Z₃ × Z₃ CROSSED PRODUCTS

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Abstract. Let A be the generic abelian crossed product with respect to Z₃ × Z₃. In this note we show that A is similar to the tensor product of 4 symbol algebras (3 of degree 9 and one of degree 3) and if A is of exponent 3 it is similar to the product of 31 symbol algebras of degree 3. We then use [9] to prove that if A is any algebra of degree 9 then A is similar to the product of 35840 symbol algebras (8960 of degree 3 and 26880 of degree 9) and if A is of exponent 3 it is similar to the product of 277760 symbol algebras of degree 3. We then show that the essential 3-dimension of the class of A is at most 6.

1. Introduction

Throughout this note we let F be a field, μ_n the group of n-th roots of 1 and ρ_n a primitive n-th root of 1. The well known Merkurjev-Suslin theorem says that: assuming F contains a primitive n-th root of 1, there is an isomorphism ψ : K_2(F)/nK_2(F) → Br(F) sending the symbol {a, b} to the symbol algebra (a, b)_{n,F}. In particular the n-th torsion part of the Brauer group is generated by symbol algebras of degree n. This means every A ∈ Br(F) is similar (denoted by ∼) to the tensor product of symbol algebras of degree n. However, their proof is not constructive. It thus raises the following questions. Let A be an algebra of degree n and exponent m. Can one explicitly write A as the tensor product of degree m symbol algebras? Also, what is the smallest number of factors needed to express A as the tensor product of degree m symbol algebras? This number is sometimes called the Merkurjev-Suslin number. These questions turn out to be quite hard in general and not much is known. Here is a short summary of some known results.

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(1) Every degree 2 algebra is isomorphic to a quaternion algebra.
(2) Every degree 3 algebra is cyclic. Thus if $\mu_3 \subset F$ it is isomorphic to a symbol algebra (Wedderburn [13]).
(3) Every degree 4 algebra of exponent 2 is isomorphic to a product of two quaternion algebras (Albert [1]).
(4) Every degree $p^n$ symbol algebra of exponent $p^m$ is similar to the product of $p^{n-m}$ symbol algebras of degree $p^m$ (Tignol [11]).
(5) Every degree 8 algebra of exponent 2 is similar to the product of four quaternion algebras (Tignol [12]).
(6) Every abelian crossed product with respect to $\mathbb{Z}_n \times \mathbb{Z}_2$ is similar to the product of a symbol algebra of degree $2n$ and a quaternion algebra, in particular, due to Albert [1], every degree 4 algebra is similar to the product of a degree 4 symbol algebra and a quaternion algebra (Lorenz, Rowen, Reichstein, Saltman [5]).
(7) Every abelian crossed product with respect to $(\mathbb{Z}_2)^4$ of exponent 2 is similar to the product of 18 quaternion algebras (Sivatski [10]).
(8) Every $p$-algebra of degree $p^n$ and exponent $p^m$ is similar to the product of $p^n - 1$ cyclic algebras of degree $p^m$ (Florence [3]).

In this paper we prove theorems 3.3 and 4.1 stating:

Let $A$ be an abelian crossed product with respect to $\mathbb{Z}_3 \times \mathbb{Z}_3$. Then

(1) $A$ is similar to the product of 4 symbol algebras (3 of degree 9 and one of degree 3).
(2) If $A$ is of exponent 3 then $A$ is similar to the product of 31 symbol algebras of degree 3.

We then use [9] to deduce the general case of an algebra of degree 9 to get theorem 5.1 stating:

Let $A$ be an $F$-central simple algebra of degree 9. Then

(1) $A$ is similar to the product of 35840 symbol algebras (8960 of degree 3 and 26880 of degree 9).
(2) If $A$ is of exponent 3 then $A$ is similar to the product of 277760 symbol algebras of degree 3.
2. \( \mathbb{Z}_p \times \mathbb{Z}_p \) ABELIAN CROSSED PRODUCTS

Let \( A \) be the generic abelian crossed product with respect to \( G = \mathbb{Z}_p \times \mathbb{Z}_p \) over \( F \), where \( p \) is an odd prime. In the notation of [2] this means: \( A = (E, G, b_1, b_2, u) = E[z_1, z_2]z_i z_i^{-1} = \sigma_i(e); z_i^p = b_1; z_i^2 = b_2; z_2 z_1 = u z_1 z_2; b_i \in E_i = E < \sigma_i >; u \in E^\times \) s.t. \( \text{N}_{E/F}(u) = 1 \) where \( \text{Gal}(E/F) = \langle \sigma_1, \sigma_2 \rangle \cong G \).

Let \( A \) be as above. Write \( E = E_1 E_2 \) where \( E_1 = F[t_1, t_1^p = f_1 \in F^\times] \) and \( E_2 = F[t_2, t_2^p = f_2 \in F^\times] \). Thus we have \( z_i t_i z_i^{-1} = \sigma_i(t_i) = t_i \) and \( z_1 t_2 = \rho_p t_2 z_1 \); \( z_2 t_1 = \rho_p t_1 z_2 \). Since \( b_i \in E_i \) we can write \( b_1 = c_0 + c_1 t_1 + ... + c_{p-1} t_1^{p-1}; b_2 = a_0 + a_1 t_2 + ... + a_{p-1} t_2^{p-1} \) where \( a_i, c_i \in F^\times \).

**Proposition 2.1.** Define \( v = e_1 z_1 + e_2 z_2 \) for \( e_i \in E \). If \( v \neq 0 \), then \([F[v^p] : F] = p\).

**Proof.** First, we compute \( vt_1 t_2 = (e_1 z_1 + e_2 z_2) t_1 t_2 = e_1 z_1 t_1 t_2 + e_2 z_2 t_1 t_2 = \rho_p t_1 t_2 e_1 z_1 + \rho_p t_1 t_2 e_2 z_2 = \rho_p t_1 t_2 (e_1 z_1 + e_2 z_2) = \rho_p t_1 t_2 v \). Thus \( v^p \) commutes with \( t_1 t_2 \) where \( v \) does not, implying \([F[v] : F[v^p]] = p\). By the definition of \( v \) we have \( v \notin F \). Thus \( \text{deg}(A) = p^2 \) implying \([F[v] : F] \in \{p, p^2\}\). If \([F[v] : F] = p\) we get that \( A \) contains the sub-algebra generated by \( t_1 t_2, v \) which is a degree \( p \) symbol over \( F \) and by the double centralizer this will imply that \( A \) is decomposable which is not true in the generic case. Thus \([F[v] : F] = p^2 \) implying \([F[v^p] : F] = p\) and we are done. \( \Box \)

The first step we take is to find a \( v \) satisfying \( \text{Tr}(v^p) = 0 \). In order to achieve that we will tensor \( A \) with an \( F \)-symbol of degree \( p \).

Define \( B = (E_1, \sigma_2, \frac{-a_0}{a_0}) \sim (E, G, 1, \frac{-a_0}{a_0}, 1) \). Now by [2] \( A \otimes B \) is similar to \( C = (E, G, b_1, \frac{-a_0}{a_0} b_2, u) \). Abusing notation we write \( z_1, z_2 \) for the new ones in \( C \).

**Proposition 2.2.** Defining \( v = z_1 + z_2 \) in \( C \) we have \( \text{Tr}(v^p) = 0 \).

**Proof.** First, notice that \( C = \sum_{i,j=0}^{p-1} E z_1^i z_2^j \). Thus \( C_0 = \{d \in C \mid \text{Tr}(d) = 0\} = E_0 + \sum_{i,j=0; (i,j) \neq (0,0)}^{p-1} E z_1^i z_2^j \) where \( E_0 = \sum_{i,j=0; (i,j) \neq (0,0)}^{p-1} F t_1 t_2 \) is the set of trace zero elements of \( E \). Now computing we see \( v^p = z_1^p + e_{p-1,1} z_1^{p-1} z_2 + ... + e_{1, p-1} z_1^{p-1} z_1 + z_2^p = b_1 + e_{p-1,1} z_1^{p-1} z_2 + ... + e_{1, p-1} z_2^{p-1} z_1 + b_2 \) where \( e_{i,j} \in E \). Define \( r = v^p - (b_1 + b_2) \). Clearly \( \text{Tr}(r) = 0 \), since the powers of \( z_1, z_2 \) in all monomial appearing in \( r \) are less then \( p \) and at least one is greater than zero. Thus, \( v^p = b_1 + b_2 + r = c_0 + c_1 t_1 + ... + c_{p-1} t_1^{p-1} + (-c_0 + \frac{c_0 a_0}{a_0} t_2 + ... + \frac{c_0 a_0}{a_0} t_2^{p-1}) + r \in C_0 \), and we are done. \( \Box \)
Proposition 2.3. $K \doteqdot F[t_1t_2, v^p]$ is a maximal subfield of $C$.

Proof. First, notice $C$ is a division algebra of degree $p^2$. Indeed, assuming it is not, then it is similar to a degree $p$ algebra, which we denote by $D$. Thus $A \otimes B$ is similar to $D$, which implies $A$ is isomorphic to $D \otimes B^{op}$. But then $A$ has exponent $p$ which is false. In the proof of 2.1 we saw that $[v^p, t_1t_2] = 0$ so we are left with showing $[K : F] = p^2$. Assuming $[K : F] = p$, we have $v^p \in F[t_1t_2]$. Let $\sigma$ be a generator of $\text{Gal}(F[t_1t_2]/F) = < \sigma >$. Clearly $z_i x = \sigma(x) z_i$ for $i = 1, 2$ and $x \in F[t_1t_2]$, hence $vx = \sigma(x)v$, that is $\sigma(x) = vxv^{-1}$. In particular, $\sigma(v^p) = vv^p v^{-1} = v^p$, implying $v^p \in F$. But then $C$ contains the subalgebra $F[t_1t_2, v]$ which is an $F$-csa of degree $p$, thus by the double centralizer $C$ would decompose into two degree $p$ algebras. This will imply that $A$ has exponent $p$, which is false. \hfill \Box

The next step is to make $K$ Galois. Let $T$ be the Galois closure of $F[v^p]$. Its Galois group is a subgroup of $S_p$ so has a cyclic $p$-Sylow subgroup, define $L$ to be the fixed subfield. Clearly $F[v^p] \otimes L$ is Galois, with group $\mathbb{Z}_p$. Thus in $C_L$ we have $K_L$ as a maximal Galois subfield with group $\mathbb{Z}_p \times \mathbb{Z}_p$. Now writing $C_L$ as an abelian crossed product we have $C_L = (K, G, b_1, b_2, u)$ where this time we have $\text{Tr}(b_2) = 0$. Thus we can write $K_L = L[t_1t_2, t_3](t_1t_2)^p = f_1 f_2; t_3^p = l \in L$, $b_1 \in L[t_1t_2]$ and $b_2 = l t_3 + ... + l_{p-1} t_3^{p-1}$.

Now we change things even more. Define $D = (f_1 f_2, (-\frac{f_1 f_2}{t_1})^{p-1}) \otimes L = (K_L, G, t_1t_2, -\frac{f_1 f_2}{t_1} (t_3)^{-1}, \rho p)$ and again by [2] we have $R \doteqdot C_L \otimes D = (K_L, G, t_1t_2 b_1, -f_1 f_2 - \frac{f_1 f_2}{t_1} t_3 - ... - \frac{f_1 f_2}{t_1^{p-2}} t_3^{p-2}, \rho p u)$. 

3. Generic $\mathbb{Z}_3 \times \mathbb{Z}_3$ abelian crossed products

From now we specialize to $p = 3$.

**Proposition 3.1.** $R$ from the end of the previous section is a symbol algebra of degree 9.

**Proof.** This proof is just as in [5]. Since we assume $\rho_9 \in F$ it is enough to find a 9-central element. Notice that in $R$ we have $z_2 t_1 t_2 = \rho_3 t_1 t_2 z_2$; $z_2^3 = -f_1 f_2 - \frac{f_1 f_2}{f_1} t_3$ and $(t_1 t_2)^3 = f_1 f_2$. Thus defining $x = t_1 t_2 + z_2$ we get $x^3 = (t_1 t_2 + z_2)^3 = (t_1 t_2)^3 + z_2^3 = -\frac{f_1 f_2}{f_1} t_3$ implying $x^9 = -(\frac{f_1 f_2}{f_1})^3 l \in L$. Thus $R = (l_3, -(\frac{f_1 f_2}{f_1})^3 l)_{9,l}$ for some $l_3 \in L$ and we are done. □

All of the above gives the following theorem:

**Theorem 3.2.** Let $A$ be a generic abelian crossed product with respect to $\mathbb{Z}_3 \times \mathbb{Z}_3$. Then after a quadratic extension $L/F$, we have that $A_L$ is similar to $R \otimes D^{-1} \otimes B^{-1}$ where $R, D, B$ are symbols as above.

In order to go down to $F$ we take corestriction. Using Rosset-Tate and the projection formula, ([4] 7.4.11 and 7.2.7), we get:

**Theorem 3.3.** Let $A$ be a generic abelian crossed product with respect to $\mathbb{Z}_3 \times \mathbb{Z}_3$. Then $A = \sum_{i=1}^4 C_i$ where $C_1, C_2, C_3$ are symbols of degree 9 and $C_4$ is a symbol of degree 3.

**Proof.** One gets $C_1, C_2$ from the corestriction of $R$ using R.T. To get $C_3$ use the corestriction of $D$ and the projection formula. Finally, $C_4$ comes from $B$. □
4. THE EXPONENT 3 CASE

In this section we will consider the case were $\exp(A) = 3$. Notice that from 3.2 $A_L \sim R \otimes D^{-1} \otimes B^{-1} = (a, b)_{9, L} \otimes (\gamma, c)_{9, L} \otimes (\alpha, \beta)_{3, L}$ where $\alpha, \beta, \gamma \in F^\times$ and $a, b, c \in L^\times$.

**Theorem 4.1.** Assume $A$ has exponent 3, then $A$ is similar to the sum of 16 degree 3 symbols over a quadratic extension and 31 degree 3 symbols over $F$.

**Proof.** The idea for this proof is credited to L.H. Rowen, U. Vishne and E. Matzri. Since $\exp(A) = 3$ we have $F \sim A^3 \sim R^3 \otimes D^{-3} \otimes B^{-3} \sim (a, b)_{3, L} \otimes (\gamma, c)_{3, L}$. Thus we get $(a, b)_{3, L} = (\gamma, c^{-1})_{3, L}$.

Now by the chain lemma for degree 3 symbols in [8] or [6] we have $x_{1, 2, 3} \in L^\times$ such that:

$$(a, b)_{3, L} = (a, x_1)_{3, L} = (x_2, x_1)_{3, L} = (x_2, x_3)_{3, L} = (\gamma, x_3)_{3, L} = (\gamma, c^{-1})_{3, L}$$

Now we write

$$(a, b)_{3, L} \otimes (\gamma, c)_{9, L} \otimes (a, b)_{3, L} \otimes (\gamma, c)_{9, L} \sim (a, b)_{9, L} \otimes (\gamma, c)_{9, L}$$

Thus $A \sim (a, b)_{9, L} \otimes (\gamma, c)_{9, L} \otimes (a, b)_{3, L} \sim (a, b)_{9, L} \otimes (\gamma, c)_{9, L} \otimes (a, b)_{9, L} \otimes (\gamma, c)_{9, L} \otimes (x_2, x_3)_{3, L} \otimes (\gamma, x_3)_{9, L} \otimes (\gamma, x_3)_{9, L} \otimes (\gamma, x_3)_{9, L} \otimes (\gamma, x_3)_{9, L}$ where now all the degree 9 symbols are of exponent 3. But by a theorem of Tignol, [11], each of these symbols is similar to the product of three degree 3 symbols. Thus we have that $A_L$ is similar to the product of 16 degree 3 symbols and over $F$ to the product of 31 symbols of degree 3 and we are done.

$\square$
5. The general case of a degree 9 algebra

In this section we combine the results of sections 2 and 3 with [9] to handle the general case of a degree 9 algebra of exponent 9 and 3. Let $A$ be a $F$-central simple algebra of degree 9.

The first step would be to follow [9] to find a field extension $P/F$ such that $A_P$ is an abelian crossed product with respect to $\mathbb{Z}_3 \times \mathbb{Z}_3$ and $[P : F]$ is prime to 3. The argument in [9] basically goes as follows: Let $K \subset A$ be a maximal subfield, i.e. $[K : F] = 9$. Now let $F \subset K \subset E$ be the normal closure of $K$ over $F$. Since we know nothing about $K$ we have to assume $G = \text{Gal}(E/F) = S_9$. Let $H < G$ be a 3-sylow subgroup and $L = E^H$, then $[L : H] = 4480$. Now extend scalars to $L$, then $KL \subset A_L$ as a maximal subfield. By Galois correspondence $KL = E^{H_1}$ for some subgroup $H_1 < H$ and $[H : H_1] = [KL : L] = 9$. Since $H$ is a 3-group we can find $H_1 \triangleleft H_2 < H$ such that $[H : H_2] = 3$ thus we have $L = E^H \subset E^{H_2} \subset KL = E^{H_1} \subset E$ and since $H_2 < H$ we know the extension $E^{H_2}/L$ is Galois with group isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$. Thus in $A_L$ we have the subfield $E^{H_2}$ which has a non trivial $L$- automorphism $\sigma$. Now let $z \in A$ be an element inducing $\sigma$ (such $z$ exists by Skolem-Noether). Consider the subfield $L[z^3]/L$, since $z^3$ commutes with $E^{H_2}$ and $z$ does not $[L[z] : L[z^3]] = 3$. In the best case scenario we have $L[z^3] = L$ which will imply $A_L$ decomposes into the tensor product of two symbols of degree 3 and we are done. In the general case we will have $[L[z^3] : L] = 3$. If $L[z^3]/L$ is Galois we are done since $E^{H_2}[z^3]$ will be a maximal subfield Galois over $L$ with group isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$, but again in general this should not be the case. However we can extend scalars to make $L[z^3]/L$ Galois, in particular consider $P = L[\text{disc}(L[z^3])]$ then, $[P : L] = 2$ and $P[z^3]/P$ is Galois and we are done. To summarize we have found an extension $P = L[\text{disc}(L[z^3])]$ with $[P : F] = 4480 \cdot 2 = 8960$ such that $A_P$ contains a maximal subfield $PE^{H_2}[z^3]/P$ Galois over $P$ with group isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$. 
Combining the above with the results of sections 2, 3 and using Rosset-Tate we get the following theorem.

**Theorem 5.1.** Let $A$ be an $F$-central simple algebra of degree 9. Then

1. $A$ is similar to the product of 35840 symbol algebras, (8960 of degree 3 and 26880 of degree 9).
2. If $A$ is of exponent 3 then $A$ is similar to the product of 277760 symbol algebras of degree 3.

### 6. Application to Essential Dimension

In [7] Merkurjev computes the essential $p$-dimension of $PGL_{p^2}$ relative to a fixed field $k$ to be $p^2 + 1$. One can interpret this result as follows: Let $F$ be a field of definition (relative to a base field $k$) for the generic division algebra of degree $p^2$. Let $E/F$ be the prime to $p$ closure of $F$. Let $l, k \subset l \subset E$, be a subfield of $E$ over which $A$ is defined. Then $l/k$ has transcendence degree at least $p^2 + 1$ (and such $l$ exists with transcendence degree exactly $p^2 + 1$). It makes sense to define the essential dimension and the essential $p$-dimension of the class of an algebra $A$ (with respect to a fixed base field $k$).

**Definition 6.1.** Let $A \in \text{Br}(F)$. Define the essential dimension and the essential $p$-dimension of the class of $A$ (with respect to a fixed base field $k$) as:

\[
edc(A) = \min \{ \text{ed}(B) | B \sim A \} \\
edc_p(A) = \text{edc}(A_E) \text{ where } E/F \text{ is the prime to } p \text{ closure of } F.
\]

Notice that [7] for $p=2$ gives $\text{ed}_2(PGL_{2^2}) = 5$ and for $p = 3$ it gives $\text{ed}_3(PGL_{3^2}) = 10$. Now assume $F$ is prime to $p$ closed. Then as proved in [9] every $F$-csa of degree $p^2$ is actually an abelian crossed product with respect to $\mathbb{Z}_p \times \mathbb{Z}_p$. Thus, in this language, in [5] they prove:

**Theorem 6.2.** Let $A$ be a generic division algebra of degree 4, then $\text{edc}(A) = \text{edc}_2(A) = 4$.

For $p = 3$ Theorem 3.3 says:

**Theorem 6.3.** Let $A$ be a generic division algebra of degree 9, then $\text{edc}_3(A) \leq 6$. 

References