The Definition by Induction Theorem

Definition Let f be a function and $A \subseteq \operatorname{Dom}(f)$. Then

$$f \restriction A = \{\langle a, f(a) \rangle \mid a \in A\}$$

$f \restriction A$ is called the *restriction* of f to A.

Note that $f \restriction A = g$ iff $\operatorname{Dom}(g) = A$ and for every $a \in A$, $f(a) = g(a)$.

Theorem "Definition by induction Theorem".
Let V be a set, $a \in V$ and $F : \mathbb{N} \times V \to V$. Then there is a unique function $G : \mathbb{N} \to V$ such that $G(0) = a$ and for every $n \in \mathbb{N}$, $G(n+1) = F(n, G(n))$.
Proof If $n \in \mathbb{N}$ denote by $\mathbb{N}^{<n}$ the set $\{k \in \mathbb{N} \mid k < n\}$. Similarly, $\mathbb{N}^{\leq n} := \{k \in \mathbb{N} \mid k \leq n\}$.

Let a, F be as in the theorem. A function g is called a *good function*, if there is $n \in \mathbb{N}$ such that $\operatorname{Dom}(g) = \mathbb{N}^{\leq n}$, $g(0) = a$ and for every $k \in \mathbb{N}^{<n}$, $g(k+1) = F(k, g(k))$.

Claim 1 For every $n \in \mathbb{N}$ there is a good function g such that $\operatorname{Dom}(g) = \mathbb{N}^{\leq n}$. **Proof** By induction. Let g be the function $\{\langle 0, a \rangle\}$. Then g is good and $\operatorname{Dom}(g) = \mathbb{N}^{\leq 0}$.

Suppose that the induction hypothesis is true for n. Let g be a good function such that $\operatorname{Dom}(g) = \mathbb{N}^{\leq n}$. Define $h = g \cup \{\langle\, n+1, F(n, g(n))\, \rangle\}$. It is left to the reader to check that h is a good function and that $\operatorname{Dom}(h) = \mathbb{N}^{\leq n+1}$. This proves Claim 1.

Claim 2 Suppose that $m, n \in \mathbb{N}$ and $m \leq n$. Suppose further that g, h are good functions, $\operatorname{Dom}(g) = \mathbb{N}^{\leq m}$ and $\operatorname{Dom}(h) = \mathbb{N}^{\leq n}$. Then $h \restriction \mathbb{N}^{\leq m} = g$.
Proof Let X be the set of all natural numbers m which have the following property.

> If g is a good function, $\operatorname{Dom}(g) = \mathbb{N}^{\leq m}$, $n \geq m$ and h is a good function such that $\operatorname{Dom}(h) = \mathbb{N}^{\leq n}$, then $h \restriction \mathbb{N}^{\leq m} = g$.

We prove that $X = \mathbb{N}$.

(1) $0 \in X$. Let g be a good function such that $\mathrm{Dom}(g) = \mathbb{N}^{\leq 0}$. Then $g = \{\langle 0, a\rangle\}$. Let $n \geq 0$ and h be a good function such that $\mathrm{Dom}(h) = \mathbb{N}^{\leq n}$. Then by the definition of goodness $h(0) = a$. So $h \restriction \mathbb{N}^{\leq 0} = g$. So $0 \in X$.

(2) Suppose that $m \in X$ and we prove that $m + 1 \in X$. Let g be a good function, $\mathrm{Dom}(g) = \mathbb{N}^{\leq m+1}$, $n \geq m + 1$ and h be a good function such that $\mathrm{Dom}(h) = \mathbb{N}^{\leq n}$.

By Claim 1, there is good function g_0 such that $\mathrm{Dom}(g_0) = \mathbb{N}^{\leq m}$. Then by the induction hypothesis $g \restriction \mathbb{N}^{\leq m} = g_0$ and $h \restriction \mathbb{N}^{\leq m} = g_0$. In particular, $g(m) = g_0(m)$ and $h(m) = g_0(m)$. So

$$g(m + 1) = F(m, g(m)) = F(m, g_0(m))$$

and

$$h(m + 1) = F(m, h(m)) = F(m, g_0(m)).$$

Hence $h(m + 1) = g(m + 1)$.

For every $i \leq m$, $h(i) = g_0(i) = g(i)$. Altogether we have that for every $i \leq m + 1$ For every $i \leq m$, $h(i) = g_0(i) = g(i)$. So $h \restriction \mathbb{N}^{\leq m+1} = g$. This implies that $m + 1 \in X$.

By the Induction Theorem, $X = \mathbb{N}$. This implies Claim 2.

We now prove that there is a function G as required in the theorem. Let $\mathcal{G}$ be the set of good functions and $G = \bigcup \mathcal{G}$.

(i) G is a function. Suppose that $\langle i, b\rangle, \langle i, c\rangle \in G$. There are good functions g and h such that $\langle i, b\rangle \in g$ and $\langle i, c\rangle \in h$. There are $m, n \in \mathbb{N}$ such that $\mathrm{Dom}(g) = \mathbb{N}^{\leq m}$ and $\mathrm{Dom}(h) = \mathbb{N}^{\leq n}$. Without loss of generality, $m \leq n$. By Claim 2, $g = h \restriction \mathbb{N}^{\leq m}$. So $b = g(i) = h(i) = c$. That is, $b = c$. So G is a function.

(ii) $\mathrm{Dom}(G) = \mathbb{N}$. Let $i \in \mathbb{N}$. By Claim 1, there is $g \in \mathcal{G}$ such that $\mathrm{Dom}(g) = \mathbb{N}^{\leq i}$. So $i \in \mathrm{Dom}(g) \subseteq \mathrm{Dom}(G)$. Hence $\mathbb{N} \subseteq \mathrm{Dom}(G)$. It is also trivial that $\mathrm{Dom}(G) \subseteq \mathbb{N}$. So $\mathrm{Dom}(G) = \mathbb{N}$.

(iii) $G(0) = a$. We know by Claim 1 that $\mathcal{G} \neq \emptyset$. So let $g \in \mathcal{G}$. Then $G(0) = g(0) = a$.

(iv) For every $n \in \mathbb{N}$, $G(n+1) = F(n, G(n))$. Let $g \in \mathcal{G}$ be such that $\mathrm{Dom}\,(g) = \mathbb{N}^{\leq n+1}$. Then $G(n+1) = g(n+1) = F(n, g(n)) = F(n, G(n))$.

Finally we prove that there is only one function satifying the requirements of the Theorem. Suppose that both G and H satisfy the requirements of the theorem, and we show that $G = H$. We prove by induction that for every $n \in \mathbb{N}$, $G(n) = H(n)$.

(i) $n = 0$. Then $g(0) = a = H(0)$.

(ii) Suppose that the induction hypothesis is true for n. That is, $G(n) = H(n)$. Then $G(n+1) = F(n, G(n)) = F(n, H(n)) = H(n+1)$.

We have shown that $G = H$. ■