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Variational model of sandpile growth

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A model describing the evolving shape of a growing pile is considered, and is shown to be equivalent to an evolutionary quasi-variational inequality. If the support surface has no steep slopes, the inequality becomes a variational one. For this case existence and uniqueness of the solution are proved.

1 Introduction

Spatially extended open dissipative systems have recently attracted much interest among physicists. These systems are capable of demonstrating almost instantaneous long-range interactions and, under the action of external forces, often tend to organize themselves into a stable or statistically stable critical state [1]. Modifications of a cellular automaton model of sandpiles [2] have often been used in simulations of these systems (see, for example, [3–6]).

More realistic continuum mechanics models have also been proposed for sandpiles [7–9], river nets [9] and type-II superconductors [10]. The continuous models contain very similar variational or quasi-variational inequalities, allowing one to describe the behaviour typical of extended dissipative systems. One of these models, the continuous deterministic model of sandpile evolution [7], is considered in this paper. The problem under consideration may be formulated as follows:

Let a cohesionless ideal granular material be poured out of a distributed source onto a given rough rigid surface. We find the shape of a growing pile.

The model [7] has been already discussed in [9], and so only a brief description is given in the first part of this paper. In the second part, we prove that this model is equivalent to an evolutionary quasi-variational inequality (the scheme of this proof has been previously outlined in [7, 9]). The last part of our work concerns a special case where the inequality becomes a variational one. Its analytical solutions, describing the growth of the real sandpiles on flat open platforms, have been found in [9]. A method for the numerical solution was proposed in [8]. Here we prove the existence and uniqueness of the solution.

2 Model of pile growth

The flow of granular material down the slope of a growing pile is usually confined to a thin boundary layer which is distinctly separated from the motionless bulk. Assuming the bulk density of the material in the pile to be constant, we can write the mass conservation law as

$$\frac{\partial h}{\partial t}+\nabla\cdot\vec{j}=w.$$

Here $h(x, t)$ is the free surface of a pile, $w(x, t)$ is the intensity of a distributed source, $\vec{j}(x, t)$ is the horizontal projection of the material flux in the surface layer, $x \in \Omega \subset R^2$, and Ω is a bounded domain with Lipschitz-continuous boundary Γ.

We assume that the surface flow is directed down the path of steepest descent,

$$\vec{j} = -m \nabla h,$$

where

$$m(x, t) \geqslant 0 \qquad (1)$$

is an unknown scalar function. We would like to stress that m is not supposed to be a local function or functional of h but is introduced as an auxiliary unknown. This is the key feature of our model. The conservation law now assumes the form

$$\frac{\partial h}{\partial t} - \nabla \cdot (m \nabla h) = w. \qquad (2)$$

At $t = 0$ the free surface coincides with the support surface,

$$h|_{t=0} = h_0(x). \qquad (3)$$

The free surface is never below the support surface,

$$h(x, t) \geqslant h_0(x), \qquad (4)$$

and wherever the free surface is above the support, it has an incline not greater than the angle of repose of the granular material,

$$h(x, t) > h_0(x) \Rightarrow |\nabla h(x, t)| \leqslant \gamma, \qquad (5)$$

where γ is the tangent of the angle of repose. No pouring occurs over the parts of the pile surface inclined at less than the angle of repose:

$$|\nabla h(x, t)| < \gamma \Rightarrow m(x, t) = 0. \qquad (6)$$

Let the granular material be allowed to leave the system freely through part Γ_1 of the domain boundary while the other part, $\Gamma_2 = \Gamma \backslash \Gamma_1$, be an impermeable wall. The corresponding boundary conditions are

$$h|_{\Gamma_1} = h_0|_{\Gamma_1} \quad \text{(a)}, \qquad m \frac{\partial h}{\partial n}\bigg|_{\Gamma_2} = 0 \quad \text{(b)}. \qquad (7)$$

The model of pile growth (1)–(7) contains two unknowns, the free surface h and an auxiliary function m. As is shown below, the latter function is a Lagrange multiplier which can be excluded in the transition to an equivalent variational formulation of this model.

3 The quasi-variational inequality

We need functional spaces $H = L^\infty(\Omega)$, $V_q = W^{1,q}(\Omega)$ with $2 \leqslant q \leqslant \infty$, and also spaces of functions of time with values in the appropriate Banach spaces, $\mathscr{H} = L^\infty(0, T; H)$ and $\mathscr{V}_q = L^q(0, T; V_q)$. We shall write simply V and $\mathscr{V}$ instead of V_∞ and $\mathscr{V}_\infty$. We denote by X' the dual to the space X, the natural pairing of elements of V_q and V_q', H and H' by $(.,.)$, and of elements of $\mathscr{V}_q$ and $\mathscr{V}_q'$, $\mathscr{H}$ and $\mathscr{H}'$ by $\langle .,. \rangle$. The notation $\|\cdot\|_X$ means the norms in the space X and we shall write $\|\cdot\|$ if $X = L^2$. We define also $Q = \Omega \times (0, T)$.

Let $h_0 \in V$. For every function $\phi \in \mathscr{V}$ we define a mapping $B_\phi : \mathscr{V} \to \mathscr{H}$ by

$$B_\phi(\psi) = \tfrac{1}{2}(|\nabla\psi|^2 - M(\phi)),$$

where

$$M(\phi)(x,t) = \begin{cases} \gamma^2 & \text{if} \quad \phi(x,t) > h_0(x), \\ \max(\gamma^2, |\nabla h_0(x)|^2) & \text{if} \quad \phi(x,t) \leqslant h_0(x). \end{cases}$$

Let us define a partial ordering on the spaces of functions, $\phi \geqslant \psi$ if this inequality holds almost everywhere (a.e.), and denote by $\mathscr{C}$ the cone of non-negative elements. We also define the family of closed convex sets

$$\mathscr{K}(\phi) = \{\psi \in \mathscr{V} \mid B_\phi(\psi) \leqslant 0, \quad \psi|_{\Gamma_1} = h_0|_{\Gamma_1} \quad \text{for a.e. } t\}.$$

We assume that $w \in \mathscr{V}'$ and $w \geqslant 0$, i.e. $\langle w, \phi \rangle \geqslant 0$ for any function $\phi \in \mathscr{V}$, $\phi \geqslant 0$. Let us consider the quasi-variational inequality

Find a function h such that

$$\exists h' = \partial h/\partial t \in \mathscr{V}_q' \quad \text{for some } 2 \leqslant q < \infty, \tag{8}$$

$$h \in \mathscr{K}(h), \tag{9}$$

$$\langle h' - w, \phi - h \rangle \geqslant 0, \quad \forall \phi \in \mathscr{K}(h), \tag{10}$$

$$h|_{t=0} = h_0. \tag{11}$$

Since $h \in \mathscr{V}$ and $h' \in \mathscr{V}_q'$, we have $h \in C([0,T]; L^2(\Omega))$ which makes sense of (11). The conditions (9)–(10) are equivalent to the following:

$$h \in \arg \min_{\substack{B_h(\phi) \leqslant 0 \\ \phi \in A}} J_h(\phi), \tag{12}$$

where the linear functional $J_h \in \mathscr{V}'$ is defined by $J_h(\phi) = \langle h' - w, \phi \rangle$ and $A \subset \mathscr{V}$ is the set of functions satisfying (7a). Below we use this formal representation of (8)–(11) as an optimization problem on $\mathscr{V}$ to prove that this quasi-variational inequality is equivalent to the model of pile growth and the auxiliary variable m is a Lagrange multiplier, related to the constraint $B_h(\phi) \leqslant 0$.

The usage of the Lagrange multiplier technique is justified if a constraint qualification hypothesis of some kind is fulfilled. The Slater condition ([11], ch. 3, §5)

$$\exists \psi_0 \in A : -B_h(\psi_0) \in \operatorname{int} \mathscr{C} \tag{13}$$

is the simplest and often the most convenient of such conditions. It is to satisfy the Slater condition we have had to define the admissible sets $\mathscr{K}(h)$ as subsets of non-reflexive space $\mathscr{V}$. Then $B_h : A \to \mathscr{H} = L^\infty(Q)$, and it is only in L^∞ of all L^p spaces the cone of non-negative elements $\mathscr{C}$ has a non-empty interior. The Lagrange multiplier belongs then to the dual space, $\mathscr{H}'$. Therefore, to show the equivalence of two problems, we need first to derive a weak formulation for the model (1)–(7), valid for $\{h, m\} \in \mathscr{V} \times \mathscr{H}'$.

The partial ordering on $\mathscr{H}$ induces a partial ordering on $\mathscr{H}'$, hence (1) may be understood as $\langle m, \phi \rangle \geqslant 0$ for any $\phi \geqslant 0$. The weak form of (2) with the boundary condition (7b) may be written as

$$\langle h' - w, \psi \rangle + \langle m, \nabla h \cdot \nabla \psi \rangle = 0 \tag{14}$$

for all $\psi \in \mathscr{V}$ such that $\psi|_{\Gamma_1} = 0$. Equations (3)–(5) and (7a) must hold almost everywhere. To make sense also to (6), let us define

$$Q^- = \{(x,t) \in Q \,|\, |\nabla h(x,t)| < \gamma \quad \text{a.e.}\}.$$

For $m \in \mathscr{H}'$, (6) can now be formulated as follows:

$$\forall \phi \in \mathscr{H}, \quad \operatorname{supp} \phi \subset Q^- \Rightarrow \langle m, \phi \rangle = 0. \tag{15}$$

Theorem 1 *Suppose* $h_0 \in V$, $w \in \mathscr{V}'$, $w \geqslant 0$, *and there exists a function* $\psi_0 \in V$ *such that*

$$\psi_0|_{\Gamma_1} = h_0|_{\Gamma_1}, \quad \| \, |\nabla \psi_0(x)| \, \|_H < \gamma.$$

Then the function h is a solution of the quasi-variational inequality (8)–(11) *if and only if there exists* $m \in \mathscr{H}'$ *such that the pair* $\{h, m\}$ *is a weak solution to problem* (1)–(7).

Proof Let us fix h in the functional J_h and mapping B_h of optimization problem (12). The continuous functional J_h is linear, B_h is convex in the sense of the partial ordering on $\mathscr{H}$, and the set A is a closed convex subset of $\mathscr{V}$. The Slater condition (13) holds for any h due to the assumption of this theorem. If the functional J_h is bounded from below in $\mathscr{K}(h)$ (we will check this later), the necessary and sufficient condition of optimality for (12) is the existence of saddle point of Lagrangian $L(\phi, p) = J_h(\phi) + \langle p, B_h(\phi) \rangle$ (see [11], ch. 3, th. 5.1). The exact formulation is: $\tilde{\phi} \in A$ is a point of minimum if and only if there exists $m \in \mathscr{H}'$, $m \geqslant 0$ such that

$$L(\tilde{\phi}, p) \leqslant L(\tilde{\phi}, m) \leqslant L(\phi, m),$$

for all $\phi \in A$, $p \in \mathscr{H}'$, $p \geqslant 0$. The condition of complementary slackness,

$$\langle m, B_h(\tilde{\phi}) \rangle = 0,$$

which means that the Langrange multiplier is zero wherever the constraint is not active, is thereby satisfied.

The solution of quasi-variational inequality, determining J_h and B_h in (12), is a point of minimum, and so must satisfy the optimality condition. Hence, substituting $\tilde{\phi} = h$, we obtain a condition characterizing solutions of quasi-variational inequality.

The function h is a solution of quasi-variational inequality (8)–(11) if and only if it satisfies the conditions (8) and (11) and there exists a functional $m \in \mathscr{H}'$, $m \geqslant 0$, such that the pair $\{h, m\}$ is a saddle point of the Lagrangian, i.e.

$$J_h(h) + \langle p, B_h(h) \rangle \leqslant J_h(h) + \langle m, B_h(h) \rangle \leqslant J_h(\phi) + \langle m, B_h(\phi) \rangle \tag{16}$$

for all $\phi \in A$, $p \geqslant 0$. The condition of complementary slackness now reads

$$\langle m, B_h(h) \rangle = 0. \tag{17}$$

Let us assume that a solution h of the quasi-variational inequality (8)–(11) exists. Then the functional J_h is bounded from below in $\mathscr{K}(h)$, and so there exists a saddle point $\{h, m\} \in \mathscr{V} \times \mathscr{H}'$. Let us show that the pair $\{h, m\}$ is a weak solution to the problem (1)–(7).

By (16), the functional $\langle h' - w, \phi \rangle + \frac{1}{2}\langle m, |\nabla\phi|^2 - M(h)\rangle$ has a minimum in A at the point $\phi = h$. Taking the variation of this functional, we obtain (14), the weak form of (2) with the boundary condition (7). Conditions (1) and (3) hold, and condition (5) also holds a.e. since $h \in \mathscr{K}(h)$.

Like any non-negative functional from $\mathscr{H}'$, the functional m can be represented [12] as

$$\langle m, \phi \rangle = \int_Q \phi \, d\mu,$$

where μ is a non-negative addition function defined on the Lebesgue-measurable subsets of Q and such that $\mu(Q) < \infty$ and

$$\forall Q_1 \subset Q, \quad \operatorname{mes} Q_1 = 0 \Rightarrow \mu(Q_1) = 0.$$

It follows from the complementary slackness condition (17) and the inequality $B_h(h) \leqslant 0$ that

$$\int_{Q^-} B_h(h) \, d\mu = 0$$

But $B_h(h) < 0$ a.e. in Q^-, hence $\mu(Q^-) = 0$ and the weak version (15) of condition (6) is true.

To prove that $\{h, m\}$ is a weak solution to (1)–(7) we need now only check that $h \geqslant h_0$. Let us denote $a^+ = \max(a, 0)$ and define

$$\phi = \begin{cases} h + (h_0 - h)^+ & \text{for} \quad 0 \leqslant t \leqslant t_0, \\ h & \text{otherwise.} \end{cases}$$

Since $\phi \in \mathscr{K}(h)$ and $\langle w, \phi - h \rangle \geqslant 0$, we obtain

$$0 \leqslant \langle h' - w, \phi - h \rangle \leqslant -\tfrac{1}{2}\|(h_0 - h(t_0))^+\|^2, \quad \forall t_0 \in [0, T],$$

which proves inequality (4).

Now let the pair $\{h, m\} \in \mathscr{V} \times \mathscr{H}'$ be a weak solution of pile growth model, i.e. $h' \in \mathscr{V}'_q$ and h, m satisfy (1), (3)–(5), (14), (15) and (7a). If $\Gamma_1 \neq \emptyset$, the set $\mathscr{K}(h)$ is bounded in $\mathscr{V}$ and so the continuous functional J_h is bounded on this set. Otherwise, (14) yields that J_h takes the same values at the functions ϕ and $\phi - (\operatorname{mes} \Omega)^{-1} \int_\Omega \phi$, and so it is sufficient to show that this functional is bounded on the set

$$\mathscr{K}_0(h) = \left\{ \phi \in \mathscr{K}(h) \,\middle|\, \int_\Omega \phi = 0 \quad \text{a.e. in } (0, T) \right\}.$$

For $\phi \in \mathscr{V}$, the condition $\int_\Omega \phi = 0$ a.e. in $(0, T)$ implies $\|\phi\|_{\mathscr{V}} \leqslant C \|\nabla \phi\|_{\mathscr{H}}$, where C is a constant independent of ϕ. For $\phi \in \mathscr{K}(h)$

$$|\nabla \phi(x, t)|^2 \leqslant M(h)(x, t) \leqslant \max(\gamma^2, \|h_0\|_V^2)$$

and J_h is bounded, since $\mathscr{K}_0(h)$ is a bounded subset of $\mathscr{V}$.

It remains to show that $\{h, m\}$ is a saddle point of the Lagrangian. By (5), $|\nabla h| \leqslant \gamma$ a.e. where $h > h_0$. On the other hand, for almost all t, the function $\phi_t(x) = h(x, t) - h_0(x)$ is differentiable a.e. with respect to x and has minima at the points of the set $\Omega_t = \{x \in \Omega | h(x, t) = h_0(x)\}$. Therefore, with $h(x, t) = h_0(x)$, we have $\nabla h(x, t) = \nabla h_0(x)$ a.e. and so $|\nabla h|^2 \leqslant M(h)$. Up to a set of measure zero, this inequality is strict only where $|\nabla h| < \gamma$, and so $\operatorname{supp} B_h(h) = Q^-$. The condition of complementary slackness (17) follows now from (15) and yields the first inequality in (16), since $\langle p, B_h(h)\rangle \leqslant 0$ for any $p \geqslant 0$.

Finally, let $\phi \in A$. Using (14) with $\psi = \phi - h$, we obtain

$$J_h(\phi) + \langle m, B_h(\phi)\rangle - J_h(h) - \langle m, B_h(h)\rangle = \tfrac{1}{2}\langle m, |\nabla(\phi - h)|^2\rangle \geqslant 0$$

which completes the proof.

Corollary *Let h be a solution of quasi-variational inequality* (8)–(11) *and* $w \geqslant 0$. *Then h is a non-decreasing function of time.*

Proof Let

$$\phi = \begin{cases} h + (h|_{t=t_1} - h)^+ & \text{for} \quad t_1 \leqslant t \leqslant t_2, \\ h & \text{otherwise.} \end{cases}$$

To show that $\phi \in \mathscr{K}(h)$ we need to check that $|\nabla\phi|^2 \leqslant M(h)$ a.e. in the set $Q_0 = \{(x,t) | t_1 \leqslant t \leqslant t_2,\ h(x,t_1) > h(x,t) \text{ a.e.}\}$, where ϕ differs from h. In this set $\phi(x,t) = h(x,t_1) > h(x,t)$ a.e. and we have already proved that $h(x,t) \geqslant h_0(x)$ a.e. for any t. Therefore, a.e. in Q_0

$$|\nabla\phi(x,t)|^2 = |\nabla h(x,t_1)|^2 \leqslant \gamma^2 \leqslant M(h)\,(x,t).$$

Substituting ϕ into (10) and taking into account that $\phi - h \geqslant 0$ and $w \geqslant 0$, we obtain $(h(x,t_1) - h(x,t_2))^+ = 0$, and so $h(x,t_2) \geqslant h(x,t_1)$ for $t_2 > t_1$.

4 A special case: a variational inequality

If the support surface has no steep slopes, i.e. $|\nabla h_0| \leqslant \gamma$, the set of admissible functions $\mathscr{K}$ is independent of the solution h and problem (8)–(11) becomes a variational inequality. We will first consider the case of homogeneous boundary condition

$$h_0|_{\Gamma_1} = 0$$

and prove the existence and uniqueness of a weak solution in this case. Let us define the Banach spaces

$$U = \{\phi \in V_4 \,|\, \phi|_{\Gamma_1} = 0\}, \quad \mathscr{U} = L^4(0,T;U),$$

and the set

$$\mathscr{K} = \{\phi \in \mathscr{U} \,|\, |\nabla\phi| \leqslant \gamma\}.$$

If $\phi \in \mathscr{K}$ is such that $\phi' \in \mathscr{U}'$, the inequality (10) yields

$$\int_0^s \{(\phi' - w, \phi - h) + (h' - \phi', \phi - h)\} \geqslant 0$$

for all $s \in [0,T]$, and so

$$\int_0^s (\phi' - w, \phi - h) \geqslant \tfrac{1}{2}\|\phi - h\|^2_{L^2(\Omega)}\Big|_{t=0}^{t=s}. \tag{18}$$

For this weak formulation of variational inequality only the continuity of h in time is needed.

Theorem 2 *Let* $w \in \mathscr{U}'$, $h_0 \in V$, $h_0|_{\Gamma_1} = 0$, *and* $|\nabla h_0| \leqslant \gamma$. *Then there exists a unique function* h *such that*

$$h \in \mathscr{K} \cap C([0,T], L^2(\Omega)),$$

$$h|_{t=0} = h_0,$$

and (18) *is fulfilled for all* $s \in [0,T]$ *and all* $\phi \in \mathscr{K}$ *such that* $\phi' \in \mathscr{U}'$.

Proof To prove the theorem we follow the penalty method developed for parabolic variational inequalities ([13], ch. 3, §6). The proof, however, has to be modified, since in our case the operator of the variational inequality is not coercive.

Existence Let us consider a boundary value problem for a parabolic equation,

$$h_\epsilon' + \frac{1}{\epsilon}\beta_\epsilon(h_\epsilon) = w, \tag{19}$$

$$h_\epsilon|_{\Gamma_1} = 0, \quad \left.\frac{\partial h_\epsilon}{\partial n}\right|_{\Gamma_2} = 0,$$

$$h_\epsilon|_{t=0} = h_0,$$

where $\epsilon > 0$ is the penalty parameter and the bounded monotone operator $\beta_\epsilon: U \to U'$ is defined by the following relation:

$$(\beta_\epsilon(h), \phi) = \int_\Omega ((|\nabla h|^2 - \gamma^2)^+ + \epsilon^2)\, \nabla h \cdot \nabla \phi, \quad \forall \phi \in U.$$

The boundary value problem has a unique solution $h_\epsilon \in \mathscr{U}$ with $h_\epsilon' \in \mathscr{U}'$ [13]. Multiplying equation (19) by h_ϵ and integrating we obtain

$$\tfrac{1}{2}\|h_\epsilon(t)\|^2 + \frac{1}{\epsilon}\int_0^t (\beta_\epsilon(h_\epsilon), h_\epsilon) = \int_0^t (w, h_\epsilon) + \tfrac{1}{2}\|h_0\|^2 \tag{20}$$

and, since $\int_0^t (\beta_\epsilon(h_\epsilon), h_\epsilon) \geqslant 0$,

$$\|h_\epsilon\|^2_{L^\infty(0,T;L^2(\Omega))} \leqslant C\|w\|_{\mathscr{U}'}\, \|h_\epsilon\|_{\mathscr{U}} + C, \tag{21}$$

$$\|h_\epsilon(t)\|^2 \leqslant C\|w\|_{\mathscr{U}'}\, \|h_\epsilon\|_{\mathscr{U}} + C. \tag{22}$$

(C will denote different constants.) From (20) and the inequality $z^4 \leqslant 2z^2\,(z^2 - \gamma^2) + \gamma^4$ it follows that

$$\int_0^T \int_\Omega |\nabla h_\epsilon|^4 \leqslant C\langle \beta_\epsilon(h_\epsilon), h_\epsilon \rangle + C \leqslant \epsilon C\|w\|_{\mathscr{U}'}\, \|h_\epsilon\|_{\mathscr{U}} + C.$$

The last estimate, inequality (22), and the inequality

$$\int_\Omega u^4 \leqslant C\left(\int_\Omega u^2\right)^2 + C\int_\Omega |\nabla u|^4$$

imply that

$$\|h_\epsilon\|_{\mathscr{U}} \leqslant C, \tag{23}$$

where C does not depend upon ϵ. Due to (21) and (23), there exists a subsequence, also denoted by h_ϵ, such that as $\epsilon \to 0$

$$h_\epsilon \to h \quad \text{in } \mathscr{U} \text{ weakly}, \tag{24}$$

$$h_\epsilon \to h \quad \text{in } L^\infty(0,T;L^2(\Omega)) \ *\text{-weakly}. \tag{25}$$

For any $\phi \in \mathscr{U}$,

$$|\langle \beta_\epsilon(h_\epsilon), \phi \rangle| \leqslant \langle (|\nabla h_\epsilon|^2 - \gamma^2)^+ + \epsilon^2, |\nabla h_\epsilon|^2 \rangle^{1/2}$$
$$\times \langle (|\nabla h_\epsilon|^2 - \gamma^2)^+ + \epsilon^2, |\nabla \phi|^2 \rangle^{1/2} \leqslant \langle \beta_\epsilon(h_\epsilon), h_\epsilon \rangle^{1/2} (C\|h_\epsilon\|_{\mathscr{U}} + C)\, \|\phi\|_{\mathscr{U}}.$$

Since $\|h_\epsilon\|_{\mathscr{U}}$ is bounded and $\langle \beta_\epsilon(h_\epsilon), h_\epsilon \rangle \leqslant C\epsilon$, this proves that $\beta_\epsilon(h_\epsilon) \to 0$ in $\mathscr{U}'$ as $\epsilon \to 0$. Denote $\beta_0 = \beta_\epsilon|_{\epsilon=0}$. Obviously, $\beta_0(h_\epsilon)$ also tends to zero in $\mathscr{U}'$. Let us show that $h \in \mathscr{K}$. Set $u = (1-\theta)\,h + \theta\phi$, where $\theta \in (0,1)$ and $\phi \in \mathscr{U}$. Due to the monotonicity of β_0,

$$0 \leqslant \langle \beta_0(h_\epsilon) - \beta_0(u), h_\epsilon - u \rangle = \langle \beta_0(h_\epsilon), h_\epsilon - h \rangle$$
$$+ \theta \langle \beta_0(h_\epsilon), h_\epsilon - \phi \rangle - \langle \beta_0(u), h_\epsilon - h \rangle - \theta \langle \beta_0(u), h - \phi \rangle.$$

All ϵ-dependent terms in this expression tend to zero as $\epsilon \to 0$, so $\langle \beta_0(u), h - \phi \rangle \leqslant 0$. Letting $\theta \to 0$ we obtain $\langle \beta_0(h), h - \phi \rangle \leqslant 0$, $\forall \phi \in \mathscr{U}$ which implies $h \in \mathscr{K}$.

Furthermore, let $\phi \in \mathscr{K}$, $\phi' \in \mathscr{U}'$. Then

$$\int_0^s (\phi' - w, \phi - h_\epsilon) = -\frac{1}{\epsilon} \int_0^s (\beta_\epsilon(h_\epsilon), \phi - h_\epsilon) + \tfrac{1}{2}\|\phi - h_\epsilon\|^2 \Big|_0^s.$$

Due to the monotonicity of β_ϵ,

$$\int_0^s (\beta_\epsilon(h_\epsilon), \phi - h_\epsilon) \leqslant \int_0^s (\beta_\epsilon(\phi), \phi - h_\epsilon) = \epsilon^2 \int_0^s \int_\Omega \nabla\phi \cdot \nabla(\phi - h_\epsilon) \leqslant \epsilon^2 (C\|\phi\|^2_{\mathscr{U}} + C).$$

Denote $\Psi = \{\psi \in C([0,T]) \mid \psi \geqslant 0\}$. For any function $\psi \in \Psi$

$$\int_0^T \psi(s) \left(\int_0^s (\phi' - w, \phi - h_\epsilon)(t) \right) \geqslant -\epsilon (C\|\phi\|^2_{\mathscr{U}} + C) \int_0^T \psi(s)$$
$$+ \frac{1}{2} \int_0^T \psi(s) \|\phi(s) - h_\epsilon(s)\|^2 - \tfrac{1}{2}\|\phi(0) - h_0\|^2 \int_0^T \psi(s).$$

Let $\epsilon \to 0$. Taking (24) and (25) into account we obtain

$$\int_0^T \psi(s) \left(\int_0^s (\phi' - w, \phi - h)(t) \right)$$
$$\geqslant \tfrac{1}{2} \liminf \int_0^T \psi(s) \|\phi(s) - h_\epsilon(s)\|^2 - \tfrac{1}{2}\|\phi(0) - h_0\|^2 \int_0^T \psi(s)$$
$$\geqslant \frac{1}{2} \int_0^T \psi(s)\, (\|\phi(s) - h(s)\|^2 - \|\phi(0) - h_0\|^2)$$

which proves that

$$\int_0^s (\phi' - w, \phi - h) \geqslant \tfrac{1}{2}\|\phi(s) - h(s)\|^2 - \tfrac{1}{2}\|\phi(0) - h_0\|^2 \quad \text{a.e.} \tag{26}$$

To show that $h \in C([0,T]; L^2(\Omega))$ we consider the problem

$$\eta u_\eta' + u_\eta = h, \quad u_\eta(0) = h_0, \quad \eta > 0. \tag{27}$$

Its solution, $u_\eta(t)$, belongs to set $\mathscr{K}$. Taking $\phi = u_\eta$ in inequality (26) we obtain

$$\int_0^s (u'_\eta, u_\eta - h) - \int_0^s (w, u_\eta - h) \geqslant \tfrac{1}{2}\|u_\eta(s) - h(s)\|^2 \quad \text{a.e.}$$

or

$$\tfrac{1}{2}\|u_\eta(s) - h(s)\|^2 + \frac{1}{\eta}\int_0^s \|h - u_\eta\|^2 \leqslant -\int_0^s (w, u_\eta - h) \quad \text{a.e..}$$

Therefore,

$$\int_0^s \|h - u_\eta\|^2 \leqslant -\eta \int_0^s (w, u_\eta - h), \tag{28}$$

$$\|u_\eta(s) - h(s)\|^2 \leqslant -2\int_0^s (w, u_\eta - h) \quad \text{a.e..} \tag{29}$$

Solutions u_η of (27) are bounded in $\mathscr{U}$ and so (28) implies that $u_\eta - h \to 0$ in $L^2(Q)$ when $\eta \to 0$. Then the sequence $u_\eta - h$ must also converge to zero weakly in $\mathscr{U}$. The integral on the right side of inequality (29) tends to zero uniformly in s when $\eta \to 0$, and this proves the continuity of the function $h: [0, T] \to L^2(\Omega)$.

Choosing in (26) the function ϕ which is equal to h_0 at $t = 0$ yields

$$\|h(s) - \phi(s)\|^2 \leqslant 2\int_0^s (\phi' - w, \phi - h)$$

and $h(s) \to h_0$ when $s \to 0$. Hence, the existence of h is proved.

Uniqueness. Let h_1 and h_2 be two solutions. Then $h = \frac{1}{2}(h_1 + h_2)$ is also a weak solution of the variational inequality. The solution of (27), u_η, belongs to $\mathscr{K}$. Substituting $\phi = u_\eta$ into the inequality (26) for h_1 and h_2 and adding these inequalities we obtain

$$2\int_0^s (u'_\eta, u_\eta - h) - 2\int_0^s (w, u_\eta - h) \geqslant \tfrac{1}{2}\|u_\eta(s) - h_1(s)\|^2 + \tfrac{1}{2}\|u_\eta(s) - h_2(s)\|^2.$$

From (27) it follows that

$$\int_0^s (u'_\eta, u_\eta - h) = -\frac{1}{\eta}\|u_\eta(s) - h(s)\|^2 \leqslant 0,$$

hence

$$\|h_1(s) - h_2(s)\|^2 \leqslant 2(\|u_\eta(s) - h_1(s)\|^2 + \|u_\eta(s) - h_2(s)\|^2) \leqslant -8\int_0^s (w, u_\eta - h).$$

The right-hand side tends to zero when $\eta \to 0$, and so the uniqueness is proved.

The case $w \in \mathscr{U}'$, $h_0|_{\Gamma_1} = 0$, considered in Theorem 2, includes the real experiment situations [3], for which the analytical solutions are known [9], in particular, those with the point sources. The theorem also shows that the solution of variational inequality can be found as the limit of solutions of nonlinear parabolic problems. This result may be important for the numerical solution of generalized model for pile evolution proposed in [8], and also sheds light on the connection between our model and the anomalous diffusion models of dissipative systems (see the discussion in [9]).

We will now prove the existence and uniqueness of a strong solution for $w \in L^2(Q)$ and the non-homogeneous boundary condition. Let us define a closed convex set

$$K = \{\phi \in V \mid |\nabla\phi| \leqslant \gamma, \quad \phi|_{\Gamma_1} = h_0|_{\Gamma_1}\}.$$

It is not difficult to show that this set is also closed in $L^2(\Omega)$. We can write the variational inequality as a Cauchy problem

$$h' + \partial I_K(h) \ni w, \quad h|_{t=0} = h_0, \tag{30}$$

where ∂ is the subdifferential and the indicator function

$$I_K(\phi) = \begin{cases} 0 & \text{if } \phi \in K, \\ \infty & \text{otherwise} \end{cases}$$

is defined on $L^2(\Omega)$. Since I_K is a convex lower-semicontinuous function and $h_0 \in K$, the problem (30) has a unique solution $h \in C(0, T; L^2(\Omega))$ such that $h(\cdot, t) \in K$ for almost all t and $h' \in L^2(Q)$ ([14], ch. 4, th. 2.1). The first two inclusions yield $h \in \mathscr{V}$. Let us show that also $h \in C^{1/4}(\bar{Q})$.

The domain Ω has a Lipschitz boundary, so the following cone property [15] is satisfied: there are $r_0 > 0$ and $\kappa > 0$ such that any $x \in \bar{\Omega}$ can be made a vertex of a sector, having radius r_0, angle κ, and lying wholly (except perhaps for the vertex) inside the domain Ω. Let $x_0 \in \bar{\Omega}$, $t_2 - t_1 = \tau > 0$, $\delta = |h(x_0, t_1) - h(x_0, t_2)|$ and U_{r_0} be the cone property sector with the vertex x_0. We define $U_r \subset U_{r_0}$ as a sector with the same angle and vertex but a smaller radius $r \leqslant r_0$. The inequality $|\nabla h| \leqslant \gamma$ yields $|h(x, t_1) - h(x, t_2)| \geqslant \delta - 2\gamma|x - x_0|$, and so

$$S = \int_{U_r} |h(x, t_1) - h(x, t_2)| \geqslant \tfrac{1}{2}\delta\kappa r^2 - \tfrac{2}{3}\gamma\kappa r^3.$$

On the other hand,

$$S \leqslant \int_{U_r}\int_{t_1}^{t_2} |h'(x, t)| \leqslant \left(\int_{U_r}\int_{t_1}^{t_2} 1\right)^{1/2} \|h'\| \leqslant Cr\tau^{1/2},$$

hence, $Cr\tau^{1/2} \geqslant \frac{1}{2}\delta\kappa r^2 - \frac{2}{3}\gamma\kappa r^3$. Taking $r = r_0(\tau/T)^{1/4}$, we obtain $\delta \leqslant C\tau^{1/4}$. Since the function h is Lipschitz continuous in x, $h \in C^{1/4}(\bar{Q})$. We have thus proved the following theorem:

Theorem 3 *Let $w \in L^2(Q)$, $h_0 \in V$, and $|\nabla h_0| \leqslant \gamma$. Then the variational inequality*

$$h(\cdot, t) \in K: \quad (h' - w, \phi - h) \geqslant 0, \quad \forall \phi \in K, \text{ a.e. in } (0, T),$$
$$h|_{t=0} = h_0$$

has a unique solution $h \in \mathscr{V} \cap C^{1/4}(\bar{Q})$ such that $h' \in L^2(Q)$.

5 Conclusion

We have considered a deterministic quasi-stationary model of the pile growth. In this model, the evolution of the pile shape is governed by the surface transport of a poured granular material, and only the direction of this transport is determined by the local conditions, i.e. the local topography of the free surface of a pile. The surface flux magnitude depends upon the solution and external source in a non-local way and is determined in this model by a Langrange multiplier, related to a condition of equilibrium.

Such a situation is typical of models of other extended dissipative systems (see for example [10]), where the relaxation is fast and the assumption that all transport occurs at the border of stability is justified.

We showed that the model is equivalent to a quasi-variational inequality and proved the existence and uniqueness of a solution in the special case where the inequality is variational. The variational formulation obtained is very useful for numerical simulation [8]. The existence of a solution in the general case is an interesting open problem.

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