ALL DIHEDRAL DIVISION ALGEBRAS OF DEGREE FIVE ARE CYCLIC

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Abstract. In [8] Rowen and Saltman proved that every division algebra which is split by a dihedral extension of degree $2n$ of the center, $n$ odd, is in fact cyclic. The proof requires roots of unity of order $n$ in the center. We show that for $n = 5$, this assumption can be removed. It then follows that $5\text{Br}(F)$, the 5-torsion part of the Brauer group, is generated by cyclic algebras, generalizing a result of Merkurjev [2] on the 2 and 3 torsion parts.

1. Mathematical background

We begin with basic notions needed for this work and refer the reader to [7] or [9] for more details.

Let $R$ be a ring and let $C(R) = \{r \in R \mid rx = xr \; \forall x \in R\}$ denote the center of $R$.

Definition 1.1. A ring $R$ will be called a simple ring if $R$ has no non-trivial two-sided ideals. In particular $R$ is a division ring if every nonzero element is invertible.

Remark 1.2. Notice that if $R$ is simple, its center is naturally a field.

Definition 1.3. An $F$-algebra $R$ is called an $F$-central simple algebra if $R$ is simple with $C(R) = F$ and $\dim_F(R) < \infty$.

Remark 1.4. Every $F$-central simple algebra $A$ has $\dim_F(A) = n^2$, and we define the degree of $A$, denoted $\text{deg}(A)$, to be $n$.

By Wedderburn’s Theorem every $F$-central simple algebra is of the form $M_n(D)$, where $D$ is a division algebra with center $F$.

The Brauer group of a field $F$, denoted $\text{Br}(F)$, is the set of isomorphism classes of $F$-central simple algebras modulo the following relation: two central simple algebras $A, B$ are equivalent if and only if there exist natural numbers $m, n$ such that $M_n(A) \cong M_m(B)$.

Proposition 1.5. Let $D$ be an $F$-central division algebra of degree $n$, and $K$ a subfield of $D$, then $K$ is a maximal subfield if and only if $[K : F] = n$.

Definition 1.6. A crossed product is an $F$-central simple algebra $A$ of degree $n$ containing a commutative $F$-subalgebra $C$ Galois over $F$, with $[C : F] = n$. Note that if $A$ is a division algebra then $C$ is a maximal subfield of $A$.

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Definition 1.7. Let $D$ be an $F$-central division algebra of degree $n$. We will say that $D$ is split by a group $G$ if $D$ contains a maximal subfield $K$ with Galois closure $E$ such that $\text{Gal}(E/F) = G$.

Theorem 1.8. Let $A$ be a crossed product where $K \subset A$ is a maximal subfield with Galois group $\text{Gal}(K/F) = G$. Then $A$ has the following description: 

$$A = \bigoplus_{\sigma \in G} Kx_{\sigma}$$

as a left $K$-vector space, and multiplication in $A$ is according to the rules:

$$x_\sigma k = \sigma(k)x_\sigma \quad \forall k \in K$$

and

$$x_\sigma x_\tau = c(\sigma, \tau)x_\tau x_\sigma$$

where $c \in H^2(G, K^\times)$ is a 2-cocycle. In this case $A$ is denoted $A = (K, G, c)$.

Remark 1.9. If $G = \langle \sigma \rangle$ we can give a simpler representation of $A$ as follows:

$$A = \bigoplus_{i=0}^{n-1} Kx^i$$

as a left $K$-vector space, where $n = \deg(A) = |G|$ and the multiplication is according to the rules:

$$xk = \sigma(k)x \quad \forall k \in K$$

and

$$x^i x^j = \begin{cases} x^{i+j}, & i + j < n \\ \beta x^{i+j-n}, & i + j \geq n \end{cases}$$

In this case, $A$ is denoted as $A = (K, \sigma, \beta)$.

Remark 1.10. If $F$ contains a primitive $n$-th root of unity $\rho$, we can give an even simpler description of $A$ (since then $K = F[x \mid x^n = \alpha = \rho]$) as follows:

$$A = F[x, y \mid x^n = \alpha; y^n = \beta; xy = \rho yx] \quad \alpha, \beta \in F$$

2. Some preliminary results

In this section we briefly repeat the arguments of Rowen and Saltman in [8] but we do not assume $F$ contains roots of unity.

The situation we will be handling is the following: $D/F$ is a central simple algebra of odd degree $n$ having a maximal subfield $K \subset D$ with Galois closure $E \supset K \supset F$, such that

$$\text{Gal}(E/F) = D_n = \langle \sigma, \tau : \sigma^n = \tau^2 = 1, \tau \sigma \tau = \sigma^{-1} \rangle,$$

and $K = E(\tau)$.

Extending scalars to $E^{(\sigma)}$, we may view $E \subset D' = D \otimes E^{(\sigma)}$. Now $\text{Gal}(E/E^{(\sigma)}) = \langle \sigma \rangle$, i.e. $D'$ is cyclic, so we have an element $\beta \in D'$ such that

$$\beta^{-1} x^\beta = \sigma(x) \quad \forall x \in E.$$  

In particular $\beta^n \in E^{(\sigma)}$. Notice that $\tau$ can be extended to $D' = D \otimes E^{(\sigma)}$ by its action on $E^{(\sigma)}$, that is, we write $\tau$ instead of $1 \otimes \tau$.

Lemma 2.1. We may assume that $\tau(\beta) = \beta^{-1}$. 

Proof. Applying $\tau$ to (1) yields

$$\tau(\beta)^{-1}\tau(x)\tau(\beta) = \sigma^{-1}(\tau(x)),$$  \quad \forall x \in E.$$

Now since $\tau$ is an automorphism of $E$, $\tau(x)$ runs over all elements of $E$, and thus

$$\tau(\beta)^{-1}y\tau(\beta) = \sigma^{-1}(y), \quad \forall y \in E$$

that is $\tau(\beta)$ acts on $E$ as $\sigma^{-1}$. Now define $\beta' = \beta^r\tau(\beta)^{-1}$, where $r = (n+1)/2$, and compute that $\tau(\beta') = \beta'^{-1}$, and $\beta'$ acts on $E$ as $\sigma$. \hfill $\square$

Let $P_t(X) = X^n + \sum_{i=1}^{n} c_i(t)X^{n-i}$ denote the characteristic polynomial of $t \in D'$. Note that $c_1(t) = -tr(t)$ and $c_n(t) = (-1)^n N(t)$ where $tr(t)$ and $N(t)$ are the reduced trace and norm of $t$.

**Lemma 2.2.** Let $t = \beta^i e$, for $e \in E$ and $0 < i < n$, $i \neq 0$. Then $tr(t) = 0$.

**Proof.** Let $d = \gcd(i,n)$.

Clearly we have $t^n/d = \beta^{ni/d} N_{\sigma_i}(e) \in E^{(\sigma^i)}$ where $N_{\sigma_i}$ is the norm from $E$ to $E^{(\sigma^i)}$.

Now $[E : E^{(\sigma^i)}] = n/d$, implying $P(X) = X^n/d - \beta^{ni/d} N_{\sigma_i}(e)$ is the characteristic polynomial of $t$, hence $tr_{E/E^{(\sigma^i)}}(t) = 0$ which implies $tr_{E/F}(t) = 0$. \hfill $\square$

**Lemma 2.3.** Let $t = (\beta + \beta^{-1})e$ for $e \in E$. Then the coefficients of $P_t(X)$ satisfy $c_i(t) = 0$ for every odd $0 < i < n$.

**Proof.** Notice that for $i$ odd, $t^i$ is a sum of elements of the form $a\beta^s$ where $a \in E$ and $s$ odd, $-n < s < n$, so by 2.2 and Newton’s identities we are done in the characteristic zero case. For the general case, we refer the reader to [8] where the main idea is that you can form a model for this situation in the form of an Azumaya algebra and then use a specialization argument. \hfill $\square$

**Corollary 2.4.** There is an element $t \in D$ such that for every $e \in E$ (and so also for $k \in K \subset E$), $c_i = 0$ for every odd $0 < i < n$ in $P_t(e)$.  

**Proof.** Since $D = D'(\tau)$ we have $t = \beta + \beta^{-1}$ is the desired element. \hfill $\square$

**Remark 2.5.** Notice that if $n = p$ is prime $\text{Char}(F) = p$, the element $t = \beta + \beta^{-1} \in D$ we found satisfies $t^p \notin F$ and $t \notin F$ and so by a theorem of Albert in the “special results” chapter of his seminal book [1], which is known as Albert’s cyclicity criterion, $D$ is cyclic (this is not a new result as J.P Tignol and P. Mammone did this for any field $F$ with $\text{Char}(F) \mid n$ in [6] using the corestriction, but it shows that the proof of Rowen and Saltman also applies to this case).
3. The case \( n = 5 \)

Now we would like to focus on the particular case where \( n = 5 \). The main tool we will be using is the following proposition taken from [3, Proposition 2.2].

**Proposition 3.1.** Let \( G(x_1, \ldots, x_n) \) be a homogeneous form of degree 3 defined over a field \( F \). If \( G \) has a solution, \( \alpha \in K^{(n)} \), defined over a quadratic extension \( K \) of \( F \), then \( G \) has a solution defined over \( F \).

**Proof.** The proof in [3] uses basic intersection theory which we will not use, instead we will give an algebraic proof (which is actually a translation of the proof in [3]) which will enable us to find an explicit solution in section 3. Since \([K:F] = 2\) the solution \( \alpha \) has the following form: \( \alpha = (\alpha_1 + \beta_1 t, \ldots, \alpha_n + \beta_n t) \) where \( \alpha_i, \beta_i \in k \), and \( t \in K \) such that \( K = F[t] \). Now specialize \( G(x_1, \ldots, x_n) \) to \( G(\alpha_1 + \beta_1 Z, \ldots, \alpha_n + \beta_n Z) \), denoting it by \( g(Z) \). Notice that the coefficient of \( Z^3 \) in \( g(Z) \) is \( G(\beta_1, \ldots, \beta_n) \) hence if \( G(\beta_1, \ldots, \beta_n) = 0 \) we have a solution defined over \( F \) else \( g(Z) \) is a degree 3 polynomial defined over \( F \). Since \( g(t) = 0 \) we get that \( g(Z) = cm_1(Z)(Z-w) \), where \( c = G(\beta_1, \ldots, \beta_n) \) and \( m_1(Z) \) is the minimal polynomial of \( t \) over \( F \). Now \( c, g(Z) \) and \( m_1(Z) \) are defined over \( F \) hence \( w \) is in \( F \) and clearly \( G(\alpha_1 + \beta_1 w, \ldots, \alpha_n + \beta_n w) = g(w) = 0 \) so we have found a solution \( \gamma = (\alpha_1 + \beta_1 w, \ldots, \alpha_n + \beta_n w) \in F^n \). \( \square \)

**Theorem 3.2.** Let \( D \) be a division algebra of degree 5 split by the group \( D_5 \) then \( D \) is cyclic.

**Proof.** In view of remark 2.5, we may assume \( \text{Char}(F) \neq 5 \). First we remark that by Albert’s cyclicity criterion it is enough to find an element \( t \in D - F \) such that \( t^5 \in F \), that is \( c_i = 0 \) for every \( 0 < i < n \). Now by 2.4 we have \( t \in D \) with the property \( c_i(te) = 0 \) for every odd \( 0 < i < n \) and \( \forall e \in E \). Now since \( P_{i-1}(x) = -N(t)^{-1}P_i(x^{-1})x^5 \) we have \( c_i(te^{-1}) = 0 \) for every even \( 0 < i < n \) and \( \forall e \in E \). Hence we are left with finding a solution for \( c_1(te^{-1}) = 0 \) (which is linear) and \( c_3(te^{-1}) = 0 \) (which is cubic) in the five dimensional vector space \( Et^{-1} \). Define \( V := \{te^{-1} \in Et^{-1} \mid c_1(te^{-1}) = 0 \} \), which is a four dimensional subspace of \( Et^{-1} \).

We have to find a solution for \( c_3(v) = 0 \) in \( V \). Let us add a fifth root of unity to \( F \), which is either a quadratic extension or a chain of two quadratic extensions. After this extension we are in the case of Rowen and Saltman where they gave an explicit element whose fifth power is in \( F \) which was \( (v + v^{-1})t^{-1} \), where \( v \in E \). This element is clearly in \( V \otimes_F F[t_5] \). Now by 3.1 since \( c_3(v) \) is homogeneous of degree 3, we have a solution after either one or two quadratic extensions. Thus, we have a solution before the extension and we are done. \( \square \)

**Remark 3.3.** If the fifth root of unity is in a quadratic extension of \( F \), we know \( D \) is cyclic by a theorem of Vishne [10, Theorem 13.6] and D. Haile, M. A. Knus, M. Rost, J. P. Tignol [5], so what actually is new is the last case of \( [F[t_5] : F] = 4 \).
4. A generic example

Fixing \( p \) let \( K = F[\rho_p] \) and denote \( \text{Gal}(K/F) = \langle \tau \rangle \). In [6, Theorem 2] Merkurjev proves that \( p\operatorname{Br}(F) \) is generated by \( F \)-central simple algebras, \( A \), of degree \( p \) such that \( A \otimes K \cong (\alpha, \beta) \) where \( K[\sqrt[p]{\alpha}] \) is cyclic over \( K \) Galois over \( F \).

In [10] Vishne calls these algebras quasi-symbols and gives more details about them including generic examples. We will show that for \( p = 5 \) these algebras are cyclic and conclude that \( \operatorname{Br}(F) \) is generated by cyclic algebras.

4.1. A generic Quasi-symbol of degree 5.

For \( p = 5 \) we have two possibilities for \( [K : F] \). The first is \( [K : F] = 2 \); in this case Vishne shows that every quasi-symbol is cyclic. The second case is \( [K : F] = 4 \); in this case every quasi-symbol \( A \) has one of the following forms (after extending scalars to \( K \)):

\begin{enumerate}
\item \( A \otimes K = (\alpha, \beta) \), where \( \alpha \in F \) and \( \tau(\beta) \equiv \beta^2 \pmod{K^{\times 5}} \).
\item \( A \otimes K = (\alpha, \beta) \), where \( \tau(\alpha) = \alpha^{-1} \) and \( \tau(\beta) \equiv \beta^{-2} \pmod{K^{\times 5}} \).
\end{enumerate}

The first kind is known to be cyclic by [10, Theorem 10.3]. So we are left with the second kind for which Vishne gives the following generic construction which we will show is cyclic. Thus every quasi-symbol of degree 5 is cyclic and hence, by [6, Theorem 2] we conclude that \( \operatorname{Br}(F) \) is generated by cyclic algebras.

Let \( k_0 \) be a field of characteristic \( \neq 5 \) and \( k = k_0[\rho] \) where \( \rho \) is a fixed primitive fifth root of unity, \( \text{Gal}(k/k_0) = \langle \tau \rangle \) where \( \tau(\rho) = \rho^2 \). Set \( K = k(a, b, \eta) \) a transcendental extension and extend \( \tau \) to \( K \) by

\[ \tau(a) = a^{-1}, \quad \tau(b) = \eta^5b^{-2}, \quad \tau(\eta) = \eta^2b^{-1}. \]

Notice that we still have \( \tau^5 = 1 \). Define \( F = K^{\langle \tau \rangle} \) and

\[ D = (a, b)_K = K[x, y \mid x^5 = a, \ y^5 = b, \ xy^{-1} = \rho x], \]

and extend \( \tau \) to \( D \) by \( \tau(x) = x^{-1} \), \( \tau(y) = \eta y^{-2} \). Notice that \( \tau^2(\eta) = \eta^{-1} \) and \( \tau^2(y) = y^{-1} \).

Now define \( D_0 = D^{\langle \tau \rangle} \); \( D_0/F \) is the generic quasi-symbol of degree 5 of the second type.

Remark 4.1. Vishne’s construction is much more general and we specialized it to the above case, for the general construction we refer the reader to [10].

Proposition 4.2. \( D_0 \) is split by \( D_5 \).

Proof. Notice that \( \text{Gal}(K[y]/F) = C_5 \ltimes C_4 = \langle \sigma \rangle \ltimes \langle \tau \rangle \) and now we will see how \( \tau \) acts on \( \sigma \). Applying \( \tau \) to \( x^{-1}tx = \sigma(t) \), which holds for every \( t \in K[y] \), yields \( \tau(\sigma(t)) = \tau(x^{-1})\tau(t)\tau(x) = x\tau(t)x^{-1} = \sigma^{-1}(\tau(t)) \) and so we get \( \tau\sigma\tau^{-1} = \sigma^{-1} \). Hence \( \tau^2 \) is a central element in \( \text{Gal}(K[y]/F) \) and it is clear that \( E = K[y]^{\langle \tau^2 \rangle} \subset K[y] \) is Galois over \( F \) with \( \text{Gal}(E/F) = D_5 = \langle \sigma \rangle \ltimes \langle \tau \rangle \) and we are done.

Corollary 4.3. \( D_0 \) is cyclic.

In [2] Merkurjev proves the following theorem:

Theorem 4.4. Let \( F \) be a field. \( n\operatorname{Br}(F) \) is generated by cyclic algebras, for \( n = 2, 3, \).
Now as a result of the above we can extend Merkurjev’s theorem to \( n = 5 \) and get

**Theorem 4.5.** \( \tilde{\text{Br}}(F) \) is generated by cyclic algebras.

*Proof.* By section 8 of [10] \( \tilde{\text{Br}}(F) \) is generated by quasi-symbols of degree 5, and so we are done. \( \square \)

### 4.2. Finding an explicit solution.

Since the above example is a generic one, it would be nice to give an explicit element with fifth power in \( F \), which is what we do now by going over the general proof.

Let \( P_1(X) = X^n + \sum_{i=1}^n c_i X^{n-i} \) denote the characteristic polynomial of \( t \in D_0 \).

\( V = (x + x^{-1})^{-1}K[y]^{(\tau)} \) is a 5-dimensional \( F \)-subspace of \( D_0 \), satisfying \( c_i(v) = c_i(v) = 0 \) for all \( v \in V \); and we want to find a solution in \( V \) for \( \text{tr}(Z) = c_1((x + x^{-1})^{-1}Z) = 0 \) and \( G(Z) = c_3((x + x^{-1})^{-1}Z) = 0 \). Extending scalars from \( F \) to \( F[\rho + \rho^{-1}] \), we have the solutions \( Z_1 = y + y^{-1} = \alpha + \beta(\rho + \rho^{-1}) \) and \( Z_2 = \tau(Z_1) = \alpha + \beta \tau(\rho + \rho^{-1}) = \alpha + \beta \tau(\rho^2 + \rho^{-2}) \) where \( \alpha = (\alpha_1, \ldots, \alpha_5), \beta = (\beta_1, \ldots, \beta_5) \in K[y]^{(\tau)} \).

Now define the following line: \( L = \{\alpha + \beta t\} = \{(\alpha_1 + \beta_1 t, \ldots, \alpha_5 + \beta_5 t)\} \) defined over \( F \).

**Proposition 4.6.** For every \( l \in L \) we have \( \text{tr}(l) = 0 \).

*Proof.* By standard linear algebra, \( L \cap \{\text{tr}(Z) = 0\} \) is either one point or the whole line \( L \); since \( Z_1, Z_2 \in L \cap \{\text{tr}(Z) = 0\} \), we get \( L \cap \{\text{tr}(Z) = 0\} = L \) and we are done. \( \square \)

Now let us study the variety \( \{G(Z) = 0\} \cap L \). First we need to compute \( G(Z) \).

In order to do that we use the representation of \( D \) induced by right multiplication on \( D = K[y] + K[y]x + K[y]x^2 + K[y]x^3 + K[y]x^4 \), namely

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & a \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

\( m \in K[y] \rightarrow \text{Diag}(m, \sigma(m), \sigma^2(m), \sigma^3(m), \sigma^4(m)) \)

Now the minimal polynomial of \( x + x^{-1} \) is

\( \lambda^5 - 5\lambda^3 + 5\lambda - (a + a^{-1}) \)

hence

\( (x + x^{-1})^{-1} = ((x + x^{-1})^4 - 5(x + x^{-1})^2 + 5)(a + a^{-1})^{-1} = (a + a^{-1})^{-1}(x^4 + x^{-4} - x^2 - x^{-2} + 1) \)

implying

\[
\begin{pmatrix}
1 & a & -1 & -a & 1 \\
\end{pmatrix}
\]

\( (a + a^{-1})^{-1} \)

\[
\begin{pmatrix}
1 & a & -1 & -a \\
a^{-1} & 1 & a & -1 \\
-1 & a^{-1} & 1 & a \\
-a^{-1} & -1 & a^{-1} & 1 \\
1 & -a^{-1} & -1 & a & -a^{-1} & 1
\end{pmatrix}
\]
Now we are left with solving for \( \alpha, \beta \) Theorem 4.7. The element we get: 

\[c_3((x + x^{-1})^{-1}m) = (a + a^{-1})^{-1} \left( m \sigma(m) \sigma^2(m) + \sigma(m) \sigma^2(m) \sigma^3(m) + \sigma^2(m) \sigma^3(m) \sigma^4(m) + \sigma^3(m) \sigma^4(m) m + \sigma^4(m) m \sigma(m) \right) = (a + a^{-1})^{-1} \operatorname{tr}_\sigma(m \sigma(m) \sigma^2(m)).\]

Yielding \( F(Z) = (a + a^{-1})^{-1} \operatorname{tr}_\sigma(Z \sigma(Z) \sigma^2(Z)) \)

Now clearly \( \{ F(Z) = 0 \} \cap L \) is defined over \( F \) by the polynomial

\[ f(t) = F(\alpha + \beta t) = (a + a^{-1})^{-1} \operatorname{tr}_\sigma(\alpha + \beta t) \sigma(\alpha + \beta t) \sigma^2(\alpha + \beta t) = (a + a^{-1})^{-1} \operatorname{tr}_\sigma(\beta \sigma(\beta) \sigma^2(\beta) t^2 + ...) = F(\beta)t^2 + ... \]

But we know two solutions for \( f(t) \), namely \( t_1 = \rho + \rho^{-1} \) and \( t_2 = \rho^2 + \rho^{-2} \), so we get \( f(t) = F(\beta)(t - t_1)(t - t_2)(t - t_3) \). Now since \( f(t) \) and \( F(\beta)(t - t_1)(t - t_2) \) are defined over \( F \), we get \( t_3 \in F \).

Explicitly \( f(0) = -t_1 t_2 t_3 F(\beta) \) implies \( t_3 = \frac{f(0)}{F(\beta)} = \frac{f(0)}{F(\beta)} = F(\alpha) \) is in \( F \). Hence we get:

**Theorem 4.7.** The element \( w = (x + x^{-1})^{-1}(\alpha + \beta \frac{F(\alpha)}{F(\beta)}) \) is in \( D_0 - F \) satisfies \( w^5 \in F \).

Now we are left with solving for \( \alpha, \beta \) from the two equations:

\[ y + y^{-1} = \alpha + \beta(\rho + \rho^{-1}) \]

\[ \eta y^{-2} + \eta^{-1} y^2 = \tau(y + y^{-1}) = \alpha + \beta(\rho^2 + \rho^{-2}) \]

Hence

\[ \beta = \frac{y + y^{-1} - \eta y^{-2} - \eta^{-1} y^2}{\rho + \rho^{-1} - \rho^2 - \rho^{-2}} \]

\[ \alpha = y + y^{-1} - \beta(\rho + \rho^{-1}) \]

### 4.3. The general case.

We will now show that the above solution for the case of quasi-symbols, where we do decent from \( F[\rho + \rho^{-1}] \) to \( F \) is valid for the general case of \( D_5 = \langle \sigma, \tau : \sigma^5 = \tau^2 = 1, \sigma \tau = \sigma^{-1} \rangle \) division algebras, where we need to decent from \( F[\rho] \otimes E(\sigma) \) to \( F \). The situation is the following: we look for a solution to \( c_3(t) = c_1(t) = 0 \) where \( c_3(t) \) are as in section 3 and \( t \in (\beta + \beta^{-1})^{-1}E(\tau) \). Let \( \operatorname{Gal}(E \otimes F[\rho]/F) = \langle \pi \rangle \); hence \( \operatorname{Gal}(E \otimes F[\rho]/F) = D_5 \times \langle \pi \rangle \) and so after extending scalars to \( F[\rho] \) we want a solution in \( (\beta + \beta^{-1})^{-1}(E \otimes F[\rho])(\tau) \times \langle \pi \rangle \), which will then be defined over \( F \).

**Proposition 4.8.** We may assume \( v + v^{-1} \in (E \otimes F[\rho])(\tau) \times \langle \pi \rangle \), for \( v \) as in the proof of theorem 3.2.
Proof. Since $v = x'\tau(x)^{-r}$, where $x$ is any eigenvector of $\sigma$ with eigenvalue $\rho$, we may write $x = \sum_{i=0}^{i=\rho-1}\sigma^i(k)$ for $k \in E(\tau) \times (\tau)$. Now $\tau(x) = \pi^2(x)$ and so $\tau(v) = \tau(x')'x'^{-r} = \pi^2(x')'x'^{-r} = \pi^2(x'\pi^2(x')^{-r}) = \pi^2(x'^{-r}) = \pi^2(v)$ implying $\tau(v + v^{-1}) = v + v^{-1}$, hence $v + v^{-1}$ is in $(E \otimes F[\rho])^\tau \times (\pi^2)$, as desired.

Now it is clear that after extending scalars to $F[\rho + \rho^{-1}]$ we have the solution $(\beta + \beta^{-1})^{-1}(v + v^{-1})$ and so we are in the same situation as in the quasi-symbol case, hence the above solution is valid for the general case too.

References