JOURNAL OF COMPUTATIONAL PHYSICS **144,** 180–193 (1998)
ARTICLE NO. CP985978

# Solution of Thin Film Magnetization Problems in Type-II Superconductivity

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Received September 25, 1997; revised March 4, 1998

A variational formulation and an efficient numerical method are derived for thin film critical-state problems. We use this method to solve problems for various film shapes with either the Bean or Kim current–voltage relation characterizing the superconducting material. 

*Key Words:* superconductivity; variational inequality; numerical solution; critical current; thin film.

## I. INTRODUCTION

The critical-state models [1, 2] provide a macroscopic phenomenological description for the magnetization of type-II superconductors in non-stationary external magnetic fields. Since the configuration of a thin superconducting platelet or film in a perpendicular field is typical of experiments with superconducting materials, solution of thin film critical-state problems is of much interest.

Analytical solutions to such problems have been found for a model with field-independent critical current (the Bean model) in thin disk [3, 4] and strip [5] geometries. These solutions differ strongly from the well-known solutions of critical-state problems in longitudinal geometry. Recently, Brandt has developed a numerical method for solution of the Bean problems for rectangular films [6, 7]. This method was also applied to inhomogeneous rectangular films [8] and generalized for problems with more general film shapes [9]. In principle, Brandt's method allows one to calculate a solution also for models with a field-dependent critical current density, such as the Kim model [2]. The numerical solutions obtained showed some very interesting features of thin film magnetization which have been observed in experiments [8–12]. These are not described by the known analytical solutions since they do not appear in disk or strip geometries.

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0021-9991/98 $25.00

In this work we propose a different numerical method for solving the thin film magnetization problems. Our method is based on the variational formulation of critical-state problems, similar to that in [13, 14] but derived for the thin film geometry, and on the finite element discretization. The algorithm proposed is better adjusted to problems with non-rectangularly shaped films than the method [6–9] which uses Fourier series for space approximation. It should also be noted that in the Bean and Kim models the current–voltage relations for the superconducting material are non-smooth and multi-valued. These constitutive relations must be approximated by a smooth function, e.g., a power law, to make the calculations [6–9] feasible. Although such approximations are sometimes introduced also to account for the thermally activated creep of magnetic flux, the ability of our method to deal with any monotone $E(J)$ dependence without its approximation is an advantage. The efficiency and universality of this method are demonstrated by examples in which we simulate the evolution of the magnetic field and current pattern in simply and multiply connected films of various shapes using the Bean or Kim models.

## II. VARIATIONAL FORMULATION

Let an external uniform nonstationary magnetic field $\mathbf{H}_e = H_e(t)e_z$ be perpendicular to a flat superconducting film. The film is assumed thin, so the model can be written in terms of a two-dimensional (2d) sheet current density $\mathbf{J}(x, t)$ defined at the film midplane $\Omega$. Here $x = \{x_1, x_2\}$ and $\mathbf{J}$ is the current density integrated across the film thickness. It is supposed that no external current is fed into the superconductor, so

$$\operatorname{div} \mathbf{J} = 0 \text{ in } \Omega, \qquad \mathbf{J} \cdot \mathbf{n} = 0 \text{ on } \Gamma, \tag{1}$$

where div is the 2d divergence operator, $\Gamma$ is the boundary of $\Omega$, and $\mathbf{n}$ is a normal to $\Gamma$. Conditions (1) should be satisfied also for the given initial current density $\mathbf{J}(x, 0) = \mathbf{J}_0(x)$.

To derive a computationally convenient formulation for the film magnetization model, we express the electric field $\mathbf{E}$ via the vector and scalar potentials, $\mathbf{A}$ and $\Phi$ [15]:

$$\mathbf{E} + \partial_t \mathbf{A} = -\nabla \Phi.$$

Let us multiply this equation by an arbitrary vector function $\mathbf{J}'(x)$ satisfying (1) and integrate it over $\Omega$. Since $\int_\Omega \nabla\Phi \cdot \mathbf{J}' = 0$, we obtain

$$(\mathbf{E} + \partial_t \mathbf{A}, \mathbf{J}') = 0. \tag{2}$$

Here and throughout $(\phi, \psi)$ denotes the scalar product $\int_\Omega \phi \cdot \psi$.

The magnetic vector potential can be represented as the sum of potential of a given external current, which induces the external magnetic field $\mathbf{H}_e$, and of the electric current in the superconductor induced by the variations of this field: $\mathbf{A} = \mathbf{A}_e + \mathbf{A}_i$. Up to the gradient of a scalar function, which is eliminated by $\mathbf{J}'$ in Eq. (2),

$$\mathbf{A}_i(x, t) = \mu_0 \int_\Omega \frac{\mathbf{J}(x', t)}{4\pi |x - x'|}\, dx',$$

where $\mu_0$ is the permeability of vacuum. Brandt [16] used the zero divergence condition (1) to introduce the stream function of the sheet current density: if $\Omega$ is simply connected, there exists a function $g(x, t)$ such that at any time moment

$$\mathbf{J} = -e_z \times \nabla g$$

and $g = 0$ on $\Gamma$. The situation is slightly more complicated if the domain $\Omega$ contains holes $\Omega_1, \ldots, \Omega_N$ (see, e.g., [17, Chap. 1, Corollary 3.1]). We still can introduce the stream function and assume $g = 0$ on $\Gamma_e$, the external boundary of $\Omega$. However, on the boundaries of holes this function takes non-zero constant values, different for different holes (these constants are time-dependent). In this case we denote by $\Omega^*$ the domain with the holes included, $\Omega^* = \Omega \cup (\cup_i \bar{\Omega}_i)$, and extend the stream function $g$ onto $\Omega^*$ continuously by setting it constant in each hole. Similarly, a stream function $g'$ can be introduced for any test function $\mathbf{J}'$ satisfying (1). It is easy to see that $\mathbf{J} \cdot \mathbf{J}' = \nabla g \cdot \nabla g'$, and so

$$(\mathbf{A}_i, \mathbf{J}') = \mu_0 \int_\Omega \int_\Omega \frac{\nabla g(x,t) \cdot \nabla g'(x')}{4\pi |x - x'|} \, dx \, dx'. \tag{3}$$

Taking into account that $\nabla \times \mathbf{A}_e = \mu_0 \mathbf{H}_e$, and using the identity $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$ and Green's formula, we obtain

$$(\mathbf{A}_e, \mathbf{J}') = \mu_0 H_e(t) \int_{\Omega^*} g'(x) \, dx. \tag{4}$$

To complete the model, we now have to specify a current–voltage characteristic of the superconducting material. This highly nonlinear constitutive relation is determined by the balance of pinning and electromagnetic driving forces acting upon the quantized superconducting vortices [18]. For thin isotropic films in a perpendicular magnetic field, the sheet current density and electric field inside the superconductor are parallel. We can write

$$\mathbf{E} = \rho \mathbf{J}, \tag{5}$$

where an auxiliary variable, the effective resistivity $\rho(x, t) \geq 0$, characterizes the energy losses accompanying the movement of vortices. We avoid the notation $\rho = \rho(J(x, t))$ with $J = |\mathbf{J}|$ since, as is discussed below, the dependence on current density is multi-valued in some critical-state models. Of course, unless $\rho$ is specified, Eq. (5) relates only the directions of vectors $\mathbf{J}$ and $\mathbf{E}$.

In the Bean model, it is assumed that the current density never exceeds some critical value, $J_c$, determined by pinning, and that the electric field is zero if $J < J_c$. ($J_c$ denotes the sheet critical current density equal to $j_c d$, where $d$ is the film thickness and $j_c$ is the bulk critical current density.) For $J = J_c$ the electric field and effective resistivity $\rho = E/J$ are not determined by this law uniquely (Fig. 1a). A similar multi-valued current–voltage law is assumed in the Kim model, where the critical current density depends on the magnetic field, $J_c = J_c(\mathbf{H})$. As has been shown in [13], the effective resistivity in these critical-state models can be regarded as a Lagrange multiplier related to the current density constraint $J \leq J_c$. Although the current density alone does not uniquely determine the electric field in such models, both these variables are determined by the complete evolutionary model of magnetization.

The constraint on current density is relaxed if, as in [6–9], the power law $E = E_c(J/J_c)^n$ (Fig. 1b) is employed as an approximation to Bean's $E(J)$ multi-valued relation. Such a

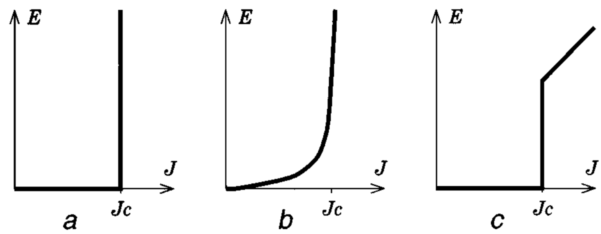


**FIG. 1.** Current–voltage relations.

model tends to the Bean model as $n$ tends to infinity [19]. Similar approximations have been used by many authors. For example, a current–voltage relation that accounts for the transition from the flux creep ($J \approx J_c$) to flux flow ($J \gg J_c$) regime (Fig. 1c) may be more realistic [20, 21]. This $E(J)$ law has been approximated by a function having neither zero nor infinite slopes in some works on magnetization of bulk superconductors [22, 23]. Although the numerical schemes based on these approximations perform sufficiently well, their convergence and stability usually become less satisfactory the closer they approximate multi-valued current–voltage relations. The variational formulation derived below is convenient for the numerical solution of thin film magnetization problems with arbitrary monotone $E(J)$ laws.

Let $E(J)$ be a monotone graph, like those shown in Fig. 1. Following Bossavit [24], we define a convex function $u = u(J)$, which may take also infinite values, as an integral

$$u(J) = \int_0^J E(s)\, ds.$$

(this function also depends on $x$ if the film is not homogeneous). It can be shown that $u(J') - u(J) \geq E(J' - J)$ for any $J, J' \geq 0$ if and only if $E$ belongs to the set $E(J)$ ($E \in E(J)$, the graph may be multi-valued). Furthermore, if $\mathbf{J}$, $\mathbf{E}$ are the current density and electric field inside the superconductor, then $E \in E(J)$ and $\mathbf{E} \parallel \mathbf{J}$. Therefore, for any vector function $\mathbf{J}'$,

$$u(J') - u(J) \geq E(J' - J) \geq \mathbf{E} \cdot (\mathbf{J}' - \mathbf{J}). \tag{6}$$

Denoting $U(J) = \int_\Omega u(J)$ and integrating the inequality (6) over $\Omega$, we obtain

$$U(J') - U(J) \geq (\mathbf{E}, \mathbf{J}' - \mathbf{J})$$

(the electric field $\mathbf{E}$ is a subgradient of the functional $U$ at a point $\mathbf{J}$). Finally, introducing the stream functions $g$ and $g'$ for $\mathbf{J}$ and $\mathbf{J}'$, respectively, and making use of Eqs. (2), (3), and (4), we arrive at the variational relation (variational inequality with a non-local operator)

$$\frac{1}{\mu_0}\{U(|\nabla g'|) - U(|\nabla g|)\} + (M\partial_t g, g' - g) + \partial_t H_e \int_{\Omega^*} (g' - g) \geq 0, \tag{7}$$

where $M$ is the linear operator defined by the symmetric bilinear form

$$(M\phi, \psi) = \int_\Omega \int_\Omega \frac{\nabla\phi(x) \cdot \nabla\psi(x')}{4\pi|x - x'|}\, dx\, dx', \tag{8}$$

which is positively definite and even coercive in a properly chosen functional space [19]. It can be shown that $(\mu_0/2)\,(M\phi, \phi)$ is the energy of magnetic field induced by the current $\mathbf{J} = -e_z \times \nabla\phi$.

The variational inequality (7) holds for all continuous test functions $g'(x)$ which are differentiable almost everywhere in $\Omega$, constant in each hole of this domain, and zero on the external boundary. The solution $g$ should also belong to this space of functions at any time moment and must satisfy the initial condition $g(x, 0) = g_0(x)$, where $g_0$ is the stream function corresponding to $\mathbf{J}_0$.

It may be noted that variational inequalities appear as the variational formulations of various physical and mechanical problems containing non-smooth constitutive relations or unilateral constraints [25]. Such formulations are very convenient for both the theoretical study and the numerical solution of these problems.

The variational formulation (7) is valid for arbitrary monotone current–voltage relation in the magnetization model and can serve as the basis for a numerical algorithm. Below, we limit our consideration to the Bean and Kim critical-state models.

## III. THE BEAN AND KIM MODELS

Let us start with the Bean model. For this model, the functional $U(J)$ is finite if and only if the condition $J \le J_c$ is fulfilled in $\Omega$. This condition is equivalent to a gradient constraint upon the stream function, $|\nabla g| \le J_c$, and it is sufficient to consider in (7) only those functions $g$ and $g'$ which satisfy this condition. On such functions the functional $U$ is zero. If the domain $\Omega$ is not simply connected, the stream functions are continuously continued by a constant inside each hole, and so their gradients in the holes are zero. Also, these functions themselves must be zero on the external boundary $\Gamma_e$ (the boundary of $\Omega^*$). Introducing the set of admissible stream functions,

$$K = \left\{ \varphi(x) \,\middle|\, \begin{array}{c} |\nabla\varphi| \le J_c \text{ in } \Omega, \\ |\nabla\varphi| = 0 \text{ in } \Omega_1, \ldots, \Omega_N, \\ \varphi = 0 \text{ on } \Gamma_e \end{array} \right\},$$

we can formulate the Bean critical-state problem as follows:

$$\textit{Find a function } g(x,t) \textit{ such that } g \in K \textit{ for all } t,\ (M\partial_t g, g' - g) + \partial_t H_e \int_{\Omega^*} (g' - g) \ge 0$$
$$\textit{for any } g' \in K,\ \textit{ and also } g(x,0) = g_0(x). \tag{9}$$

The existence of a unique solution to this problem is shown in [19].

The formulation (9) can be extended for the Kim model where the critical current density is field-dependent. In the case of thin film magnetization it is assumed $J_c = J_c(H_z)$, where $H_z$ is the normal to the film surface component of magnetic field (this component of $\mathbf{H}$ has no jump at the film mid plane $z = 0$). Using the Biot–Savart law we can express $H_z$ in terms of the stream function:

$$\begin{aligned} H_z &= H_e + e_z \cdot \frac{1}{4\pi} \int_\Omega \nabla\left(\frac{1}{|x - x'|}\right) \times \mathbf{J}(x', t)\, dx' \\ &= H_e - \frac{1}{4\pi} \int_\Omega \nabla\left(\frac{1}{|x - x'|}\right) \cdot \nabla g(x')\, dx'. \end{aligned} \tag{10}$$

Thus $J_c = J_c(H_z[g])$ and the set of admissible stream functions in the variational inequality (9) depends on the unknown solution itself:

$$K = K(g).$$

This is an additional nonlinearity. Problems of this kind are called quasivariational inequalities. Computationally, we resolve this nonlinearity by means of an additional cycle of iterations (see below).

## IV. CRITICAL STATES IN THE BEAN MODEL

According to the Bean model, a stationary critical state with $J \equiv J_c$ is established in a superconductor placed into a growing external field when the field becomes sufficiently strong. This solution is readily found analytically if the domain $\Omega$ is simply connected and the film is homogeneous.

Let $\partial_t H_e$ be constant and, say, negative. Then the stationary form of (9) can be written as follows:

$$\textit{Find } g \in K \textit{ such that } \int_\Omega (g' - g) \leq 0 \textit{ for all } g' \in K.$$

This is equivalent to the well-known problem of completely plastic torsion of a beam [26] and also to the maximal sandpile shape problem [27]:

$$\max_{g \in K} \int_\Omega g.$$

The solution to this problem is

$$g(x) = J_c\, dist(x, \Gamma), \tag{11}$$

where *dist* is the distance function. The current density $\mathbf{J} = -e_z \times \nabla g$ is discontinuous and abruptly changes its direction at the ridges, also called $d^+$ lines [10], of domain $\Omega$ (a point $x \in \Omega$ belongs to a ridge if there exist at least two different points on $\Gamma$, $x_1$ and $x_2$, such that $|x - x_1| = |x - x_2| = dist(x, \Gamma)$; see [26, 28]). Although the stationary solutions described by (11) have been found for some film shapes in works on thin film magnetization, this simple general formula has not been presented there.

## V. COMPUTATIONAL SCHEME

Numerical methods for solution of variational inequalities are well developed [29]. For the Kim model, where $K = K(g)$, the finite difference discretization of (9) in time leads to the stationary quasivariational inequalities at each time layer:

$$\textit{Find } g(x) \textit{ such that } g \in K(g) \textit{ and } (Mg - M\hat{g}, g' - g) + (H_e - \hat{H}_e) \int_{\Omega^*} (g' - g) \geq 0$$
$$\textit{for any } g' \in K(g)$$

(here "ˆ" means the value from the previous time layer; the operator $M$ is defined by (8)). These inequalities are equivalent to the following optimization problems with an implicit constraint,

$$\mathcal{F}(g) = \min_{\varphi\in K(g)} \mathcal{F}(\varphi), \tag{12}$$

where

$$\mathcal{F}(\varphi) = \frac{1}{2}(M\varphi,\varphi) - (M\hat{g},\varphi) + (H_e - \hat{H}_e)\int_{\Omega^*}\varphi$$

is a quadratic functional. Since, as we noted above, the form $(M\varphi, \varphi)$ is coercive, the functional $\mathcal{F}$ is strictly convex. To solve (12) one can use an iterative scheme, e.g.,

$$\mathcal{F}(g^{k+1}) = \min_{\varphi\in K(g^k)} \mathcal{F}(\varphi) \tag{13}$$

and discretize (13) in space using piecewise linear finite elements. Finally, we solve the resulting problems of convex programming by the augmented Lagrangian method [30, 31].

Note that to set properly the constraints at each iteration (13), it is necessary to determine the magnetic field $H_z[g^k]$ by evaluating the integral in (10) numerically. Such iterations are not needed for the Bean model, where the critical current does not depend on the magnetic field. However, also in this case, computing $H_z$ is often needed to compare the simulation results with results of magneto-optical measurements of the flux density.

We will now describe the main steps of the numerical algorithm in more detail.

*Finite Element Approximation*

We triangulate the domain $\Omega$ and approximate the stream functions by continuous functions, linear inside each finite element and zero at the nodes belonging to the external boundary $\Gamma_e$. If $\Omega$ is not simply connected, the finite element mesh is extended also inside the holes. The linear elements are convenient for approximating the gradient constraints implied by the condition $\varphi \in K(g^k)$, since the gradients become constant inside each finite element. The only difficulty in approximating the functional $\mathcal{F}$ is that the scalar products containing the operator $M$ lead to the integrals

$$\int_\Omega\int_\Omega \frac{\nabla\phi_i(x)\cdot\nabla\phi_j(x')}{4\pi|x-x'|}\,dx\,dx',$$

some of which are singular (here $\phi_l$ is the piecewise linear basis function, equal to one at the mesh node $l$ and zero in all other nodes). Since the gradients are constant in each finite element, one only has to calculate

$$q_{m,n} = \int_{\Delta_m}\int_{\Delta_n} \frac{1}{|x-x'|}\,dx\,dx'$$

for all pairs of elements $\Delta_m$, $\Delta_n$. For $m \neq n$ we set $q_{m,n} = |\Delta_m|\,|\Delta_n|/|x_m^0 - x_n^0|$, where $x_l^0$ is the center of $\Delta_l$. The integrals $q_{m,m}$ can be approximated as

$$q_{m,m} \approx |\Delta_m|\int_{\Delta_m}\frac{1}{|x_m^0 - x'|}\,dx' = |\Delta_m|\int_0^{2\pi} r(\psi)\,d\psi,$$

where $\{r, \psi\}$ are the polar coordinates with the center at $x_m^0$ and $r = r(\psi)$ is the boundary of $\Delta_m$. The last integral is regular and a simple quadrature formula can be used for its evaluation.

*Solution of the Constrained Optimization Problems*

At each iteration (13), the constraint $\varphi \in K(g^k)$ can be written as

$$|\nabla\varphi(x)| \leq J_c(x),$$

where $J_c(x)$ is known: it is zero in the holes of $\Omega$ if the domain is multiply connected, constant inside $\Omega$ in the Bean model, or, if the Kim model is used, is determined by the magnetic field (10) with $g = g^k$. Inside each finite element the gradient of $\varphi$ is constant and we approximate the constraint inside element $\Delta_l$ by the condition

$$f_l = |\Delta_l|\left(|\nabla\varphi|^2|_{\Delta_l} - J_c^2\left(x_l^0\right)\right) \leq 0 \tag{14}$$

(the areas of finite elements, $|\Delta_l|$, are convenient normalizing coefficients).

The functional and constraints of the discretized optimization problem depend on the vector $\varphi$ of $\varphi$-values at the mesh nodes and we can write this problem as

$$\min_{\{f_l(\varphi)\leq 0\}} \mathcal{F}(\varphi). \tag{15}$$

To solve this problem numerically, we used the augmented Lagrangian technique ([30, 31]; see also [32, 33]), which is a combination of the penalty and duality methods. This combination has advantages over each of the two approaches: the algorithm converges faster than the pure duality methods, and the convergence takes place without the necessity of an infinite growth of the penalty parameter causing instability of the penalty methods. At each iteration of this algorithm, given the vector of Lagrange multipliers $\rho^i$ we find vector $\varphi^i$ minimizing the augmented Lagrangian

$$L_r(\varphi, \rho) = \mathcal{F}(\varphi) + \frac{1}{4r}\sum_l \left\{[(\rho_l + 2r\, f_l(\varphi))^+]^2 - \rho_l^2\right\},$$

after which the new Lagrange multipliers are found in accordance with

$$\rho_l^{i+1} = \left(\rho_l^i + 2r\, f_l(\varphi^i)\right)^+.$$

Here $r > 0$ is a constant and $u^+$ means max$(u, 0)$. For the unconstrained minimization of $L_r$, needed at each iteration, we used the point relaxation method solving at each node $j$ the nonlinear equations

$$\partial L_r(\varphi^i, \rho^i)/\partial\varphi^{i,j} = 0$$

by Newton's method (here $\varphi^{i,j}$ is the $j$th coordinate of vector $\varphi^i$, $i$ is the iteration number).

Note that it is not necessary to perform the optimization of the Lagrangian in $\varphi$ with high accuracy at each iteration: this accuracy may be increased gradually in accordance with the convergence of the Lagrange multipliers.

*Magnetic Field Calculation*

To calculate the critical current density $J_c(H_z)$ in the Kim model, it is necessary to evaluate the integral in Eq. (10) at the center of each finite element $\Delta_l$. This can be done as

$$\int_\Omega \nabla \frac{1}{|x_l^0 - x'|} \cdot \nabla g(x') \approx \sum_m \int_{\Delta_m} \nabla \frac{1}{|x_l^0 - x'|} \cdot \nabla g(x')$$
$$= \sum_m \left( \oint_{\partial\Delta_m} \frac{\mathbf{n}}{|x_l^0 - x'|} \right) \cdot \nabla g \Big|_{\Delta_m},$$

where $\partial\Delta_m$ is the boundary of $m$th finite element, $\mathbf{n}$ is the unit outward normal to this boundary. The line integrals in the last sum were calculated by means of the Simpson quadrature applied on each side of $\Delta_m$.

## VI. NUMERICAL RESULTS

The numerical procedure described above has been realized in Matlab [34]. We used Matlab PDE Toolbox functions for visualization and domains triangulation, and Matlab Compiler to accelerate the calculations. The penalty parameter $r$ of the optimization procedure was $10^5$. The computation of a typical example took from several minutes to an hour on IBM RS6000/370. In all examples below we assumed the virgin initial state ($g_0 = 0$). We will first present the simulation results for the Bean model.

As a test for our computational scheme, the numerical solution for a thin disk was compared with the analytical solution [3] (see Fig. 2). In this example we used a rather fine finite element mesh to recover the current density $J = |\nabla g|$. A much cruder mesh is usually quite sufficient to determine only the pattern of current contours, which are the level contours of the stream function $g$.

As was already demonstrated by Brandt [6, 7], the magnetic field penetrates a rectangle (Fig. 3) from its sides, and not from the corners as might be naively expected. The magnetic

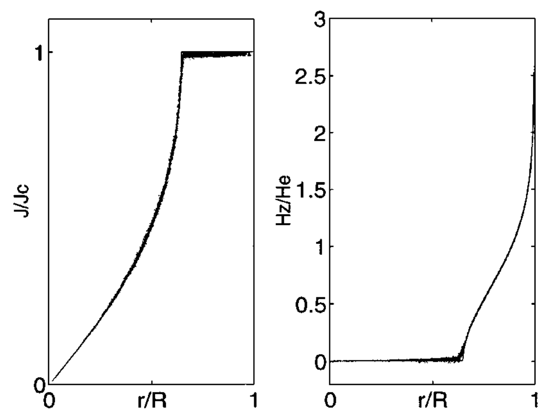


**FIG. 2.** The analytical and numerical solutions for a thin disk: the current density (left) and normal component of magnetic field (right). The values of $J$ and $H_z$ in each finite element are plotted against the radial coordinate of the element center as distinct points. These points are close to the solid lines representing the analytical solutions. The external field is $H_e/J_c = 0.5$. The finite element mesh contained 3416 nodes and 6670 elements.

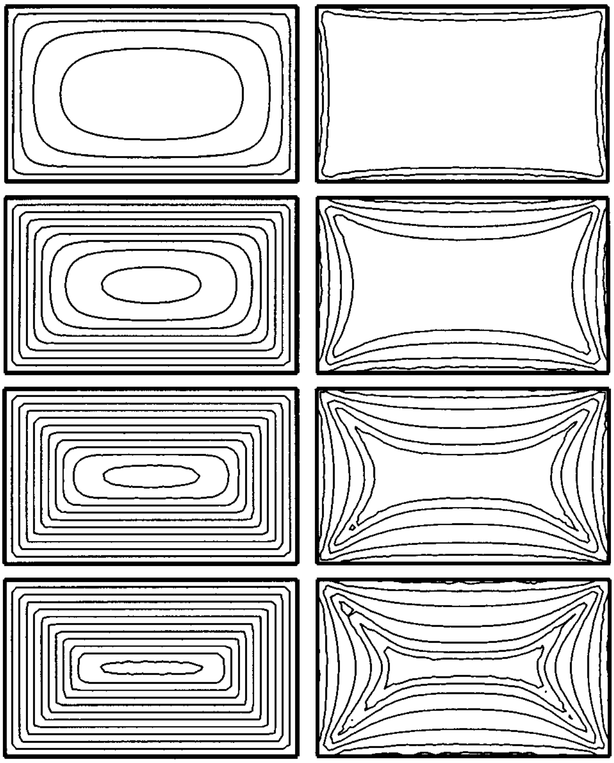

**FIG. 3.** Magnetization of rectangular film. The current contours (left) and the level contours of magnetic field at the film midplane (right). The external magnetic field $H_e/J_c$ (from top to bottom): 0.25, 0.50, 0.75, 1. Finite element mesh: 1424 nodes, 2728 elements.

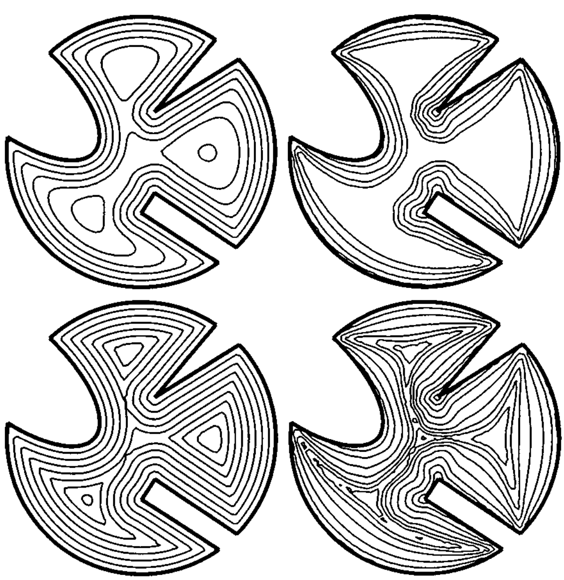

**FIG. 4.** Magnetization of an irregularly shaped film. The current contours (left) and level contours of magnetic field (right). The external magnetic field $H_e/J_c = 0.5$, 1.

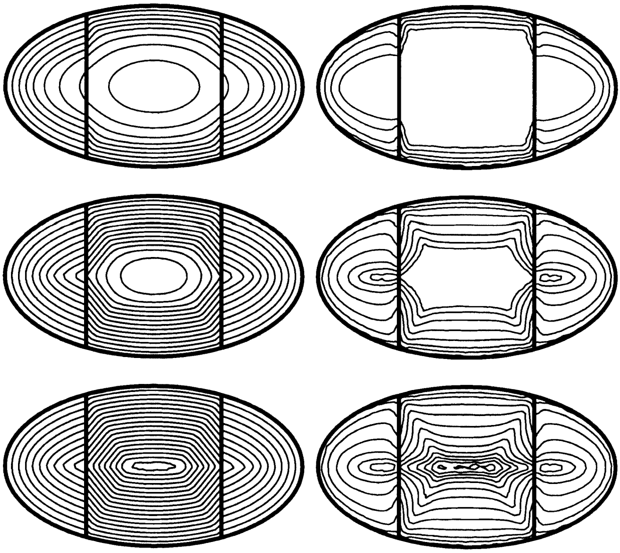

**FIG. 5.** Magnetization of an inhomogeneous film. The current contours (left) and level contours of magnetic field (right). The critical current density: $J_c$ in the left and right parts, $1.5J_c$ in the middle part of the film. The external magnetic field $H_e/J_c = 0.5$, 1, 1.5.

field is zero in the region where the sheet current density is less than critical. This zero field core shrinks with the growth of external field and the development of a steady-state current density solution described by the formula (11) is clearly seen in this, as well as in the next example (Fig. 4).

The steady-state solution is more complicated if the film is not homogeneous. In the example in Fig. 5, the critical current density is 1.5 times higher in the central part of the film than in the two other parts.

If the film is multiply connected, the finite element mesh should be extended into the holes (see Fig. 6). The zero current condition inside the holes implies there the constraint $|\nabla g| = 0$. Magnetization of such a film (simulated on a finer mesh) is shown in Fig. 7.

As the last example, let us consider the magnetization of a rectangular film in a non-monotonic external field. We now assume the Kim model current–voltage relation

$$J_c = \frac{J_{c_0}}{1 + |H_z|/H_0},$$

where $H_0$ and $J_{c_0}$ are constants, and solve the quasivariational inequality. It can be seen from Fig. 3 and the first part of Fig. 8 (increasing field) that qualitatively the current patterns and magnetic fields are similar for the Bean and Kim models and that in the latter case the

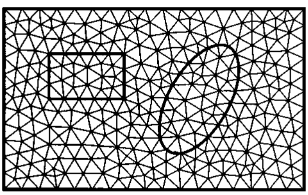

**FIG. 6.** Triangulation of a film with two holes.

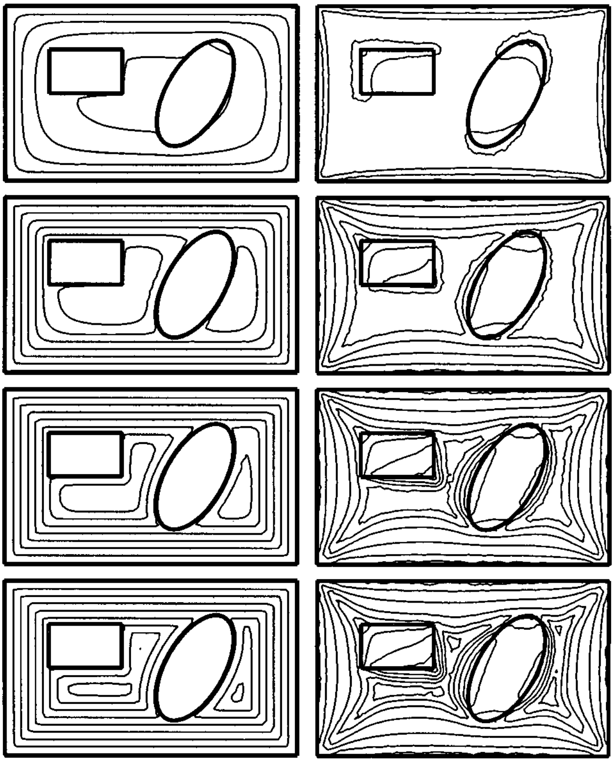

**FIG. 7.** Magnetization of a multiply connected film. The current contours (left) and level contours of magnetic field (right). The external magnetic field $H_e/J_c = 0.25, 0.50, 0.75, 1$.

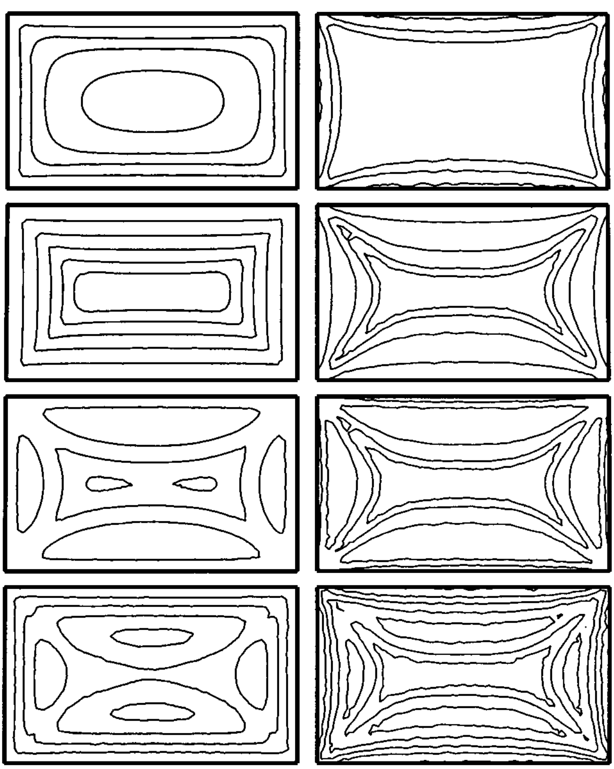

**FIG. 8.** Magnetization of a film in a non-monotone external field (Kim model). The critical current density: $J_c = J_{c_0}/(1 + |H_z|/H_0)$, where $H_0/J_{c_0} = 0.5$. The external magnetic field: $H_e/J_{c_0} = 0.25, 0.50, 0.25, 0$.

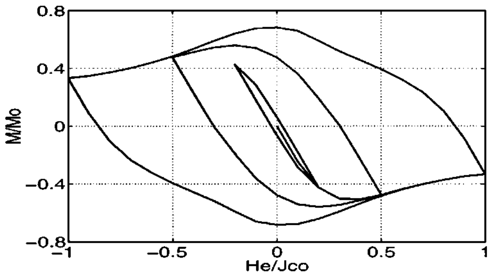


**FIG. 9.** Hysteresis loops (Kim model). Normalized magnetic moment $M/M_0$ against external magnetic field, $M_0 = |\Omega| J_{c_0}/4$.

magnetic field penetrates further because the shielding current decreases as the field grows. A similar conclusion was made in [35], where a semi-analytical procedure, generalizing the method [3] for the Kim model, has been developed for modeling magnetization of an infinite thin strip. The magnetic moment of a film can be presented as an integral of the stream function,

$$\mathbf{M} = \frac{1}{2}\int_\Omega \mathbf{r} \times \mathbf{J} = e_z \int_\Omega g.$$

This integral is evaluated with high accuracy even if a crude finite element mesh is used. The hysteresis loops calculated for the film from the previous example are presented in Fig. 9. These loops are far more similar to those observed in experiments than the loops which can be calculated using the Bean model: the magnetic moment of a film in a strong field becomes smaller because the critical current density decreases.

## ACKNOWLEDGMENTS

The author is grateful to A. Bossavit for helpful and stimulating discussions. This work was supported by the Feinberg Fellowship.

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