Symmetrically Indivisible Structures - Not what one would expect

Nadav Meir

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Let $\mathcal{M}$ be a countable $\mathcal{L}$-structure.

**Definition**

$\mathcal{M}$ is *indivisible* if for every colouring of $\mathcal{M}$ in two colours, there is a monochromatic $\mathcal{M}' \subseteq \mathcal{M}$ such that $\mathcal{M}'$ is isomorphic to $\mathcal{M}$.
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**Example**

An infinite set (with no structure).

$\langle \mathbb{N}, \leq \rangle$.

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Let $\mathcal{M}$ be a symmetrically indivisible structure in a language $\mathcal{L}$. Let $\mathcal{L}_0 \subseteq \mathcal{L}$. Is $\mathcal{M} | \mathcal{L}_0$ symmetrically indivisible?

We will show that the answer is no, so symmetric indivisibility doesn't behave quite as one would expect.

Additionally there are many more symmetrically indivisible structures than you might think.
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Introduction

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Additionally there are many more symmetrically indivisible structures than you might think.
Examples for many symmetrically indivisible structures

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Let $\mathcal{L}$ be a first-order language. $\mathcal{M}, \mathcal{N}$ $\mathcal{L}$-structures. We say that $\mathcal{M} \preccurlyeq \mathcal{N}$ if there is an embedding $e : \mathcal{M} \hookrightarrow \mathcal{N}$.

Let $\mathcal{L}$ be a first-order language, $\mathcal{M}$, $\mathcal{N}$ $\mathcal{L}$-structures. We say that $\mathcal{M} \preccurlyeq_s \mathcal{N}$ if there is a symmetric embedding $e : \mathcal{M} \hookrightarrow \mathcal{N}$.

Let $\mathcal{L}$ be a first-order language, $\mathcal{M}$, $\mathcal{N}$ $\mathcal{L}$-structures. We say that $\mathcal{M} \preccurlyeq_e \mathcal{N}$ if there is an elementary embedding $e : \mathcal{M} \hookrightarrow \mathcal{N}$.
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Recall:

**Definition**

A *preorder* is a binary relation which is reflexive and transitive.

Every preorder induces an equivalence relation:

**Observation**

If \( \preceq \) is a preorder on a class \( A \), for each \( a, b \in A \) define \( a \sim b \) if \( a \preceq b \) and \( b \preceq a \).
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If $\preceq$ is a preorder on a class $A$, for each $a, b \in A$ define $a \sim b$ if $a \preceq b$ and $b \preceq a$. Then $\sim$ is an equivalence relation on $A$. 
Recall:

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**Observation**

If $\preceq$ is a preorder on a class $A$, for each $a, b \in A$ define $a \sim b$ if $a \preceq b$ and $b \preceq a$. Then $\sim$ is an equivalence relation on $A$ and $\preceq$ is a well defined partial order on the quotient space $A/\sim$. 
It is easy to verify that $\sim$, $\sim_s$ and $\sim_e$ are preorders on the class of $\mathcal{L}$-structures, therefore, we can define the equivalence relation induced by these 3 relations:

**Definition**

Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{L}$-structures. We say that $\mathcal{M} \sim \mathcal{N}$ if $\mathcal{M} \preceq \mathcal{N}$ and $\mathcal{N} \preceq \mathcal{M}$. Similarly, we define $\sim_s$ and $\sim_e$. 

**Proposition**

If $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures then:

- If $\mathcal{M} \sim \mathcal{N}$ then $\mathcal{M}$ is indivisible if and only if $\mathcal{N}$ is indivisible.
- If $\mathcal{M} \sim_s \mathcal{N}$ then $\mathcal{M}$ is symmetrically indivisible if and only if $\mathcal{N}$ is symmetrically indivisible.
- If $\mathcal{M} \sim_e \mathcal{N}$ then $\mathcal{M}$ is elementarily indivisible if and only if $\mathcal{N}$ is elementarily indivisible.
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**Definition**

Let \( M, N \) be \( \mathcal{L} \)-structures. We say that \( M \sim N \) if \( M \preceq N \) and \( N \preceq M \).

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**Proposition**

If \( M \) and \( N \) are \( \mathcal{L} \)-structures then:

- if \( M \sim N \) then \( M \) is indivisible if and only if \( N \) is indivisible.
- If \( M \sim_s N \) then \( M \) is **symmetrically indivisible** if and only if \( N \) is **symmetrically indivisible**.
- If \( M \sim_e N \) then \( M \) is **elementarily indivisible** if and only if \( N \) is **elementarily indivisible**.
We will prove the proposition for symmetric indivisibility. A similar proof works for indivisibility and elementary indivisibility.

(sketch).

Suppose $\mathcal{M}$ is symmetrically indivisible, and let $c : \mathcal{N} \to \{\text{red, blue}\}$

We need to produce a monochromatic substructure that is symmetrically embedded.
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- $\mathcal{M} \lesssim \mathcal{N}$, so $c$ induces a colouring $c_\mathcal{M}$ on $\mathcal{M}$. 
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- $\mathcal{M} \preceq \mathcal{N}$, so $c$ induces a colouring $c_\mathcal{M}$ on $\mathcal{M}$.
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- $\mathcal{N} \preceq_s \mathcal{M} \Rightarrow \mathcal{N} \preceq_s \mathcal{M}'$ so we have a symmetrically embedded substructure (copy of $\mathcal{N}$) $\hat{\mathcal{N}} \subseteq \mathcal{M}$. 
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- \( \mathcal{M} \preceq \mathcal{N} \), so \( c \) induces a colouring \( c_{\mathcal{M}} \) on \( \mathcal{M} \).
- \( \mathcal{M} \) is symmetrically indivisible, so there is a symmetrically embedded monochromatic \( \mathcal{M}' \subseteq \mathcal{M} \) isomorphic to \( \mathcal{M} \).
- \( \mathcal{N} \preceq_{s} \mathcal{M} \Rightarrow \mathcal{N} \preceq_{s} \mathcal{M}' \) so we have a symmetrically embedded substructure (copy of \( \mathcal{N} \)) \( \hat{\mathcal{N}} \subseteq \mathcal{M} \).
- Let \( \mathcal{N}' \) be the copy of \( \hat{\mathcal{N}} \) in \( \mathcal{N} \).
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Suppose \( \mathcal{M} \) is symmetrically indivisible, and let

\[ c : \mathcal{N} \to \{ \text{red, blue} \} \]

We need to produce a monochromatic substructure that is symmetrically embedded.

- \( \mathcal{M} \not\lesssim \mathcal{N} \), so \( c \) induces a colouring \( c_\mathcal{M} \) on \( \mathcal{M} \).
- \( \mathcal{M} \) is symmetrically indivisible, so there is a symmetrically embedded monochromatic \( \mathcal{M}' \subseteq \mathcal{M} \) isomorphic to \( \mathcal{M} \).
- \( \mathcal{N} \not\lesssim_s \mathcal{M} \Rightarrow \mathcal{N} \not\lesssim_s \mathcal{M}' \) so we have a symmetrically embedded substructure (copy of \( \mathcal{N} \)) \( \hat{\mathcal{N}} \subseteq \mathcal{M} \).
- Let \( \mathcal{N}' \) be the copy of \( \hat{\mathcal{N}} \) in \( \mathcal{N} \).
- \( \mathcal{N}' \) is a monochromatic symmetrically embedded copy of \( \mathcal{N} \) in \( \mathcal{N} \).
In this talk, we work with countable linear orders and countable undirected graphs.
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**Definition (The Random Graph)**

Rado’s random graph is the unique (up to isomorphism) countable graph $\Gamma$ with the property that for any finite, disjoint $U, V \subseteq \Gamma$ there is a vertex $v$ joined to every element in $U$ and disjoint from every element in $V$. 
Theorem (M. Kojman, S. Geschke - 2008)

The random graph $\Gamma$ is symmetrically indivisible.

We will use these two structures to construct, using the equivalence relation $\sim_s$, many symmetrically indivisible structures.
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We will use these two structures to construct, using the equivalence relation $\sim_s$, many symmetrically indivisible structures.
Theorem (Henson, 71)

For every countable graph $G$, $G \lesssim_s \Gamma$.

Corollary

Every countable graph which symmetrically embeds $\Gamma$ is symmetrically indivisible.

Proof.

Let $G$ be a countable graph which symmetrically embeds $\Gamma$, then by definition, $G \sim_s \Gamma$. Thus because $\Gamma$ is symmetrically indivisible, so are $G$ and $\Gamma$ by the previous proposition.
Theorem (Henson, 71)

For every countable graph $G$, $G \preceq_s \Gamma$.

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For \((\mathbb{Q}, <)\) we have a similar result to that of Henson:

**Lemma**

*For every countable linear order \(A, A \preceq_s \mathbb{Q}\)*

Proof.

Let \(A[\mathbb{Q}]\) be the linear order whose universe is \(A \times \mathbb{Q}\) with the lexicographical order. \(A[\mathbb{Q}]\) |\(=\) DLO and is countable, thus by \(\aleph_0\)-categoricity of DLO, we have \(A[\mathbb{Q}] \cong \mathbb{Q}\) and \(e: A \rightarrow \mathbb{Q}\) defined as \(e(a) := (a, 0)\) is a symmetric embedding of \(A\) into \(A[\mathbb{Q}] \cong \mathbb{Q}\).

and thus the exact same proof of the corollary works for linear orders:

**Corollary**

*Every countable linear order which symmetrically embeds \((\mathbb{Q}, <)\) is symmetrically indivisible.*
For \((\mathbb{Q}, <)\) we have a similar result to that of Henson:

**Lemma**

*For every countable linear order \(A\), \(A \preceq_s \mathbb{Q}\)*

**Proof.**

Let \(A[\mathbb{Q}]\) be the linear order whose universe is \(A \times \mathbb{Q}\) with the lexicographical order. \(A[\mathbb{Q}] \models DLO\) and is countable, thus by \(\aleph_0\)-categoricity of \(DLO\), we have \(A[\mathbb{Q}] \simeq \mathbb{Q}\) and \(e : A \rightarrow \mathbb{Q}\) defined as \(e(a) := (a, 0)\) is a symmetric embedding of \(A\) into \(A[\mathbb{Q}] \simeq \mathbb{Q}\).
For \((\mathbb{Q}, <)\) we have a similar result to that of Henson:

**Lemma**

For every countable linear order \(A\), \(A \preccurlyeq_s \mathbb{Q}\)

**Proof.**

Let \(A[\mathbb{Q}]\) be the linear order whose universe is \(A \times \mathbb{Q}\) with the lexicographical order. \(A[\mathbb{Q}] \models DLO\) and is countable, thus by \(\aleph_0\)-categoricity of DLO, we have \(A[\mathbb{Q}] \simeq \mathbb{Q}\) and \(e : A \to \mathbb{Q}\) defined as \(e(a) := (a, 0)\) is a symmetric embedding of \(A\) into \(A[\mathbb{Q}] \simeq \mathbb{Q}\).

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**Corollary**

Every countable linear order which symmetrically embeds \((\mathbb{Q}, <)\) is symmetrically indivisible.
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Corollary 1: There are many symmetrically indivisible linear orders.

Corollary 2: There are many symmetrically indivisible graphs.

This can be extended to hypergraphs, $K_n$-free graphs, and more...
Exercise

1. Show a 1-1 correspondence between countable linear orders (upto isomorphism) and countable linear orders symmetrically embedding \( \mathbb{Q} \).

2. Show a 1-1 correspondence between countable graphs (up-to isomorphism) and countable graphs symmetrically embedding \( \Gamma \).

so from the exercise, combined with the previous two corollaries we get:

Corollary

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Exercise

1. Show a 1-1 correspondence between countable linear orders (upto isomorphism) and countable linear orders symmetrically embedding $\mathbb{Q}$.

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so from the exercise, combined with the previous two corollaries we get:

Corollary

1. There are many symmetrically indivisible linear orders.

2. There are many symmetrically indivisible graphs.

3. All of this can be extended to hypergraphs, $K_n$-free graphs, and more . . .
Next we will set the ground for answering the question:

**Question**

Let $\mathcal{M}$ be a symmetrically indivisible structure in a language $\mathcal{L}$. Let $\mathcal{L}_0 \subseteq \mathcal{L}$. Is $\mathcal{M} \upharpoonright \mathcal{L}_0$ symmetrically indivisible?

To answer this question, we will need to analyze a specific graph.
Γ*- an indivisible structure which is not symmetrically indivisible

Recall that Rado’s random graph is the unique (up to isomorphism) countable graph Γ with the property that for any finite, disjoint $U, V \subseteq \Gamma$ there is a vertex $v$ joined to every element in $U$ and disjoint from every element in $V$. 

Definition

We define $\Gamma^*$ as follows:

Let $c : \Gamma \to \omega$ be an enumeration of $\Gamma$ with $|\Gamma^*| = |\Gamma| \cup \{(a, b) \in \omega \times \omega | a \leq b\}$

$E_{\Gamma^*} = E_{\Gamma} \cup \{(a_1, b_1), (a_2, b_2) \in \omega \times \omega | b_1 = b_2\} \cup \{(g, (a, b)) \in |\Gamma| \times (\omega \times \omega) | c(g) = b\}$

In words, if $\{g_n\}_{n \in \omega}$ enumerates the vertices of $\Gamma$, then for each $n \in \omega$ we add a clique $K_n$ and connect it only to $g_n$. 
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**Definition**

We define $\Gamma^*$ as follows:
Let $c : \Gamma \to \omega$ be an enumeration of $\Gamma$

$$|\Gamma^*| := |\Gamma| \cup \{ (a, b) \in \omega \times \omega \mid a \leq b \}$$

$$E^{\Gamma^*} := E^\Gamma \cup \{ \langle (a_1, b_1), (a_2, b_2) \rangle \in (\omega \times \omega)^2 \mid b_1 = b_2 \} \cup \{ \langle g, (a, b) \rangle \in |\Gamma| \times (\omega \times \omega) \mid c(g) = b \}$$
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In words, if $\{g_n\}_{n \in \omega}$ enumerates the vertices of $\Gamma$, then for each $n \in \omega$ we add a clique $K_n$ and connect it only to $g_n$. 

Γ*- an indivisible structure which is not symmetrically indivisible
Lemma

\( \Gamma^* \) is not rigid (i.e. \( \Gamma^* \) has non-trivial automorphisms)

Proof.

consider for example a non-trivial automorphism of \( K_3 \)

Lemma

If \( \sigma : \Gamma^* \to \Gamma^* \) is an automorphism of \( \Gamma^* \) then \( \sigma \upharpoonright \Gamma = Id_{\Gamma} \)

Proof.
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Lemma

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• σ(Γ) = Γ.
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Lemma

If $\sigma : \Gamma^* \rightarrow \Gamma^*$ is an automorphism of $\Gamma^*$ then $\sigma | \Gamma = \text{Id}_\Gamma$

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- $\sigma(\Gamma) = \Gamma$.
- For each $K_n$, $\sigma(K_n) = K_n$. 

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- $\sigma(\Gamma) = \Gamma$.
- for each $K_n$, $\sigma(K_n) = K_n$.
- If $\sigma(g_n) = g_m$ then $\sigma(K_n) = K_m$. 
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Lemma

If \( \sigma : \Gamma^* \rightarrow \Gamma^* \) is an automorphism of \( \Gamma^* \) then \( \sigma \upharpoonright \Gamma = Id_\Gamma \)

Proof.

1. \( \sigma(\Gamma) = \Gamma \).
2. for each \( K_n \), \( \sigma(K_n) = K_n \).
3. If \( \sigma(g_n) = g_m \) then \( \sigma(K_n) = K_m \), so \( n = m \) and \( \sigma(g_n) = g_n \).
Proposition

$\Gamma^*$ is indivisible but not symmetrically indivisible.

Proof.

To show $\Gamma^*$ is not symmetrically indivisible, let $c : \Gamma^* \to \{\text{red}, \text{blue}\}$ the colouring of $\Gamma^*$ defined by $c(x) = \begin{cases} \text{red} & \text{if } x \in \Gamma \\ \text{blue} & \text{if } x \not\in \Gamma \end{cases}$

By vertex degree considerations, there is no blue substructure isomorphic to $\Gamma^*$. Suppose $G \subseteq \Gamma$ is a red substructure isomorphic to $\Gamma^*$, then it has a non-trivial automorphism. This isomorphism does not extend to an automorphism of $\Gamma^*$, since it is not $\text{Id}$ on $\Gamma$.

$G$ was arbitrary, so $\Gamma^*$ is not symmetrically indivisible.
Proposition

$\Gamma^*$ is indivisible but not symmetrically indivisible.

Proof.

First, $\Gamma^*$ is indivisible, since clearly $\Gamma \preccurlyeq \Gamma^*$ and as we said earlier, for every countable graph $G$, $G \preccurlyeq_s \Gamma$, in particular $\Gamma^* \preccurlyeq \Gamma$ thus $\Gamma^* \sim \Gamma$ and because $\Gamma$ is indivisible so is $\Gamma^*$.

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Symmetrically Indivisible Structures - Not what one would expect
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- By vertex degree considerations, there is no blue substructure isomorphic to \( \Gamma^* \).
- Suppose \( G \subseteq \Gamma \) is a red substructure isomorphic to \( \Gamma^* \), then it has a non-trivial automorphism.
- This isomorphism does not extend to an automorphism of \( \Gamma^* \), since it is it is not \( Id \) on \( \Gamma \).
- \( G \) was arbitrary, so \( \Gamma^* \) is not symmetrically indivisible.
Definition

Let $\mathcal{L}_{OG} = \{E, <\}$ be the language of ordered graph. We say that an $\mathcal{L}_{OG}$-structure $G$ is an enumerated graph and $G \upharpoonright E$ is a graph and $(G \upharpoonright <) \simeq (\mathbb{N}, <)$. We say that an $\mathcal{L}_{OG}$-structure $\Gamma$ is an enumerated Rado graph if $\Gamma \upharpoonright E$ is (isomorphic to) The Rado graph and $\Gamma$ is enumerated. Note that since $(\mathbb{N}, <)$ is rigid, every enumerated graph is rigid as well, thus in context of enumerated graphs, symmetrically indivisibility and indivisibility coincide.
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Note that since $(\mathbb{N}, <)$ is rigid, every enumerated graph is rigid as well, thus in context of enumerated graphs, symmetrically indivisibility and indivisibility coincide.
The same (even simpler) ideas from earlier about Rado’s graph, can be modified to enumerated Rado graphs to show:

**Proposition**

If \((\Gamma, <)\) is an enumerated Rado graph, then \((\Gamma, \leq)\) embeds every enumerated graph.

**Proposition**

If \((\Gamma, \leq)\) is an enumerated Rado graph, then \((\Gamma, \leq)\) is indivisible.
If $(\Gamma^*, <)$ is $\Gamma^*$ from earlier with any enumeration on it, then it embeds an emurated Rado’s graph $(\Gamma, <)$.
If \((\Gamma^*, <)\) is \(\Gamma^*\) from earlier with any enumeration on it, then it embeds an emurated Rado’s graph \((\Gamma, <)\). so with the previous propositions we have \((\Gamma^*, <) \sim (\Gamma, <)\).
If \((\Gamma^*, <)\) is \(\Gamma^*\) from earlier with any enumeration on it, then it embeds an emurated Rado’s graph \((\Gamma, <)\). so with the previous propositions we have \((\Gamma^*, <) \sim (\Gamma, <)\), and \((\Gamma, <)\) is indivisible.
If \((\Gamma^*, <)\) is \(\Gamma^*\) from earlier with any enumeration on it, then it embeds an emurated Rado’s graph \((\Gamma, <)\). So with the previous propositions we have \((\Gamma^*, <) \sim (\Gamma, <)\), and \((\Gamma, <)\) is indivisible, so \((\Gamma^*, <)\) is indivisible and thus by rigidness it is symmetrically indivisible.
If \((\Gamma^*, <)\) is \(\Gamma^*\) from earlier with any enumeration on it, then it embeds an emurated Rado’s graph \((\Gamma, <)\). So with the previous propositions we have \((\Gamma^*, <) \sim (\Gamma, <)\), and \((\Gamma, <)\) is indivisible, so \((\Gamma^*, <)\) is indivisible and thus by rigidness it is symmetrically indivisible. But earlier we saw \(\Gamma^*\), the reduct of \((\Gamma, <)\) is not symmetrically indivisible.
If \((\Gamma^*, <)\) is \(\Gamma^*\) from earlier with any enumeration on it, then it embeds an emurated Rado's graph \((\Gamma, <)\). So with the previous propositions we have \((\Gamma^*, <) \sim (\Gamma, <)\), and \((\Gamma, <)\) is indivisible, so \((\Gamma^*, <)\) is indivisible and thus by rigidness it is symmetrically indivisible. But earlier we saw \(\Gamma^*\), the reduct of \((\Gamma, <)\) is not symmetrically indivisible, so \((\Gamma^*, <)\) is a counterexample for the question presented.
elementarily indivisible structures

Nadav Meir

Symmetrically Indivisible Structures - Not what one would expect
Questions?
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Thank you!
Assaf Hasson, Menachem Kojman and Alf Onshuus. 
On symmetric indivisibility of countable structures 
Available at the URL http://www.math.bgu.ac.il/~hassonas/papers/final.pdf

C. Ward Henson. 
A family of countable homogeneous graphs. 

Menachem Kojman and Stefan Geschke. 
Symmetrized induced ramsey theorems. 
Available on 


Someone.

some book in algebra