r. \tag{4.23b}$$

The bound (4.18a) follow immediately from (4.23a,b), (4.19) and (3.11).

Similarly to the proof of Theorem 3.1, we have that that there exists a unique $u_r^h \in S_M^h$ satisfying (4.6a). In addition, we have from (4.17), (4.6a) and (4.5) that

$$|u_r^h|_{0,p,\Omega} \leq 2\,\mu_r \sup_{\underline{v}^h \in \underline{V}_0^r(\Omega)} \frac{(\,|\underline{Q}_r^h|^{r-2}\underline{Q}_r^h, \underline{v}^h)^h}{\|\underline{v}^h\|_{\underline{V}^r(\Omega)}} \leq C_r\,|\underline{Q}_r^h|_{0,r,\Omega}^{r-1}\,. \tag{4.24}$$

Choosing $v^{\mp,h} \equiv 0$ in (4.22a), we have that $v^{\pm,h} = \mp\underline{\nabla}\,.\,\underline{v}^h$; and hence, on noting (4.6a), that $\ell\,|q_r^{\pm,h}|^{r-2}\,q_r^{\pm,h} \mp u_r^h$ are constant on $\Omega^\pm$. On choosing

$$\lambda_r^{\pm,h} = -\ell\,|q_r^{\pm,h}|^{r-2}\,q_r^{\pm,h} \pm u_r^h\,, \tag{4.25}$$

we obtain that (4.6b,c) hold for general $v^{\pm,h} \in S_\pm^h$. Therefore, we have proved that there exists a unique solution to $(\mathcal{P}_r^h)$, (4.6a–e), which is equivalent to the minimization problem (4.11). Furthermore, we have from (4.25) that

$$|\lambda_r^{\pm,h}| \leq |\Omega^\pm|^{-\frac{r-1}{r}}\left[\ell\,|q_r^{\pm,h}|_{0,r,\Omega^\pm}^{r-1} + |u_r^h|_{0,p,\Omega}\right]. \tag{4.26}$$

Combining (4.24), (4.18a) and (4.26) yields the desired result (4.18b). □

THEOREM 4.2 *Let the Assumptions (A1) and (A2) hold. For all $r \in (1, r_0)$, the unique solution $(\underline{Q}_r^h, q_r^{+,h}, q_r^{-,h}, u_r^h, \lambda_r^{+,h}, \lambda_r^{-,h})$ of $(\mathcal{P}_r^h)$ is such that as $h \to 0$*

$$\underline{Q}_r^h \to \underline{Q}_r \qquad \text{weakly in } [L^r(\Omega)]^n, \tag{4.27a}$$
$$\underline{\nabla} \cdot \underline{Q}_r^h \to \underline{\nabla} \cdot \underline{Q}_r \qquad \text{weakly in } L^r(\Omega), \tag{4.27b}$$
$$q_r^{\pm,h} \to q_r^{\pm} \qquad \text{weakly in } L^r(\Omega^{\pm}), \tag{4.27c}$$
$$u_r^h \to u_r \qquad \text{weakly in } L^p(\Omega), \tag{4.27d}$$
$$\lambda_r^{\pm,h} \to \lambda_r^{\pm}\,; \tag{4.27e}$$

where $(\underline{Q}_r, q_r^+, q_r^-, u_r, \lambda_r^+, \lambda_r^-)$ is the unique solution of $(\mathcal{P}_r)$.

Proof. The results (4.27a–e) follow immediately for a subsequence $\{(\underline{Q}_r^{h_j}, q_r^{\pm,h_j}, u_r^{h_j}, \lambda_r^{\pm,h_j})\}_{h_j>0}$ of $\{(\underline{Q}_r^h, q_r^{\pm,h}, u_r^h, \lambda_r^{\pm,h})\}_{h>0}$ from (4.18a,b).

It follows from (4.27c) that we may may pass to the limit in the h_j version of (4.6e) to obtain (3.3e), on recalling (4.2). For any $\eta \in L^p(\Omega)$, we choose $\eta^h = P^h\eta \in S^h$ in the h_j version of (4.6d). Noting (4.27b,c), and (4.8), we obtain the desired result (3.3d).

For any $\underline{\xi} \in [C_0^\infty(\Omega)]^n$ and $\xi^\pm \in C_0^\infty(\Omega)$, we choose $\underline{v}^h = \underline{I}^{h_j}\underline{\xi} - \underline{Q}_r^{h_j}$ and $v^{\pm,h} = P_\pm^h \xi^\pm$ in the h_j versions of (4.6a–c). Hence we have, on noting (3.1), that

$$\begin{aligned}
(u_r^{h_j}, \underline{\nabla} \cdot (\underline{Q}_r^{h_j} - \underline{I}^{h_j}\underline{\xi}))_\Omega &= (|\underline{Q}_r^{h_j}|^{r-2}\underline{Q}_r^{h_j}, \underline{Q}_r^{h_j} - \underline{I}^{h_j}\underline{\xi})^{h_j} \\
&\geq \frac{1}{r}\,(|\underline{Q}_r^{h_j}|^r - |\underline{I}^{h_j}\underline{\xi}|^r, 1)^{h_j}\,, \qquad (4.28a)\\
(u_r^{h_j} - \lambda_r^{+,h_j}, q_r^{+,h_j} - P_+^{h_j}\xi^+)_{\Omega^+} &= \ell\,(|q_r^{+,h_j}|^{r-2}q_r^{+,h_j}, q_r^{+,h_j} - P_+^{h_j}\,\xi^+)_{\Omega^+} \\
&\geq \frac{\ell}{r}\,(|q_r^{+,h_j}|^r - |P_+^{h_j}\,\xi^+|^r, 1)_{\Omega^+}\,, \qquad (4.28b)\\
-(u_r^{h_j} + \lambda_r^{-,h_j}, q_r^{-,h_j} - P_-^{h_j}\xi^-)_{\Omega^-} &= \ell\,(|q_r^{-,h_j}|^{r-2}q_r^{-,h_j}, q_r^{-,h_j} - P_-^{h_j}\,\xi^-)_{\Omega^-} \\
&\geq \frac{\ell}{r}\,(|q_r^{-,h_j}|^r - |P_-^{h_j}\,\xi^-|^r, 1)_{\Omega^-}\,. \qquad (4.28c)
\end{aligned}$$

Summing the above, and noting (4.6d) and (4.13), yields that

$$\begin{aligned}
&-(u_r^{h_j}, \underline{\nabla} \cdot \underline{\xi})_\Omega + (f_r^+ - P_+^{h_j}\xi^+, u_r^{h_j})_{\Omega^+} - (f_r^- - P_-^{h_j}\xi^-, u_r^{h_j})_{\Omega^-} \\
&\qquad - (\lambda_r^{+,h_j}, q_r^{+,h_j} - P_+^{h_j}\xi^+)_{\Omega^+} - (\lambda_r^{-,h_j}, q_r^{-,h_j} - P_-^{h_j}\xi^-)_{\Omega^-} \\
&= (u_r^{h_j}, \underline{\nabla} \cdot (\underline{Q}_r^{h_j} - \underline{I}^{h_j}\underline{\xi}))_\Omega + (u_r^{h_j} - \lambda_r^{+,h_j}, q_r^{+,h_j} - P_+^{h_j}\xi^+)_{\Omega^+} \\
&\qquad - (u_r^{h_j} + \lambda_r^{-,h_j}, q_r^{-,h_j} - P_-^{h_j}\xi^-)_{\Omega^-} \\
&\geq \frac{1}{r}\big[(|\underline{Q}_r^{h_j}|^r - |\underline{I}^{h_j}\underline{\xi}|^r, 1)^{h_j} + \ell\,(|q_r^{+,h_j}|^r - |P_+^{h_j}\,\xi^+|^r, 1)_{\Omega^+} \\
&\qquad + \ell\,(|q_r^{-,h_j}|^r - |P_-^{h_j}\,\xi^-|^r, 1)_{\Omega^-}\big].
\end{aligned} \tag{4.29}$$

We have from (4.5), (3.1) and (4.27a,c) that

$$\liminf_{h_j\to 0}\,(|\underline{Q}_r^{h_j}|^r,1)^{h_j} \geq (|\underline{Q}_r|^r,1)_\Omega \quad\text{and}\quad \liminf_{h_j\to 0}\,(|q_r^{\pm,h_j}|^r,1)_{\Omega^\pm} \geq (|q_r^{\pm}|^r,1)_{\Omega^\pm}\,. \tag{4.30}$$

Passing to the limit $h_j \to 0$ in (4.29), and noting (4.27a–e), (4.8), (4.15) and (4.30), yields that

$$\begin{aligned}
&-(u_r,\underline{\nabla}\,.\,\underline{\xi})_\Omega + (f_r^+ - \xi^+, u_r)_{\Omega^+} - (f_r^- - \xi^-, u_r)_{\Omega^-} \\
&\qquad\qquad\qquad - (\lambda_r^+, q_r^+ - \xi^+)_{\Omega^+} - (\lambda_r^-, q_r^- - \xi^-)_{\Omega^-} \\
&\qquad \geq \frac{1}{r}\left[(|\underline{Q}_r|^r - |\underline{\xi}|^r,1)_\Omega + \ell\,(|q_r^+|^r - |\xi^+|^r,1)_{\Omega^+} + \ell\,(|q_r^-|^r - |\xi^-|^r,1)_{\Omega^-}\right] \\
&\qquad\qquad\qquad \forall\ (\underline{\xi},\xi^+,\xi^-) \in [C_0^\infty(\Omega)]^n \times C_0^\infty(\Omega^+) \times C_0^\infty(\Omega^-)\,.
\end{aligned} \tag{4.31}$$

Recalling that $\underline{V}_0^r(\Omega)$ and $L^r(\Omega^\pm)$ are the strong closures of $[C_0^\infty(\Omega)]^n$ and $C_0^\infty(\Omega^\pm)$ in the norms $\|\cdot\|_{\underline{V}^r(\Omega)}$ and $|\cdot|_{0,r,\Omega^\pm}$, respectively, we have that (4.31) remains true for all $(\underline{\xi},\xi^+,\xi^-) \in \underline{V}_0^r(\Omega) \times L^r(\Omega^+) \times L^r(\Omega^-)$.

Therefore choosing $\underline{\xi} = \underline{Q}_r \pm \varepsilon\,\underline{v}$ and $\xi^\pm = q_r^\pm$ with $\underline{v} \in \underline{V}_0^r(\Omega)$ in (4.31), and noting (3.1) and (3.3d), yields the desired result (3.3a), on letting $\varepsilon \to 0$. Similarly, choosing $\underline{\xi} = \underline{Q}_r$ and $\xi^\pm = q_r^\pm \pm \varepsilon\, v^\pm$ with $v^\pm \in L^r(\Omega^\pm)$ in (4.31), and noting (3.1) and (3.3d), yields the desired results (3.3b,c), on letting $\varepsilon \to 0$. Hence we have that $(\underline{Q}_r, q_r^+, q_r^-, u_r, \lambda^+, \lambda^-) \in \underline{V}_0^r(\Omega) \times L^r(\Omega^+) \times L^r(\Omega^-) \times L^p(\Omega) \times \mathbb{R} \times \mathbb{R}$ solves $(\mathcal{P}_r)$. As the solution of $(\mathcal{P}_r)$ is unique, the whole sequence $\{(\underline{Q}_r^h, q_r^{\pm,h}, u_r^h, \lambda_r^{\pm,h})\}_{h>0}$ converges in (4.27a–e). □

Remark 4.1 *Clearly, on combining Theorems 3.2 and 4.2 we have the subsequence convergence of* $\{(\underline{Q}_r^h, q_r^{+,h}, q_r^{-,h}, u_r^h, \lambda_r^{+,h}, \lambda_r^{-,h})\}_{(r>1,h>0)}$ *as* $r_j \to 1$ *and* $h_j \to 0$ *to* $(\underline{Q}, q^+, q^-, u, \lambda^+, \lambda^-) \in \underline{V}_0^{\mathcal{M}}(\Omega) \times \mathcal{M}(\overline{\Omega^+}) \times \mathcal{M}(\overline{\Omega^-}) \times C_M(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$, *which is a solution of* $(\widehat{\mathcal{P}})$, *(3.23a–e).*

4.2 Approximation of $(\mathcal{U}_r)$

Similarly to subsection 3.2, we assume that $f := f^+ - f^- \in L^{r_0}(\Omega)$ for some $r_0 > 1$. Then for a given $r \in (1, r_0)$, we consider the following fully practical approximation of $(\mathcal{U}_r)$ by $\underline{V}_0^h$:

$(\mathcal{U}_r^h)$ Find $\underline{Q}_r^h \in \underline{V}_0^h$ such that

$$(|\underline{Q}_r^h|^{r-2}\,\underline{Q}_r^h, \underline{v}^h)^h + \ell\,(|\underline{\nabla}\,.\,\underline{Q}_r^h - P^h f|^{r-2}\,(\underline{\nabla}\,.\,\underline{Q}_r^h - P^h f), \underline{\nabla}\,.\,\underline{v}^h)_\Omega = 0 \qquad \forall\ \underline{v}^h \in \underline{V}_0^h\,. \tag{4.32}$$

$(\mathcal{U}_r^h)$ is the Euler-Lagrange equation for the strictly convex minimization problem: Find $\underline{Q}_r^h \in \underline{V}_0^h$ such that

$$E_\ell^{r,h}(\underline{Q}_r^h) \leq E_\ell^{r,h}(\underline{v}^h) := \frac{1}{r}\left[(|\underline{v}^h|^r,1)^h + \ell\,(|\underline{\nabla}\,.\,\underline{v}^h - P^h f|^r,1)_\Omega\right] \qquad \forall\ \underline{v}^h \in \underline{V}_0^h\,. \tag{4.33}$$

Hence, there exists a unique solution $\underline{Q}_r^h \in \underline{V}_0^h$ to $(\mathcal{U}_r^h)$; and moreover, we have, on noting (4.5) and (4.8), that

$$|\underline{Q}_r^h|_{0,r,\Omega}^r + \ell\,|\underline{\nabla}\,.\,\underline{Q}_r^h - P^h f|_{0,r,\Omega}^r \le C_{\ell,r}\,|f|_{0,r,\Omega}^r\,. \qquad (4.34)$$

It follows from (3.36) that there exists $\underline{Q}_r \in \underline{V}_0^r(\Omega)$ such that there exists a subsequence $\{\underline{Q}_r^{h_j}\}_{h_j>0}$, where $\underline{Q}_r^h$ is the unique solution of $(\mathcal{U}_r^h)$, such that (4.27a,b) hold as $h_j \to 0$.

For any $\underline{\xi} \in [C_0^\infty(\Omega)]^n$, we have from (4.33) that $E_\ell^{r,h_j}(\underline{Q}_r^{h_j}) \le E_\ell^{r,h_j}(\underline{I}^{h_j}\underline{\xi})$. Passing to the $h_j \to 0$ limit in this, on noting (4.15), (4.8), (4.14b), (4.30), (4.27b) and (3.1), yields that

$$(|\underline{Q}_r|^r + \ell\,|\underline{\nabla}\,.\,\underline{Q}_r - f|^r, 1)_\Omega \le (|\underline{\xi}|^r + \ell\,|\underline{\nabla}\,.\,\underline{\xi} - f|^r, 1)_\Omega \qquad \forall\, \underline{\xi} \in [C_0^\infty(\Omega)]^n\,. \qquad (4.35)$$

Recalling that $\underline{V}_0^r(\Omega)$ is the strong closure of $[C_0^\infty(\Omega)]^n$ in the norm $\|\cdot\|_{\underline{V}^r(\Omega)}$ we have that (4.35) remains true for all $\underline{\xi} \in \underline{V}_0^r(\Omega)$, which is equivalent to $(\mathcal{U}_r)$, (3.34). As the solution of $(\mathcal{U}_r)$ is unique, the whole sequence $\{\underline{Q}_r^h\}_{h>0}$ converges in (4.27a,b). Finally, the analogue of Remark 4.1 holds for $(\mathcal{U}_r^h)$.

4.3 Approximation of $(\mathcal{PM}_r)$

Similarly to subsection 3.3, we assume that the non-negative $f^{(i)} \in L^{r_0}(\Omega)$, $i = 1 \to I$, for some $r_0 > 1$. In addition, for given constants $m \in (0, \min_{i=1\to I}\{\int_\Omega f^{(i)}\}]$ and $G \ge \frac{m}{|\Omega_g|}$, we introduce the convex sets

$$K_G^h(m) := \{\eta^h \in K_G^h : \int_{\Omega_g} \eta^h \, d\underline{x} = m\}\,, \quad \text{where} \quad K_G := \{\eta^h \in S_g^h : 0 \le \eta^h \le G\}\,. \qquad (4.36)$$

For a given $r \in (1, r_0)$, we then consider the following fully practical approximation of $(\mathcal{PM}_r)$ by $\underline{V}_0^h$ and S^h :

$(\mathcal{PM}_r^h)$ Find $\underline{Q}_r^{(i),h} \in \underline{V}_0^h$, $i = 1 \to I$, and $g_r^h \in K_G^h(m)$ such that

$$\begin{aligned}&(|\underline{Q}_r^{(i),h}|^{r-2}\,\underline{Q}_r^{(i),h}, \underline{v}^h)^h \\ &\qquad + \ell\,(|\underline{\nabla}\,.\,\underline{Q}_r^{(i),h} - P^h f^{(i)} + g_r^h|^{r-2}\,(\underline{\nabla}\,.\,\underline{Q}_r^{(i),h} - P^h f^{(i)} + g_r^h), \underline{\nabla}\,.\,\underline{v}^h)_\Omega = 0 \\ &\qquad\qquad \forall\, \underline{v}^h \in \underline{V}_0^h\,,\ i = 1 \to I, \qquad (4.37a)\end{aligned}$$

$$\begin{aligned}&\sum_{i=1}^{I}(|\underline{\nabla}\,.\,\underline{Q}_r^{(i),h} - P^h f^{(i)} + g_r^h|^{r-2}\,(\underline{\nabla}\,.\,\underline{Q}_r^{(i),h} - P^h f^{(i)} + g_r^h), \eta^h - g_r^h)_{\Omega_g} \ge 0 \\ &\qquad\qquad \forall\, \eta^h \in K_G^h(m)\,. \qquad (4.37b)\end{aligned}$$

$(\mathcal{PM}_r^h)$ is the Euler-Lagrange system for the strictly convex minimization problem: Find

$\underline{Q}_r^{(i),h} \in \underline{V}_0^h$, $i = 1 \to I$, and $g_r^h \in K_G^h(m)$ such that

$$\begin{aligned} & E_\ell^{r,h}(\{\underline{Q}_r^{(i),h}\}_{i=1}^I, g_r^h) \\ & \qquad \leq E_\ell^{r,h}(\{\underline{v}^{(i),h}\}_{i=1}^I, \eta^h) := \frac{1}{r} \sum_{i=1}^I \left[(|\underline{v}^{(i),h}|^r, 1)^h + \ell\, (|\underline{\nabla} \,.\, \underline{v}^{(i),h} - P^h f^{(i)} + \eta^h|^r, 1)_\Omega \right] \\ & \qquad\qquad \forall\ \underline{v}^{(i),h} \in \underline{V}_0^h,\ i = 1 \to I, \quad \forall\ \eta^h \in K_G^h(m)\,. \end{aligned} \tag{4.38}$$

Hence, there exists a unique solution $(\{\underline{Q}_r^{(i),h}\}_{i=1}^I, g_r^h)$ to $(\mathcal{PM}_r^h)$; and moreover, we have, on noting (4.5) and (4.8), that

$$\sum_{i=1}^I \left[|\underline{Q}_r^{(i),h}|_{0,r,\Omega}^r + \ell\, |\underline{\nabla} \,.\, \underline{Q}_r^{(i),h} - P^h f^{(i)} + g_r^h|_{0,r,\Omega}^r \right] \leq C_{\ell,r} \left[\frac{I\, m^r}{|\Omega_g|^{r-1}} + \sum_{i=1}^I |f^{(i)}|_{0,r,\Omega}^r \right]. \tag{4.39}$$

It follows from (4.39) that there exist $\underline{Q}_r^{(i)} \in \underline{V}_0^r(\Omega)$, $i = 1 \to I$, and $g_r \in K_G(m)$ such that there exists a subsequence $\{\{\underline{Q}_r^{(i),h_j}\}_{=1}^I, g_r^{h_j}\}_{h_j>0}$, where $(\{\underline{Q}_r^{(i),h}\}_{i=1}^I, g_r^h)$ is the unique solution of $(\mathcal{PM}_r^h)$, such that

$$\underline{Q}_r^{(i),h_j} \to \underline{Q}_r^{(i)} \qquad \text{weakly in } [L^r(\Omega)]^n\,, \qquad i = 1 \to I, \tag{4.40a}$$

$$\underline{\nabla} \,.\, \underline{Q}_r^{(i),h_j} \to \underline{\nabla} \,.\, \underline{Q}_r^{(i)} \qquad \text{weakly in } L^r(\Omega), \qquad i = 1 \to I, \tag{4.40b}$$

$$g_r^{h_j} \to g_r \qquad \text{vaguely in } L^\infty(\Omega_g) \tag{4.40c}$$

hold as $h_j \to 0$.

For any $\underline{\xi}^{(i)} \in [C_0^\infty(\Omega)]^n$, $i = 1 \to I$, and $\eta \in K_G(m)$, we have from (4.38) that $E_\ell^{r,h_j}(\{\underline{Q}_r^{(i),h_j}\}_{i=1}^I, g_r^{h_j}) \leq E_\ell^{r,h_j}(\{\underline{I}^{h_j} \underline{\xi}^{(i)}\}_{i=1}^I, P^{h_j}\eta)$. Passing to the $h_j \to 0$ limit in this, on noting (4.15), (4.8), (4.14b), (4.40a–c) and (3.1), yields that

$$\begin{aligned} \sum_{i=1}^I \left[(|\underline{Q}_r^{(i)}|^r + \ell\, |\underline{\nabla} \,.\, \underline{Q}_r^{(i)} - f^{(i)} + g_r|^r, 1)_\Omega \right] \leq \sum_{i=1}^I \left[(|\underline{\xi}^{(i)}|^r + \ell\, |\underline{\nabla} \,.\, \underline{\xi}^{(i)} - f^{(i)} + \eta|^r, 1)_\Omega \right] \\ \forall\ \underline{\xi}^{(i)} \in [C_0^\infty(\Omega)]^n, \quad i = 1 \to I, \quad \forall\ \eta \in K_G(m)\,. \end{aligned} \tag{4.41}$$

Recalling that $\underline{V}_0^r(\Omega)$ is the strong closure of $[C_0^\infty(\Omega)]^n$ in the norm $\|\cdot\|_{\underline{V}^r(\Omega)}$ we have that (4.41) remains true for all $\underline{\xi} \in \underline{V}_0^r(\Omega)$, which is equivalent to $(\mathcal{PM}_r)$, (3.46a,b). As the solution of $(\mathcal{PM}_r)$ is unique, the whole sequence $\{\{\underline{Q}_r^{(i),h}\}_{i=1}^I, g_r^h\}_{h>0}$ converges in (4.40a–c). Finally, the analogue of Remark 4.1 holds for $(\mathcal{PM}_r^h)$.

5 Augmented Lagrangian Methods for $(\mathcal{P}_r^h)$ and $(\mathcal{PM}_r^h)$

In this section we introduce, and prove convergence of, algorithms to solve the problems $(\mathcal{P}_r^h)$, (4.6a–e), and $(\mathcal{PM}_r^h)$, (4.37a,b); which involve constraints, and hence Lagrange multipliers, as opposed to $(\mathcal{U}_r^h)$, (4.32).

5.1 $(\mathcal{P}_r^h)$

For a given constant $\mu > 0$, we consider the following iterative method for solving the constrained problem (4.6a–e):

For $k \geq 0$, given $u_r^{h,k} \in S^h$ and $\lambda_r^{\pm,h,k} \in \mathbb{R}$; find $\underline{Q}_r^{h,k+1} \in \underline{V}_0^h$ and $q_r^{h,\pm,k+1} \in S_\pm^h$ such that

$$\begin{aligned}
&(|\underline{Q}_r^{h,k+1}|^{r-2}\underline{Q}_r^{h,k+1}, \underline{v}^h)^h + \mu\,(\underline{\nabla}\,.\,\underline{Q}_r^{h,k+1} - [f_r^+ - q_r^{+,h,k+1}] + [f_r^- - q_r^{-,h,k+1}], \underline{\nabla}\,.\,\underline{v}^h)_\Omega \\
&\qquad = (u_r^{h,k}, \underline{\nabla}\,.\,\underline{v}^h)_\Omega \qquad \forall\ \underline{v}^h \in \underline{V}_0^h, \qquad (5.1a)\\
&\ell\,(|q_r^{+,h,k+1}|^{r-2}q_r^{+,h,k+1}, v^{+,h})_{\Omega^+} + \mu\,(\underline{\nabla}\,.\,\underline{Q}_r^{h,k+1} - [f_r^+ - q_r^{+,h,k+1}] + [f_r^- - q_r^{-,h,k+1}], v^{+,h})_{\Omega^+}\\
&\quad - \mu\,[\,(f_r^+ - q_r^{+,h,k+1}, 1)_{\Omega^+} - m]\,(v^{+,h},1)_{\Omega^+} = (u_r^{h,k} - \lambda_r^{+,h,k}, v^{+,h})_{\Omega^+} \qquad \forall\ v^{+,h} \in S_+^h, \qquad (5.1b)\\
&\ell\,(|q_r^{-,h,k+1}|^{r-2}q_r^{-,h,k+1}, v^{-,h})_{\Omega^-} - \mu\,(\underline{\nabla}\,.\,\underline{Q}_r^{h,k+1} - [f_r^+ - q_r^{+,h,k+1}] + [f_r^- - q_r^{-,h,k+1}], v^{-,h})_{\Omega^-}\\
&\quad - \mu\,[\,(f_r^- - q_r^{-,h,k+1}, 1)_{\Omega^-} - m]\,(v^{-,h},1)_{\Omega^-} = (-u_r^{h,k} - \lambda_r^{-,h,k}, v^{-,h})_{\Omega^-} \qquad \forall\ v^{-,h} \in S_-^h. \qquad (5.1c)
\end{aligned}$$

Then for a given constant $\rho > 0$, set

$$u_r^{h,k+1} = u_r^{h,k} - \rho\left[\underline{\nabla}\,.\,\underline{Q}_r^{h,k+1} - [P_+^h f_r^+ - q_r^{+,h,k+1}] + [P_-^h f_r^- - q_r^{-,h,k+1}]\right] \in S^h\,, \qquad (5.2a)$$

$$\lambda_r^{\pm,h,k+1} = \lambda_r^{\pm,h,k} - \rho\,[\,(f_r^\pm - q_r^{\pm,h,k+1}, 1)_{\Omega^\pm} - m] \in \mathbb{R}\,. \qquad (5.2b)$$

We introduce the augmented Lagrangian associated with the minimization problem, (4.11), which is equivalent to $(\mathcal{P}_r^h)$:

$$\begin{aligned}
&\mathcal{L}_{\ell,\mu}^{r,h}(\underline{v}^h, v^{+,h}, v^{-,h}, \eta^h, \lambda^+, \lambda^-)\\
&:= \frac{1}{r}\,\left[(|\underline{v}^h|^r, 1)^h + \ell\,(|v^{+,h}|^r, 1)_{\Omega^+} + \ell\,(|v^{-,h}|^r, 1)_{\Omega^-}\right] - \lambda^+\,[\,(f_r^+ - v^{+,h}, 1)_{\Omega^+} - m]\\
&\quad - \lambda^-\,[\,(f_r^- - v^{-,h}, 1)_{\Omega^-} - m] - (\underline{\nabla}\,.\,\underline{v}^h - [f_r^+ - v^{+,h}] + [f_r^- - v^{-,h}], \eta^h)_\Omega\\
&\quad + \frac{\mu}{2}\,\left[\,[\,(f_r^+ - v^{+,h}, 1)_{\Omega^+} - m]^2 + [\,(f_r^- - v^{-,h}, 1)_{\Omega^-} - m]^2\,\right]\\
&\quad + \frac{\mu}{2}\,|\underline{\nabla}\,.\,\underline{v}^h - [f_r^+ - v^{+,h}] + [f_r^- - v^{-,h}]\,|_{0,\Omega}^2\\
&\qquad\qquad \forall\ (\underline{v}^h, v^{+,h}, v^{-,h}, \eta^h, \lambda^+, \lambda^-) \in \underline{V}_0^h \times S_+^h \times S_-^h \times S^h \times \mathbb{R} \times \mathbb{R}\,. \qquad (5.3)
\end{aligned}$$

Then (5.1a–c) is the Euler-Lagrange system for the minimization of $\mathcal{L}_{\ell,\mu}^{r,h}(\underline{v}^h, v^{+,h}, v^{-,h}, u_r^{h,k}, \lambda_r^{+,h,k}, \lambda_r^{-,h,k})$ over $\underline{v}^h \in \underline{V}_0^h$ and $v^{\pm,h} \in S_\pm^h$ for given $u_r^{h,k} \in S^h$ and $\lambda_r^{\pm,h,k} \in \mathbb{R}$. As this is a strictly convex minimization problem, we have existence and uniqueness of $\underline{Q}_r^{h,k+1} \in \underline{V}_0^h$ and $q_r^{\pm,h,k+1} \in S_\pm^h$ solving (5.1a–c).

THEOREM 5.1 *For any $\mu > 0$, and any $u_r^{h,0} \in S^h$ and $\lambda_r^{\pm,h,0} \in \mathbb{R}$; if $\rho \in (0, 2\,\mu)$, then the sequence $\{(\underline{Q}_r^{h,k}, q_r^{+,h,k}, q_r^{-,h,k}, u_r^{h,k}, \lambda_r^{+,h,k}, \lambda_r^{-,h,k})\}_{k\geq 0}$ generated by the algorithm (5.1a–c) and (5.2a,b) is such that*

$$\underline{Q}_r^{h,k} \to \underline{Q}_r^h, \qquad q_r^{\pm,h,k} \to q_r^{\pm,h} \qquad as \quad k \to \infty\,, \qquad (5.4)$$

where $(\underline{Q}_r^h, q_r^{\pm,h}, u_r^h, \lambda_r^{\pm,h})$ *is the unique solution of* $(\mathcal{P}_r^h)$, *(4.6a–e).*

Proof. Let $\overline{\underline{Q}}^k := \underline{Q}_r^h - \underline{Q}_r^{h,k}$, $\overline{q}^{\pm,k} := q_r^{\pm,h} - q_r^{\pm,h,k}$, $\overline{u}^k := u_r^h - u_r^{h,k}$ and $\overline{\lambda}^{\pm,k} := \lambda_r^{\pm,h} - \lambda_r^{\pm,h,k}$. Then we have from (4.6a–e) and (5.1a–c) that

$$\begin{aligned}
&(|\underline{Q}_r^h|^{r-2}\underline{Q}_r^h - |\underline{Q}_r^{h,k+1}|^{r-2}\underline{Q}_r^{h,k+1}, \overline{\underline{Q}}^{k+1})^h + \mu\,|\underline{\nabla}\,.\,\overline{\underline{Q}}^{k+1} + \overline{q}^{+,k+1} - \overline{q}^{-,k+1}|_{0,\Omega}^2 \\
&\quad + \ell\,(|q_r^{+,h}|^{r-2}q_r^{+,h} - |q_r^{+,h,k+1}|^{r-2}q_r^{+,h,k+1}, \overline{q}^{+,k+1})_{\Omega^+} + \mu\,[\,(\overline{q}^{+,k+1},1)_{\Omega^+}\,]^2 \\
&\quad + \ell\,(|q_r^{-,h}|^{r-2}q_r^{-,h} - |q_r^{-,h,k+1}|^{r-2}q_r^{-,h,k+1}, \overline{q}^{-,k+1})_{\Omega^-} + \mu\,[\,(\overline{q}^{-,k+1},1)_{\Omega^-}\,]^2 \\
&\qquad = (\overline{u}^k, \underline{\nabla}\,.\,\overline{\underline{Q}}^{k+1} + \overline{q}^{+,k+1} - \overline{q}^{-,k+1})_\Omega - (\overline{\lambda}^{+,k}, \overline{q}^{+,k+1})_{\Omega^+} - (\overline{\lambda}^{-,k}, \overline{q}^{-,k+1})_{\Omega^-}\,; \qquad (5.5)
\end{aligned}$$

and from (4.6d,e) and (5.2a,b) that

$$\begin{aligned}
|\overline{u}^{k+1}|_{0,\Omega}^2 &= |\overline{u}^k|_{0,\Omega}^2 - 2\,\rho\,(\overline{u}^k, \underline{\nabla}\,.\,\overline{\underline{Q}}^{k+1} + \overline{q}^{+,k+1} - \overline{q}^{-,k+1})_\Omega \\
&\qquad + \rho^2\,|\underline{\nabla}\,.\,\overline{\underline{Q}}^{k+1} + \overline{q}^{+,k+1} - \overline{q}^{-,k+1}|_{0,\Omega}^2\,, \qquad (5.6a) \\
|\overline{\lambda}^{\pm,k+1}|^2 &= |\overline{\lambda}^{\pm,k}|^2 + 2\,\rho\,\overline{\lambda}^{\pm,k}\,(\overline{q}^{\pm,k+1},1)_{\Omega^\pm} + \rho^2\,[\,(\overline{q}^{\pm,k+1},1)_{\Omega^\pm}\,]^2\,. \qquad (5.6b)
\end{aligned}$$

Combining (5.5) and (5.6a,b) yields that

$$\begin{aligned}
&|\overline{u}^{k+1}|_{0,\Omega}^2 + |\overline{\lambda}^{+,k+1}|^2 + |\overline{\lambda}^{-,k+1}|^2 + 2\,\rho\,(|\underline{Q}_r^h|^{r-2}\underline{Q}_r^h - |\underline{Q}_r^{h,k+1}|^{r-2}\underline{Q}_r^{h,k+1}, \overline{\underline{Q}}^{k+1})^h \\
&\quad + 2\,\rho\,\ell\,(|q_r^{+,h}|^{r-2}q_r^{+,h} - |q_r^{+,h,k+1}|^{r-2}q_r^{+,h,k+1}, \overline{q}^{+,k+1})_{\Omega^+} \\
&\quad + 2\,\rho\,\ell\,(|q_r^{-,h}|^{r-2}q_r^{-,h} - |q_r^{-,h,k+1}|^{r-2}q_r^{-,h,k+1}, \overline{q}^{-,k+1})_{\Omega^-} \\
&\quad + (2\,\rho\,\mu - \rho^2)\,\Big[|\underline{\nabla}\,.\,\overline{\underline{Q}}^{k+1} + \overline{q}^{+,k+1} - \overline{q}^{-,k+1}|_{0,\Omega}^2 + [\,(\overline{q}^{+,k+1},1)_{\Omega^+}\,]^2 + [\,(\overline{q}^{-,k+1},1)_{\Omega^-}\,]^2\Big] \\
&\qquad = |\overline{u}^k|_{0,\Omega}^2 + |\overline{\lambda}^{+,k}|^2 + |\overline{\lambda}^{-,k}|^2\,. \qquad (5.7)
\end{aligned}$$

Hence for $\rho \in (0, 2\,\mu)$, we have that the sequence $\{|\overline{u}^k|_{0,\Omega}^2 + |\overline{\lambda}^{+,k}|^2 + |\overline{\lambda}^{-,k}|^2\}_{k\geq 1}$ is monotonically decreasing and bounded below. Therefore it must converge to a limit, and hence the desired result (5.4) holds on noting (3.2). □

5.2 $(\mathcal{PM})_r$

For given constants μ, $\rho > 0$, we consider the following iterative method for solving the constrained problem (4.37a,b):

For $k \geq 0$, given $F_{0,r}^{h,k}$, $F_{G,r}^{h,k} \in S_g^h$ and $\lambda_r^{h,k} \in \mathbb{R}$; find $\underline{Q}_r^{(i),h,k+1} \in \underline{V}_0^h$, $i = 1 \to I$, and

$g_r^{h,k+1} \in S_g^h$ such that

$$(|\underline{Q}_r^{(i),h,k+1}|^{r-2}\, \underline{Q}_r^{(i),h,k+1}, \underline{v}^h)^h$$
$$+ \ell\, (|\underline{\nabla} \cdot \underline{Q}_r^{(i),h,k+1} - P^h f^{(i)} + g_r^{h,k+1}|^{r-2}\, (\underline{\nabla} \cdot \underline{Q}_r^{(i),h,k+1} - P^h f^{(i)} + g_r^{h,k+1}), \underline{\nabla} \cdot \underline{v}^h)_\Omega = 0$$
$$\forall\, \underline{v}^h \in \underline{V}_0^h, \quad i = 1 \to I, \quad (5.8a)$$

$$\ell \sum_{i=1}^{I} (|\underline{\nabla} \cdot \underline{Q}_r^{(i),h,k+1} - P^h f^{(i)} + g_r^{h,k+1}|^{r-2}\, (\underline{\nabla} \cdot \underline{Q}_r^{(i),h,k+1} - P^h f^{(i)} + g_r^{h,k+1}), \eta^h)_{\Omega_g}$$
$$- ([F_{0,r}^{h,k} - 2\,\rho\, g_r^{h,k+1}]_+ - [F_{G,r}^{h,k} + 2\,\rho\,(g_r^{h,k+1} - G)]_+, \eta^h)_{\Omega_g}$$
$$+ \mu\,[(g_r^{h,k+1}, 1)_{\Omega_g} - m]\,(\eta^h, 1)_{\Omega_g} = -(\lambda_r^{h,k}, \eta^h)_{\Omega_g} \qquad \forall\, \eta^h \in S_g^h\,. \qquad (5.8b)$$

Then set

$$F_{0,r}^{h,k+1} = [F_{0,r}^{h,k} - 2\,\rho\, g_r^{h,k+1}]_+ \in S_g^h\,, \qquad F_{G,r}^{h,k+1} = [F_{G,r}^{h,k} + 2\,\rho\,(g_r^{h,k+1} - G)]_+ \in S_g^h\,, \quad (5.9a)$$
$$\lambda_r^{h,k+1} = \lambda_r^{h,k} + \rho\,[\,(g_r^{h,k+1}, 1)_{\Omega_g} - m] \in \mathbb{R}\,. \qquad (5.9b)$$

We introduce an augmented Lagrangian associated with the minimization problem, (4.38):

$$\mathcal{L}_{\ell,\mu,\rho}^{r,h}(\{\underline{v}^{(i),h}\}_{i=1}^I, \eta^h, F_0^h, F_G^h, \lambda)$$
$$:= \frac{1}{r} \sum_{i=1}^{I} \left[(|\underline{v}^{(i),h}|^r, 1)^h + \ell\,(|\underline{\nabla} \cdot \underline{v}^{(i),h} - P^h f^{(i)} + \eta^h|^r, 1)_\Omega\right] + \lambda\,[\,(\eta^h, 1)_{\Omega_g} - m]$$
$$+ \frac{\mu}{2}\, \left[(\eta^h, 1)_{\Omega_g} - m\right]^2 + \frac{1}{4\,\rho}\,([F_0^h - 2\,\rho\,\eta^h]_+^2 + [F_G^h + 2\,\rho\,(\eta^h - G)]_+^2, 1)_{\Omega_g}$$
$$\forall\, \underline{v}^{(i),h} \in \underline{V}_0^h,\; i = 1 \to I, \quad \forall\, \eta^h,\, F_0^h,\, F_G^h \in S_g^h, \quad \forall\, \lambda \in \mathbb{R}\,. \quad (5.10)$$

Then (5.8a,b) is the Euler-Lagrange system for the minimization of $\mathcal{L}_{\ell,\mu,\rho}^{r,h}(\{\underline{v}^{(i),h}\}_{i=1}^I, \eta^h, F_{0,r}^{h,k}, F_{G,r}^{h,k}, \lambda_r^{h,k})$ over $\underline{v}^{(i),h} \in \underline{V}_0^h$, $i = 1 \to I$, and $\eta^h \in S_g^h$ for given $F_{0,r}^{h,k}$, $F_{G,r}^{h,k} \in S_g^h$ and $\lambda_r^{h,k} \in \mathbb{R}$. As this is a strictly convex minimization problem, we have existence and uniqueness of $\underline{Q}_r^{(i),h,k+1} \in \underline{V}_0^h$, $i = 1 \to I$, and $g_r^{h,k+1} \in S_g^h$ solving (5.8a,b).

THEOREM 5.2 *For any $\mu > 0$, and any $F_{0,r}^{h,0}$, $F_{G,r}^{h,0} \in S_g^h$ and $\lambda_r^{h,0} \in \mathbb{R}$; if $\rho \in (0, 2\,\mu)$, then the sequence $\{(\{\underline{Q}_r^{(i),h,k}\}_{i=1}^I, g_r^{h,k}, F_{0,r}^{h,k}, F_{G,r}^{h,k}, \lambda_r^{h,k})\}_{k\geq 0}$ generated by the algorithm (5.8a,b) and (5.9a,b) is such that*

$$\underline{Q}_r^{(i),h,k} \to \underline{Q}_r^{(i),h}, \quad i = 1 \to I, \qquad g_r^{h,k} \to g_r^h \qquad \text{as} \quad k \to \infty\,, \qquad (5.11)$$

where $(\{\underline{Q}_r^{(i),h}\}_{i=1}^I, g_r^h)$ is the unique solution of $(\mathcal{PM}_r^h)$, (4.37a,b).

Proof. Firstly, we note that the unique solution $(\{\underline{Q}_r^{(i),h}\}_{i=1}^I, g_r^h)$ of $(\mathcal{PM}_r^h)$, (4.37a,b) solves (4.37a) and

$$\ell \sum_{i=1}^{I} (|\underline{\nabla} \cdot \underline{Q}_r^{(i),h} - P^h f^{(i)} + g_r^h|^{r-2}\, (\underline{\nabla} \cdot \underline{Q}_r^{(i),h} - P^h f^{(i)} + g_r^h), \eta^h)_{\Omega_g}$$
$$= (F_{0,r}^h - F_{G,r}^h - \lambda_r^h, \eta^h)_{\Omega_g} \qquad \forall\, \eta^h \in S_g^h \qquad (5.12)$$

for some $\lambda_r^h \in \mathbb{R}$, and where

$$F_{0,r}^h = [F_{0,r}^h - 2\,\rho\, g_r^h]_+ \in S_g^h\,, \qquad F_{G,r}^h = [F_{G,r}^h + 2\,\rho\,(g_r^h - G)]_+ \in S_g^h\,. \tag{5.13}$$

Let $Z_r^{(i),h} := \underline{\nabla}\,.\,\underline{Q}_r^{(i),h} - P^h f^{(i)} + g_r^h \in S^h$ and $Z_r^{(i),h,k} := \underline{\nabla}\,.\,\underline{Q}_r^{(i),h,k} - P^h f^{(i)} + g_r^{h,k} \in S^h$. In addition, let $\overline{\underline{Q}}^{(i),k} := \underline{Q}_r^{(i),h} - \underline{Q}_r^{(i),h,k}$, $\overline{Z}^{(i),k} := Z_r^{(i),h} - Z_r^{(i),h,k}$, $\overline{g}^k := g_r^h - g_r^{h,k}$, $\overline{F}_0^k := F_{0,r}^h - F_{0,r}^{h,k}$, $\overline{F}_G^k := F_{G,r}^h - F_{G,r}^{h,k}$ and $\overline{\lambda}^k := \lambda_r^h - \lambda_r^{h,k}$. Then we have from (4.37a), (5.12) and (5.8a,b) that

$$\begin{aligned}
&\sum_{i=1}^{I}(|\underline{Q}_r^{(i),h}|^{r-2}\underline{Q}_r^{(i),h} - |\underline{Q}_r^{(i),h,k+1}|^{r-2}\underline{Q}_r^{(i),h,k+1}, \overline{\underline{Q}}^{(i),k+1})^h \\
&\qquad + \ell \sum_{i=1}^{I}\Big((|Z_r^{(i),h}|^{r-2}Z_r^{(i),h} - |Z_r^{(i),h,k+1}|^{r-2}Z_r^{(i),h,k+1}, \overline{Z}^k\Big)_\Omega + \mu\,[(\overline{g}^{k+1},1)_{\Omega_g}]^2 \\
&\qquad\qquad = (\overline{F}_0^{k+1} - \overline{F}_G^{k+1} - \overline{\lambda}^k, \overline{g}^{k+1})_{\Omega_g}\,;
\end{aligned} \tag{5.14}$$

and from (5.13) and (5.9a,b) that

$$|\overline{F}_0^{k+1}|^2_{0,\Omega_g} \leq (\overline{F}_0^{k+1}, \overline{F}_0^k - 2\,\rho\,\overline{g}^{k+1})_{\Omega_g}\,, \qquad |\overline{F}_G^{k+1}|^2_{0,\Omega_g} \leq (\overline{F}_G^{k+1}, \overline{F}_G^k + 2\,\rho\,\overline{g}^{k+1})_{\Omega_g}\,, \tag{5.15a}$$

$$|\overline{\lambda}^{k+1}|^2 = |\overline{\lambda}^k|^2 + 2\,\rho\,\overline{\lambda}^k\,(\overline{g}^{k+1},1)_{\Omega_g} + \rho^2\,[\,(\overline{g}^{k+1},1)_{\Omega_g}\,]^2\,. \tag{5.15b}$$

Combining (5.14) and (5.15a,b) yields that

$$\begin{aligned}
&|\overline{F}_0^{k+1}|^2_{0,\Omega_g} + |\overline{F}_G^{k+1}|^2_{0,\Omega_g} + |\overline{\lambda}^{k+1}|^2 + (2\,\rho\,\mu - \rho^2)\,[\,(\overline{g}^{k+1},1)_{\Omega_g}\,]^2 \\
&\quad + 2\,\rho \sum_{i=1}^{I}(|\underline{Q}_r^{(i),h}|^{r-2}\underline{Q}_r^{(i),h} - |\underline{Q}_r^{(i),h,k+1}|^{r-2}\underline{Q}_r^{(i),h,k+1}, \overline{\underline{Q}}^{(i),k+1})^h \\
&\quad + 2\,\rho\,\ell \sum_{i=1}^{I}\Big((|Z_r^{(i),h}|^{r-2}Z_r^{(i),h} - |Z_r^{(i),h,k+1}|^{r-2}Z_r^{(i),h,k+1}, \overline{Z}^{k+1}\Big)_\Omega \\
&\qquad\qquad \leq |\overline{\lambda}^k|^2 + (\overline{F}_0^{k+1}, \overline{F}_0^k)_{\Omega_g} + (\overline{F}_G^{k+1}, \overline{F}_G^k)_{\Omega_g} \\
&\qquad\qquad \leq |\overline{\lambda}^k|^2 + \tfrac{1}{2}\,\Big[\,|\overline{F}_0^{k+1}|^2_{0,\Omega_g} + |\overline{F}_G^{k+1}|^2_{0,\Omega_g} + |\overline{F}_0^k|^2_{0,\Omega_g} + |\overline{F}_G^k|^2_{0,\Omega_g}\Big]\,.
\end{aligned} \tag{5.16}$$

Hence for $\rho \in (0, 2\,\mu)$, we have that the sequence $\{|\overline{F}_0^k|^2_{0,\Omega_g} + |\overline{F}_G^k|^2_{0,\Omega_g} + 2\,|\overline{\lambda}^k|^2\}_{k\geq 1}$ is monotonically decreasing and bounded below. Therefore it must converge to a limit, and hence the desired result (5.11) holds on noting (3.2). □

6 Solution of the Nonlinear Algebraic Systems

All the resulting nonlinear algebraic systems obtained above for $(\mathcal{P}_r^h)$, $(\mathcal{U}_r^h)$ and $(\mathcal{PM}_r^h)$ contain the same type of nonlinearity, $|v|^{r-2}v$ or its vector form, and this is solved iteratively. This is in addition to the external iterative loops, described in Section 5 above, to satisfy the constraints in the case of the partial MK, $(\mathcal{P}_r^h)$, and optimal matching $(\mathcal{PM}_r^h)$, problems. Clearly, replacing this nonlinear term as $|v^j|^{r-2}v^{j+1}$ at the $(j+1)^{\rm th}$ iteration is not possible as $r<2$ and $|v^j|$ can be zero. Our iterative scheme is based instead on the following regularized representation of this nonlinear term,

$$|v^j|_\varepsilon^{r-2}(v^{j+\frac{1}{2}}-v^j)+|v^j|^{r-2}v^j\,, \tag{6.1}$$

where $|v|_\varepsilon := \sqrt{|v|^2+\varepsilon^2}$ for $0<\varepsilon\ll 1$; and so all terms are well defined. In addition, over-relaxation $v^{j+1}=\alpha\, v^{j+\frac{1}{2}}+(1-\alpha)\, v^j$ with $\alpha\geq 1$ is used to accelerate the convergence.

Below we present such an iterative procedure for the solution of the nonlinear system, (5.1a–c), arising at the $(k+1)^{\rm th}$ iteration of the augmented Lagrangian method applied to $(\mathcal{P}_r^h)$. Given approximations to the Lagrange multipliers $u_r^{h,k}\in S^h$ and $\lambda_r^{\pm,h,k}\in\mathbb{R}$, and on setting $\underline{Q}_r^{h,k+1,0}=\underline{Q}_r^{h,k}\in\underline{V}_0^h$ and $\underline{q}_r^{\pm,h,k+1,0}=\underline{q}_r^{\pm,h,k}\in S_\pm^h$, we then apply the following iterative loop for $j\geq 0$; where to simplify the notation, we will omit the indices involving k,r and h on the iterates:

Given $\underline{Q}^j\equiv\underline{Q}_r^{h,k+1,j}\in\underline{V}_0^h$ and $q^{\pm,j}\equiv q_r^{\pm,h,k+1,j}\in S_\pm^h$; find $\underline{Q}^{j+\frac{1}{2}}\in\underline{V}_0^h$ and $q^{\pm,j+\frac{1}{2}}\in S_\pm^h$ such that

$$\begin{aligned}
&(|\underline{Q}^j|_\varepsilon^{r-2}\underline{Q}^{j+\frac{1}{2}},\underline{v}^h)^h+\mu\,(\underline{\nabla}\,.\,\underline{Q}^{j+\frac{1}{2}}+q^{+,j+\frac{1}{2}}-q^{j+\frac{1}{2}},\underline{\nabla}\,.\,\underline{v}^h)_\Omega\\
&\qquad=(u+\mu\, f^+-\mu\, f^-,\underline{\nabla}\,.\,\underline{v}^h)_\Omega+(\,[\,|\underline{Q}^j|_\varepsilon^{r-2}-|\underline{Q}^j|^{r-2}]\,\underline{Q}^j,\underline{v}^h)^h \qquad \forall\ \underline{v}^h\in\underline{V}_0^h, \quad \text{(6.2a)}\\
&\ell\,(|q^{+,j}|_\varepsilon^{r-2}q^{+,j+\frac{1}{2}},v^{+,h})_{\Omega^+}+\mu\,(\underline{\nabla}\,.\,\underline{Q}^{j+\frac{1}{2}}+q^{+,j+\frac{1}{2}}-q^{-,j+\frac{1}{2}},v^{+,h})_{\Omega^+}\\
&\qquad=(u+\mu\, f^+-\mu\, f^--\lambda^+,v^{+,h})_{\Omega^+}+\ell\,(\,[\,|q^{+,j}|_\varepsilon^{r-2}-|q^{+,j}|^{r-2}]\,q^{+,j},v^{+,h})_{\Omega^+}\\
&\qquad\qquad+\mu\,[\,(f^+-q^{+,j},1)_{\Omega^+}-m]\,(v^{h,+},1)_{\Omega^+} \qquad \forall\ v^{h,+}\in S_+^h, \quad \text{(6.2b)}\\
&\ell\,(|q^{-,j}|_\varepsilon^{r-2}q^{-,j+\frac{1}{2}},v^{-,h})_{\Omega^-}-\mu\,(\underline{\nabla}\,.\,\underline{Q}^{j+\frac{1}{2}}+q^{+,j+\frac{1}{2}}-q^{-,j+\frac{1}{2}},v^{-,h})_{\Omega^-}\\
&\qquad=(-u-\mu\, f^++\mu\, f^--\lambda^-,v^{-,h})_{\Omega^-}+\ell\,(\,[\,|q^{-,j}|_\varepsilon^{r-2}-|q^{-,j}|^{r-2}]\,q^{-,j},v^{-,h})_{\Omega^-}\\
&\qquad\qquad+\mu\,[\,(f^--q^{-,j},1)_{\Omega^-}-m]\,(v^{-,h},1)_{\Omega^-} \qquad \forall\ v^{-,h}\in S_-^h\,. \quad \text{(6.2c)}
\end{aligned}$$

Then set $\underline{Q}^{j+1}=\alpha\,\underline{Q}^{j+\frac{1}{2}}+(1-\alpha)\,\underline{Q}^j$ and $q^{\pm,j+1}=\alpha\, q^{\pm,j+\frac{1}{2}}+(1-\alpha)\, q^{\pm,j}$, and repeat the above iteration until the difference of iterates satisfy a given tolerance before updating the Lagrange multipliers. Note that the terms $[\,(f^\pm-q^{\pm,j},1)_{\Omega^\pm}-m]$ are lagged in order to maintain the sparsity of the linear system (6.2a–c).

The unbalanced problem $(\mathcal{U}_r^h)$ is unconstrained, and was solved using an iterative scheme similar to (6.2a–c), without any exterior Lagrange multiplier loop.

The additional nonlinearity, $\xi(t):=[t]_+$, had to be dealt with in the partial matching problem, $(\mathcal{PM}_r^h)$, because the augmented Lagrangian algorithm was employed also for the inequality constraints $g_r^h\geq 0$ and $g_r^h\leq G$. This non-differentiable nonlinearity was regularized as $\xi_\varepsilon(t):=|[t]_+|_\varepsilon-\varepsilon\equiv\sqrt{[t]_+^2+\varepsilon^2}-\varepsilon$ for $0<\varepsilon\ll 1$; and linearized at each iteration, similarly to (6.1).

7 Numerical Experiments

To test our numerical schemes we solved several two-dimensional examples. Solutions to MK and optimal matching problems are often singular; so to approximate such solutions and to determine more accurately the free boundaries of the transportation domains we used an adaptive finite element mesh; see [5] for the details of the refinement/coarsening algorithm. Coarse initial meshes with 1300–1800 triangles have been adaptively refined twice in most examples, yielding final meshes containing several thousand triangles. In the partial MK problems, $(\mathcal{P}_r^h)$, the mesh was refined wherever the fluxes $\underline{Q}_r^h$ and $q_r^{\pm,h}$ changed rapidly. Typically, these rapid changes of the auxiliary fluxes $q_r^{\pm,h}$ occur on the free boundaries of transportation domains; we used this observation to determine the free boundaries efficiently. Although the auxiliary fluxes are eliminated from the unbalanced and optimal matching problems, $(\mathcal{U}_r^h)$ and $(\mathcal{PM}_r^h)$, we calculated them when solving these problems for the purposes of mesh adaption and the determination of the free boundary.

In all the examples below we chose $r = 1 + 10^{-7}$ and $\varepsilon = 10^{-7}$. The domain Ω was either a unit square, in which case we chose $\ell = \frac{1}{2}\sqrt{2}$, or a unit square with a section removed; in the latter case the value $\ell = 2$ was sufficient to satisfy the necessary condition (2.1). We note that, provided (2.1) was satisfied, the value of ℓ did not influence the solution and the auxiliary fluxes were always non-negative. However, the value of ℓ did influence the convergence of the iterations and, although for the smaller value of ℓ we used over-relaxation with $\alpha = 1.5$, which led to a moderate acceleration in the partial and unbalanced MK problems; computations with $\ell = 2$ were performed with $\alpha = 1$ (no over-relaxation). Similarly, no over-relaxation was used for the partial optimal matching problems.

The Matlab Partial Differential Equations toolbox was used for the domain triangulation, and subdomains, $\Omega^{\pm}$ and Ω_g, with curved boundaries were approximated by a union of triangles. We refer to [4] for the Matlab realization of the lowest order Raviart-Thomas element in $\mathbb{R}^2$. With probability one any delta functions used as sources or sinks lied in the interior of a triangle, so no smoothing of these discrete sources/sinks was necessary.

The parameters for the augmented Lagrangian method for the equality and the inequality constraints in all examples were $\mu = \rho = 1$. The iterations have been performed until all constraints were satisfied up to a given tolerance ε_{AL}. Thus, for the partial MK problems, the conditions were

$$\begin{gathered} m^{-1} |\, \underline{\nabla} \cdot \underline{Q}_r^{h,k} - [P_+^h f_r^+ - q_r^{+,h,k}] + [P_-^h f_r^- - q_r^{-,h,k}]\, |_{0,1,\Omega} \leq \varepsilon_{AL}\,, \\ m^{-1} |(f_r^{\pm} - q_r^{\pm,h,k}, 1)_{\Omega^{\pm}} - m| \leq \varepsilon_{AL}\,; \end{gathered} \tag{7.1}$$

and, for the partial matching problems,

$$m^{-1}|(g_r^{h,k}, 1)_{\Omega_g} - m| \leq \varepsilon_{AL}, \qquad -\varepsilon_{AL} \leq g_r^{h,k} \leq G\,(1 + \varepsilon_{AL})\,. \tag{7.2}$$

On the k^{th} iteration of the augmented Lagrangian algorithm, the solution of the nonlinear algebraic system (5.1a–c) was computed until the tolerance $\varepsilon_{NL}^k = \varepsilon_{NL}/\sqrt{k}$ using (6.2a–c), so that the accuracy increased gradually with k. More precisely, let $\mathcal{E}^h$ be the set of all internal edges and $\underline{\phi}_e^h(\underline{x})$ the basis vector function associated with the edge $e \in \mathcal{E}^h$ in the Raviart-Thomas finite element space $\underline{V}_0^h$, see [4]. Then any $\underline{v}^h \in \underline{V}_0^h$ can be represented as $\sum_{e\in\mathcal{E}^h} v_e^h \, \underline{\phi}_e$ and we define the norm $\|\underline{v}^h\|_{\mathcal{E}^h} := \sum_{e\in\mathcal{E}^h} |e| \, |v_e^h|$. Our stopping criterion was

$$\frac{\|\underline{Q}^{j+1} - \underline{Q}^j\|_{\mathcal{E}^h}}{\|\underline{Q}^{j+1}\|_{\mathcal{E}^h}} \le \varepsilon_{NL}^k \qquad \text{and} \qquad \frac{|q^{\pm,j+1} - q^{\pm,j}|_{0,1,\Omega^\pm}}{|q_r^{\pm,j+1}|_{0,1,\Omega^\pm}} \le \varepsilon_{NL}^k \tag{7.3}$$

in the partial MK problems, (6.2a–c); only the first quantity, with $\varepsilon_{NL}^k = \varepsilon_{NL}$, in (7.3) for the unbalanced problems, and

$$\max_{i=1\to I} \left\{ \frac{\|\underline{Q}^{(i),j+1} - \underline{Q}^{(i),j}\|_{\mathcal{E}^h}}{\|\underline{Q}^{(i),j+1}\|_{\mathcal{E}^h}} \right\} \le \varepsilon_{NL}^k \qquad \text{and} \qquad \frac{|g^{j+1} - g^j|_{0,1,\Omega_g}}{|g^{j+1}|_{0,1,\Omega_g}} \le \varepsilon_{NL}^k \tag{7.4}$$

for the partial matching problems. It should be noted that it was better to update the multipliers after a few iterations of the nonlinear loop without necessarily satisfying the stopping criteria (7.3) and (7.4); and demand the fulfilment of these conditions only near the convergence of the augmented Lagrangian algorithm, Finally, we noticed that in all the partial matching examples that we solved, the optimal distributions g_r^h were non-negative even if the constraint $g_r^h \ge 0$ was omitted, and not taken into account in the numerical procedure. We do not know, however, whether this condition is automatically satisfied for all optimal partial matching problems.

Example 1. Partial MK problem, two ellipses. Let $f^+ = 1$ and $f^- = 2$ inside their supports $\Omega^\pm$, the left and right ellipses, respectively (Fig. 2, left), with semiaxes $a^+ = 2\,a^- = 0.2$, $b^+ = b^- = 0.45$, and the distance $d = 0.6$ between their axes of vertical symmetry. We chose $m = \frac{1}{2}\int_\Omega f^+ = \frac{1}{2}\int_\Omega f^- = \pi\, a^- \, b^-$. In this case it is cheapest to move the mass along horizontal paths from the right half of the left ellipse to the left half of the right one, with the axes of vertical symmetry of the ellipses as the exact free boundaries. Elementary integration gives the total cost of such transportation: $\mathcal{C} = a^- b^- (\pi d - 4\,a^-) = 0.06682$. The auxiliary fluxes $q^\pm$ are zero inside the regions from/to where the mass is transported and are equal to $f^\pm$ outside. Numerical calculations reproduced well this behavior and, since these jumps occur at the free boundaries, in our numerical simulations these boundaries are shown as the level contours $q_r^{\pm,h} = \frac{1}{2}\max q_r^{\pm,h}$. The approximate total cost was calculated on an adapted mesh of about nine thousand triangles as $\int_\Omega |\underline{Q}| \approx \sum_{\sigma\in\mathcal{T}^h} |\sigma| |\underline{Q}_r^h(\underline{o}_\sigma)| = 0.06671$, where $\underline{o}_\sigma$ denotes the centre of triangle σ. In this example $\varepsilon_{AL} = \varepsilon_{NL} = 10^{-4}$.

Example 2. Partial MK problem, two ellipses and an obstacle. Similar to the previous example but with Ω being the unit square with a section removed, that leads to a change in the transport paths (see Fig. 2, middle). The transport density becomes singular near the tip of the obstacle, with the approximate total cost being 0.07776. The exact solution is unknown; the numerical results where obtained with $\varepsilon_{AL} = 2 \cdot 10^{-5}$, $\varepsilon_{NL} = 10^{-4}$ and a mesh containing about five thousand triangles (Fig. 2, right).

Example 3. Partial MK problem with discrete sinks. Let f^+ be constant inside its support Ω^+, the circle with radius $R_0 = 0.4$, centre $(0.5, 0.5)$, and with total mass $\int_\Omega f^+ =$

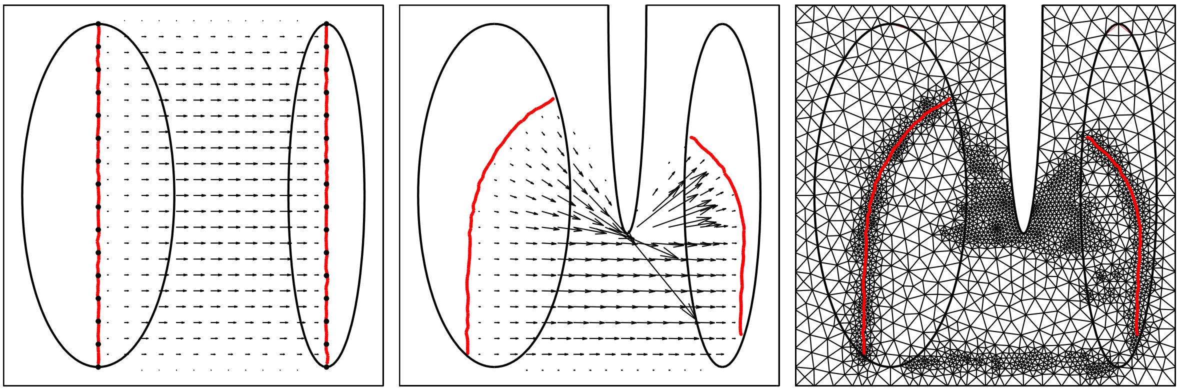

Figure 2: Partial MK problem. The left and right ellipses are supports of constant f^+ and f^-, respectively, and $m = \frac{1}{2}\int_\Omega f^+ = \frac{1}{2}\int_\Omega f^-$. Left – Ω is a unit square; middle – same with a section removed leading to an obstacle to transportation; right – the adapted mesh. Shown: arrows - the computed flux, red lines - the computed free boundary, black dots in the left plot indicate the known position of the exact free boundaries.

1; let f^- be the sum of three unit delta functions placed at $(0.1, 0.8)$, $(0.4, 0.5)$ and $(0.7, 0.7)$, (Fig. 3). We chose $m = \frac{1}{2}$. It is easy to see that mass from each point of the transportation domain should in this case be transported to the closest δ-sink (each of them remains unsaturated) and all points on the boundary of this transportation domain must be at the same unknown distance r_0 from the closest sink. Taking into account all possible intersections of the main circle and three circles of radius r centred at the sinks, one can calculate the integral of f^+ over such a domain as a function $s(r)$. Solving the nonlinear algebraic equation $s(r_0) = m$, we found that $r_0 = 0.2034$ and were able to plot the exact free boundary together with the boundary determined numerically, which is represented by the level contour $q_r^{+,h} = \frac{1}{2}\max q_r^{+,h}$. The approximate masses transported to each sink were, respectively, 0.0461, 0.2484 and 0.2055. In this example $\varepsilon_{AL} = 0.5\cdot 10^{-4}$, $\varepsilon_{NL} = 10^{-4}$ and; to clarify the solution structure, we plotted not the optimal flux itself but the directions of optimal transportation, the vector field $\underline{Q}_r^h / |\underline{Q}_r^h|_\varepsilon$.

Example 4. Unbalanced MK problem, two ellipses. The same as examples 1 and 2, except $f^- = 1$ inside the right ellipse, so $m = \frac{1}{2}\int_\Omega f^+ = \int_\Omega f^-$ and so the problems are unbalanced. In the first case (Fig. 4, left) Ω is the unit square and the optimal plan is again to move the mass from the right half of the left ellipse along the horizontal paths; the only free boundary is the vertical axis of symmetry of this ellipse, and the auxiliary flux $q^+ = f^+ - \underline{\nabla}\,.\,\underline{Q}$ has a jump on this boundary. The total cost of transportation, found by means of elementary integration, is $\mathcal{C} = a^- b^- (\pi\, d - \frac{8}{3}\, a^-) = 0.07282$. The numerical solution of $(\mathcal{U}_r^h)$ in this example was obtained on a crude mesh adapted only once and containing about three thousand triangles with $\varepsilon_{NL} = 4 \cdot 10^{-6}$. Nevertheless, we obtained an accurate total cost estimate, $\sum_{\sigma\in\mathcal{T}^h} |\sigma||\underline{Q}_r^h(\underline{o}_\sigma)| = 0.07270$. The free boundary (red line), approximated by the level contour $q_r^{+,h} = \frac{1}{2}\max q_r^{+,h}$ is also close to the exact boundary (black dots). The problem with an obstacle (Fig. 4, right) was solved on a mesh with about eight thousand triangles, with $\varepsilon_{NL} = 5 \cdot 10^{-5}$, yielding an approximate

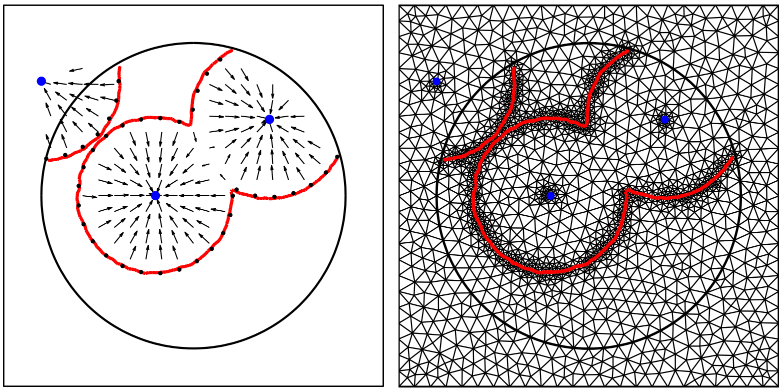

Figure 3: Partial MK problem. f^+ is distributed uniformly in the circle and has the total mass $\int_\Omega f^+ = 1$; f^- is the sum of three point sinks each of mass one (the blue dots). Left – the optimal transportation plan for $m = \frac{1}{2}$; right – the adapted mesh, about four thousand triangles. Shown: arrows - the computed directions of optimal transportation, red lines - the computed free boundary, black dots in the left plot indicate the position of the exact boundary.

cost 0.08971.

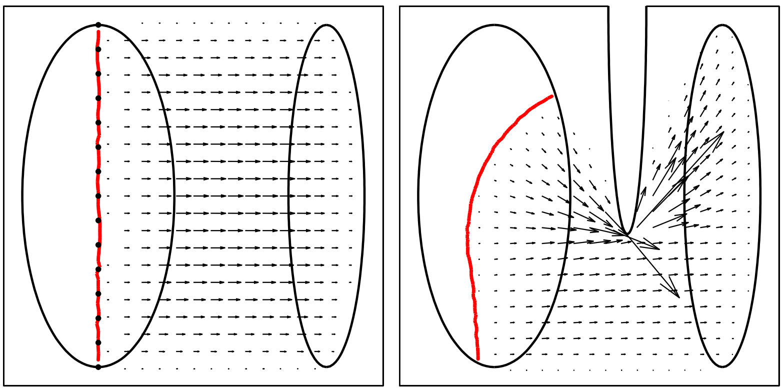

Figure 4: Unbalanced MK problem. As in Fig. 2 but with $m = \frac{1}{2}\int_\Omega f^+ = \int_\Omega f^-$.

Example 5. Unbalanced and balanced MK problems with discrete sinks. If, in the configuration of example 3, we take f^- to be the sum of three delta functions at the same locations as before, but now each with its weight equal to the mass transported to that sink in example 3, then the problem is now an unbalanced one with the same solution. Solving the unbalanced problem, $(\mathcal{U}_r^h)$, on a mesh of approximately five thousand triangles with $\varepsilon_{NL} = 3 \cdot 10^{-5}$ we obtained free boundaries that are visually identical to those obtained in example 3; and the total cost estimates differed by 0.6%. We present two more examples with the same source and three equal point sinks in the same locations as before. In the first one (Fig. 5, left) the delta sinks each have the weight $\frac{1}{6}$; this is again an unbalanced problem with $m = \frac{1}{2}$. In the second the weights are $\frac{1}{3}$, so $m = \int_\Omega f^+ = \int_\Omega f^-$ and the problem is balanced; and hence there is no free boundary as both auxiliary fluxes are zero. The solution (Fig. 5, right) obtained from $(\mathcal{U}_r^h)$ with about thirteen thousand triangles, and $\varepsilon_{NL} = 10^{-5}$. Here the support of f^+ is divided into three attraction domains corresponding to the three sinks. On the boundaries of the attraction domains $\underline{Q}$ is zero and, approximately, these boundaries are represented by the level contour $|\underline{Q}_r^h| = 2 \cdot 10^{-3} \max |\underline{Q}_r^h|$ (green line). We note that in the balanced MK problem $\underline{\nabla} \cdot \underline{Q} = f^+ - f^-$ and, numerically, we obtained that $|P^h(f^+ - f^-) - \underline{\nabla} \cdot \underline{Q}_r^h|_{0,\infty,\Omega} = 2 \cdot 10^{-6}$.

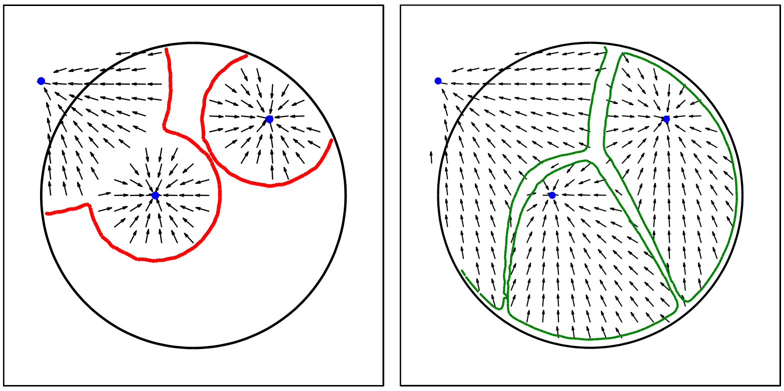

Figure 5: Unbalanced and balanced MK problem with three discrete sources; as in the Fig. 3, except the sink weights. Left – each sink is a delta function with weight $\frac{1}{6}$, hence $m = \int_\Omega f^- = \frac{1}{2}\int_\Omega f^+$. Right – each sink has weight $\frac{1}{3}$ and hence the problem is balanced, $m = \int_\Omega f^- = \int_\Omega f^+$. The green lines are the approximate boundaries of the sink attraction domains.

Example 6. Partial matching problem, two ellipses. The ellipses in Fig. 6 are as in example 1. We set $f^{(1)} = 1$ in the left ellipse, $f^{(2)} = 2$ in the right one; the matching mass $m = \frac{1}{2}\int_\Omega f^{(1)} = \frac{1}{2}\int_\Omega f^{(2)} = 0.045\,\pi$, and Ω_g is the rectangle between the two ellipses having width 0.2. We chose first $G = 1$ (Fig. 6, left). Since $G\,|\Omega_g| > m$, a solution to $(\mathcal{PM})$ exists. In this case, the transport to Ω_g along horizontal paths from the right half of the left ellipse and the left half of the right one is admissible; this is the cheapest matching transportation. Note, however, that for each pair of the meeting transport rays the matching with any distribution of g upon their common segment lying in $\overline{\Omega_g}$ has the same total cost, so there are infinitely many optimal matching plans satisfying $g \in [0, G]$. However, a selection is made by our regularized formulation, $(\mathcal{PM}_r)$. The total cost of optimal matching is equal to the transportation cost in example 1, $\mathcal{C} = 0.06682$. Solving $(\mathcal{PM}_r^h)$ using an adapted mesh with about eight thousand triangles and $\varepsilon_{AL} = 10^{-4}$, $\varepsilon_{NL} = 10^{-3}$, we obtained $\mathcal{C} \approx \sum_{\sigma\in\mathcal{T}^h} |\sigma| \left[|\underline{Q}_r^{(1),h}(\underline{o}_\sigma)| + |\underline{Q}_r^{(2),h}(\underline{o}_\sigma)| \right] = 0.06671$.

For $G = m/|\Omega_g|$ (Fig. 6, right) the optimal matching plan also exists; in this case g must be equal to G in the whole set $\overline{\Omega_g}$ and transportation along only horizontal rays is now not possible. A numerical solution was obtained on a mesh with about seven thousand triangles, $\varepsilon_{AL} = \varepsilon_{NL} = 10^{-4}$, and the estimated cost was $\mathcal{C} \approx 0.06821$.

Example 7. Partial matching problem, two rectangles. Let $f^{(1)} = f^{(2)} = 1$ in their supports, the left and right rectangles, respectively; and the circle be the matching domain Ω_g (see Fig. 7). We set $m = \frac{1}{3}\int_\Omega f^{(1)} = \frac{1}{2}\int_\Omega f^{(2)}$. Solving $(\mathcal{PM}_r^h)$ with $G = 2\,m/|\Omega_g|$ (Fig. 7, left), we obtain, within the chosen tolerances, $g_r^h = G$ in half of Ω_g and $g_r^h = 0$

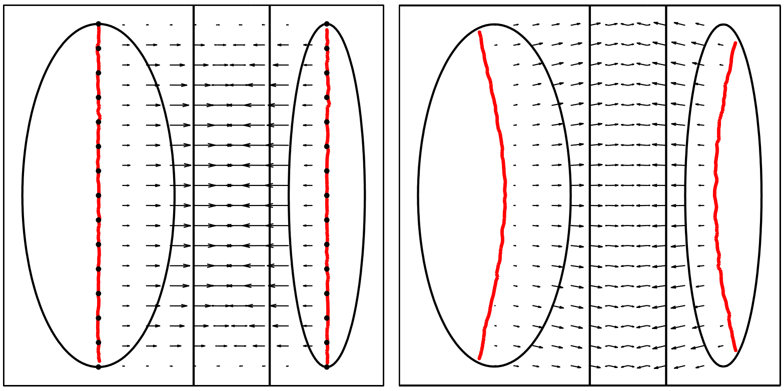

Figure 6: Partial matching problems. Left – $G = 1$, right – $G = m/|\Omega_g|$. Shown: the boundaries of the domain Ω_g and the supports of $f^{(1)}$ and $f^{(2)}$ (black lines); the optimal fluxes $\underline{Q}^h_{1,r}$ and $\underline{Q}^h_{2,r}$ (arrows); the computed boundaries of the transportation domains (red lines). The exact boundaries are indicated by the black dots (left).

in the other half; in this case the mesh contained about five thousand triangles. The matching domain shrinks as the value of G increases; in the limit $G \to \infty$ (not covered by our theoretical analysis) the optimal matching occurs at the leading edge of the boundary of Ω_g. Solving the same problem with $G = 10^{10}$ and an adapted mesh of about 3500 triangles (Fig. 7, right), we obtained that g^h_r is supported in a single layer of triangles along the leading border of Ω_g; the constraint $g \leq G$ was inactive in this case. In both of these examples we set $\varepsilon_{AL} = 10^{-4}$ and $\varepsilon_{NL} = 10^{-3}$.

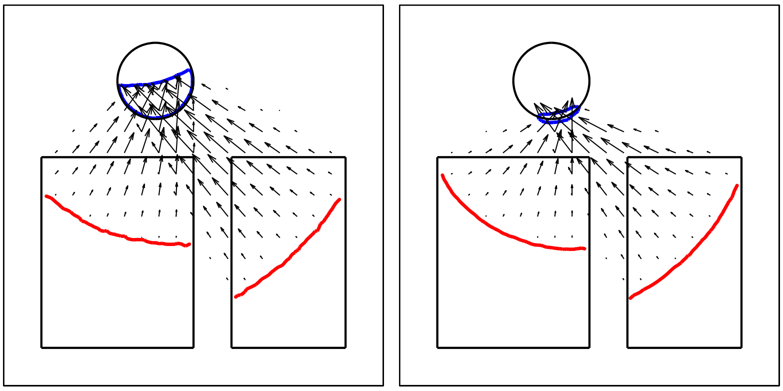

Figure 7: Partial matching problems. Left – $G = 2\,m/|\Omega_g|$, right – $G = \infty$. Shown: the boundaries of the domain Ω_g and the supports of $f^{(1)}$ and $f^{(2)}$ (black lines); the optimal fluxes $\underline{Q}^h_{1,r}$ and $\underline{Q}^h_{2,r}$ (arrows); the computed boundaries of the transportation domains (red lines) and of the matching domains (blue lines).

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