Getting a model of (ZF − Fnd) ∪ {¬Fnd} from a model of ZF

The set of axioms below is called ZFA (Zermelo Frankel with atoms).

Let $\mathcal{L}$ be the language $\{\in, =, U, S\}$, $\in$ is a binary relation symbol, $=$ is the equality symbol, and U, S are unary relation symbols. The theory ZF with urelements (also called atoms) will be defined below. It is a theory in the language $\mathcal{L}$. It is denoted by ZFA. Its axioms are.

(1) $\forall x(U(x) \vee S(x)) \wedge \forall x \neg(U(x) \wedge S(x))$.

(2) $\forall x \forall y(x \in y \rightarrow S(y))$.

(3) Extensionality.

$$\forall x \forall y((S(x) \wedge S(y) \wedge \forall u(u \in x \leftrightarrow u \in y)) \rightarrow x = y).$$

(4) Union.

$$\forall x(S(x) \rightarrow \exists y(S(y) \wedge \forall u(u \in y \leftrightarrow \exists v(v \in x \wedge u \in v))).$$

The formula $x \subseteq y$ abbreviates $\forall z(z \in x \rightarrow z \in y)$.

(5) Powerset.

$$\forall x(S(x) \rightarrow \exists y(S(y) \wedge \forall u(u \in y \leftrightarrow u \subseteq x))).$$

(6) Replacement.

$$\forall \vec{x}(Func_{v,u}(\varphi(u, v, \vec{x})) \rightarrow$$
$$\forall x(S(x) \rightarrow \exists y(S(y) \wedge \forall v(v \in y \leftrightarrow \exists u(u \in x \wedge \varphi(u, v, \vec{x})))))).$$

The formula $x = \emptyset$ abbreviates $S(x) \wedge \forall y(y \notin x)$.

(7) Infinity.

$$\exists x(\emptyset \in x \wedge \forall y(y \in x \rightarrow y \cup \{y\} \in x)).$$

(8) Foundation.

$$\forall x(\exists y(y \in x) \rightarrow \exists y(y \in x \wedge \forall z(z \in x \rightarrow z \notin y))).$$

We start with a universe V which satisfies ZFA. Let $\in_{\mathrm{U}}$ be an extesional narrow from the left relation on U. We define a class Q and a relation $\in'$ on Q such that $\langle Q, \in' \rangle$ satisfies (ZF − Fnd) ∪ {¬Fnd}.

Let $Q_0 = U$. For every $\alpha \in On$ let

$Q_{\alpha+1} = Q_\alpha \cup \{a \subseteq Q_\alpha \mid \text{there is no } b \in U \text{ such that } a = \{x \mid x \in_{\mathrm{U}} b\}\}$.

If α is a limit ordinal, define $Q_\alpha = \bigcup_{\beta<\alpha} Q_\beta$. Let $Q = \bigcup_{\alpha \in On} Q_\alpha$. Define

$\in' \; = \; \in^Q \cup \in_{\mathrm{U}}$.

Theorem $\langle Q, \in' \rangle$ is a model of ZF - Fnd.

Note that Q is a transitive class. We also use the fact $U \subseteq Q$.

(1) Extensionality.

Let $x, y \in Q$ and suppose that for every $u \in Q$, $u \in' x$ iff $u \in' y$. That is,

$$\{u \mid u \in' x\} = \{u \mid u \in' y\}.$$

Case 1 $x \in U$.

Then

$\{u \mid u \in' x\} = \{u \mid u \in_{\mathrm{U}} x\}$. Suppose by contradiction that $y \notin U$. Then $\{u \mid u \in' y\} = \{u \in Q \mid u \in y\}$. Since Q is transitive,

$\{u \in Q \mid u \in y\} = \{u \mid u \in y\}$. Hence

(1) $$\{u \mid u \in y\} = \{u \mid u \in_{\mathrm{U}} x\}.$$

For some α, $y \in Q_{\alpha+1}$. But (1) contradicts the definition of $Q_{\alpha+1}$. So $y \in U$. Hence (using the fact that $U \subseteq Q$), $\{u \mid u \in' y\} = \{u \mid u \in_{\mathrm{U}} y\}$.

Now, $\{u \mid u \in_{\mathrm{U}} x\} = \{u \mid u \in' x\} = \{u \mid u \in' y\} = \{u \mid u \in_{\mathrm{U}} y\}$. Since $\in_{\mathrm{U}}$ is extensional $x = y$.

Case 2 $x \notin U$.

It follows from Case 1 applied to y that $y \notin U$. Hence (using the transitivity of Q),

$\{u|u \in x\} = \{u \in Q|u \in x\} = \{u|u \in' x\} = \{u|u \in' y\} = \{u \in Q|u \in y\} = \{u \mid u \in y\}$. By Extensionality, $x = y$.

(2) Union.

Let $x \in Q$.

Case 1 $x \in U$.

Let $y = \{u \mid (\exists v \in_{\mathrm{U}} x)(u \in_{\mathrm{U}} v)\}$. By the left narrowness of $\in_{\mathrm{U}}$ and the Replacement Axiom, y is a set. If there is $z \in U$ such that $y = \{u \mid u \in_{\mathrm{U}} z\}$, then it is easy to check that $(z = \bigcup x)^{(\langle Q, \in' \rangle)}$. If there is no such z, then $y \in Q_1$ and $(y = \bigcup x)^{(\langle Q, \in' \rangle)}$.

Case 2 $x \notin U$.

Let $y_1 = \{u \mid (\exists v \in x)(u \in_{\mathrm{U}} v)\}$ and $y_2 = \{u \mid (\exists v \in x)(u \in v)\}$. By the left narrowness of $\in_{\mathrm{U}}$ and the Replacement Axiom, y_1 is a set. By the union axiom y_2 is a set. Since $U \subseteq Q$, $y_1 \subseteq Q$. By the transitivity of Q, $y_2 \subseteq Q$. So $y := y_1 \cup y_2 \subseteq Q$.

If there is $z \in U$ such that $y = \{u \mid u \in_{\mathrm{U}} z\}$, then it is easy to check that $(z = \bigcup x)^{(\langle Q, \in' \rangle)}$. If there is no such z, then $y \in Q_1$ and $(y = \bigcup x)^{(\langle Q, \in' \rangle)}$.

It is left to the students to check that the other axioms hold in $\langle Q, \in' \rangle$.

Corollary Let A be a class and $\widetilde{\in}$ be an extensional left narrow relation on A. Then there are a model $\langle U, \in' \rangle$ of ZF $-$ Fnd and a class $A' \subseteq U$ such that A' is transitive and $\langle A', \in' \rangle \cong \langle A, \widetilde{\in} \rangle$.

Let $\langle V, \in, U, S \rangle$ be a universe satisfying ZFA + "$U \neq \emptyset$". Then there a relation $\langle Q, \in' \rangle$ on U such that $\langle Q, \in' \rangle$ is extensional left narrow and non well founded. (Take for example $\{\langle u, u \rangle \mid u \in U\}$). We thus obtain a universe satisfying (ZF $-$ Fnd) $\cup$ $\{\neg$Fnd$\}$.

Example It may happen that $P(a) \in a$.

Let $\in'$ be the following relation on $\{a, b, u, v, w\}$.

$$a, u, v, w \in' b, \quad (1)$$

$$a, b \in' a, \quad (2)$$

$$a \in' u, \quad (3)$$

$$b \in' v. \quad (4)$$

Then $\in'$ is extensional.

So by the above corollary there is a model U of ZF $-$ Fnd such that $\{a, b, u, v, w\}$ is a transitive set in U. So $\emptyset = w$, $\{a, b\} = a$, $\{a\} = u$ and $\{b\} = v$. It follows that

$$b = \{a, u, v, w\} = \{a, \{a\}, \{b\}, \emptyset\}.$$

Also, $P(a) = \{\emptyset, a, \{a\}, \{b\}\}$. So $P(a) = b \in' a$.

Remark Since by Cantor's Theorem, there is no function from a onto $P(a)$, it cannot happen that $P(a) \subseteq a$.