Rigidity and the Lower Bound Theorem for Doubly Cohen-Macaulay Complexes

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Abstract

We prove that for $d \geq 3$, the 1-skeleton of any $(d-1)$-dimensional doubly Cohen-Macaulay (abbreviated 2-CM) complex is generically d-rigid. This implies that Barnette's lower bound inequalities for boundary complexes of simplicial polytopes ([4],[3]) hold for every 2-CM complex of dimension ≥ 2 (see Kalai [8]). Moreover, the initial part (g_0, g_1, g_2) of the g-vector of a 2-CM complex (of dimension ≥ 3) is an M-sequence. It was conjectured by Björner and Swartz [14] that the entire g-vector of a 2-CM complex is an M-sequence.

1 Introduction

The g-theorem gives a complete characterization of the f-vectors of boundary complexes of simplicial polytopes. It was conjectured by McMullen in 1970 and proved by Billera and Lee [5] (sufficiency) and by Stanley [13] (necessity) in 1980. A major open problem in f-vector theory is the g-conjecture, which asserts that this characterization holds for all homology spheres. The open part of this conjecture is to show that the g-vector of every homology sphere is an M-sequence, i.e. it is the f-vector of some order ideal of monomials. Based on the fact that homology spheres are doubly Cohen-Macaulay (abbreviated 2-CM) and that the g-vector of some other classes of 2-CM complexes is known to be an M-sequence (e.g. [14]), Björner and Swartz [14] recently suspected that

Conjecture 1.1 *([14], a weakening of Problem 4.2.) The g-vector of any 2-CM complex is an M-sequence.*

We prove a first step in this direction, namely:

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Theorem 1.2 *Let K be a $(d-1)$-dimensional 2-CM simplicial complex (over some field) where $d \geq 4$. Then $(g_0(K), g_1(K), g_2(K))$ is an M-sequence.*

This theorem follows from the following theorem, combined with an interpretation of rigidity in terms of the face ring (Stanley-Reisner ring), due (implicitly) to Lee [10].

Theorem 1.3 *Let K be a $(d-1)$-dimensional 2-CM simplicial complex (over some field) where $d \geq 3$. Then K has a generically d-rigid 1-skeleton.*

Kalai [8] showed that if a simplicial complex K of dimension ≥ 2 satisfy the following conditions then it satisfies Barnette's lower bound inequalities:

(a) K has a generically $(dim(K)+1)$-rigid 1-skeleton.

(b) For each face F of K of codimension > 2, its link $lk_K(F)$ has a generically $(dim(lk_K(F))+1)$-rigid 1-skeleton.

(c) For each face F of K of codimension 2, its link $lk_K(F)$ (which is a graph) has at least as many edges as vertices.

Kalai used this observation to prove that Barnette's inequalities hold for a large class of simplicial complexes.

Observe that the link of a vertex in a 2-CM simplicial complex is 2-CM, and that a 2-CM graph is 2-connected. Combining it with Theorem 1.3 and the above result of Kalai we conclude:

Corollary 1.4 *Let K be a $(d-1)$-dimensional 2-CM simplicial complex where $d \geq 3$. For all $0 \leq i \leq d-1$ $f_i(K) \geq f_i(n,d)$ where $f_i(n,d)$ is the number of i-faces in a (equivalently every) stacked d-polytope on n vertices. (Explicitly, $f_{d-1}(n,d) = (d-1)n-(d+1)(d-2)$ and $f_i(n,d) = \binom{d}{i}n - \binom{d+1}{i+1}i$ for $1 \leq i \leq d-2$.)* $\square$

Theorem 1.3 is proved by decomposing K into a union of minimal $(d-1)$-cycle complexes (Fogelsanger's notion [6]). Each of these pieces has a generically d-rigid 1-skeleton ([6]), and the decomposition is such that gluing the pieces together results in a complex with a generically d-rigid 1-skeleton. The decomposition is detailed in Theorem 3.4.

This paper is organized as follows: In Section 2 we give the necessary background from rigidity theory, explain the connection between rigidity and the face ring, and reduce the results mentioned in the Introduction to Theorem 3.4. In Section 3 we give the necessary background on 2-CM complexes, prove Theorem 3.4 and discuss related problems and results.

2 Rigidity

The presentation of rigidity here is based mainly on the one in Kalai [8]. Let $G = (V, E)$ be a graph. A map $f : V \to \mathbb{R}^d$ is called a $d-$*embedding*. It is $rigid$ if any small enough perturbation of it which preserves the lengths of the edges is induced by an isometry of $\mathbb{R}^d$. Formally, f is called $rigid$ if there exists an $\varepsilon > 0$ such that if $g : V \to \mathbb{R}^d$ satisfies $d(f(v), g(v)) < \varepsilon$ for every $v \in V$ and $d(g(u), g(w)) = d(f(u), f(w))$ for every $\{u, w\} \in E$, then $d(g(u), g(w)) = d(f(u), f(w))$ for every $u, w \in V$ (where $d(a, b)$ denotes the Euclidean distance between the points a and b).

G is called $generically\ d - rigid$ if the set of its rigid d-embeddings is open and dense in the topological vector space of all of its d-embeddings.

Let $V = [n]$, and let $Rig(G, f)$ be the $dn \times |E|$ matrix which is defined as follows: for its column corresponding to $\{v < u\} \in E$ put the vector $f(v) - f(u)$ (resp. $f(u) - f(v)$) at the entries of the d rows corresponding to v (resp. u) and zero otherwise. G is generically d-rigid iff $Im(Rig(G, f)) = Im(Rig(K_V, f)$ for a generic f, where K_V is the complete graph on V. $Rig(G, f)$ is called the *rigidity matrix* of G (its rank is independent of the generic f that we choose).

Let G be the 1-skeleton of a $(d-1)$-dimensional simplicial complex K. We define d generic degree-one elements in the polynomial ring $A = \mathbb{R}[x_1, .., x_n]$ as follows: $\Theta_i = \sum_{v\in[n]} f(v)_i x_v$ where $f(v)_i$ is the projection of $f(v)$ on the i-th coordinate, $1 \leq i \leq d$. Then the sequence $\Theta = (\Theta_1, .., \Theta_d)$ is an l.s.o.p. for the face ring $\mathbb{R}[K] = A/I_K$ (I_K is the ideal in A generated by the monomials whose support is not an element of K). Let $H(K) = \mathbb{R}[K]/(\Theta) = H(K)_0 \oplus H(K)_1 \oplus ...$ where (Θ) is the ideal in A generated by the elements of Θ and the grading is induced by the degree grading in A. Consider the multiplication map $\omega : H(K)_1 \longrightarrow H(K)_2$, $m \to \omega m$ where $\omega = \sum_{v\in[n]} x_v$. Lee [10] proved that

$$dim_{\mathbb{R}} Ker(Rig(G, f)) = dim_{\mathbb{R}} H(K)_2 - dim_{\mathbb{R}} \omega(H(K)_1). \qquad (1)$$

Assume that G is generically d-rigid. Then $dim_{\mathbb{R}} Ker(Rig(G, f)) = f_1(K) - rank(Rig(K_V, f)) = g_2(K) = dim_{\mathbb{R}} H(K)_2 - dim_{\mathbb{R}} H(K)_1$. Combining with (1), the map ω is injective, and hence $dim_{\mathbb{R}}(H(K)/(\omega))_i = g_i(K)$ for $i = 2$; clearly this holds for $i = 0, 1$ as well. Hence $(g_o(K), g_1(K), g_2(K))$ is an M-sequence. We conclude that Theorem 1.3 implies Theorem 1.2, via the following algebraic result:

Theorem 2.1 *Let K be a $(d-1)$-dimensional 2-CM simplicial complex (over some field) where $d \geq 3$. Then the multiplication map $\omega : H(K)_1 \longrightarrow H(K)_2$ is injective.* $\square$

In order to prove Theorem 1.3, we need the concept of minimal cycle complexes, introduced by Fogelsanger [6]. We summarize his theory below.

Fix a field k (or more generally, any abelian group) and consider the formal chain complex on a ground set $[n]$, $C = (\oplus\{kT : T \subseteq [n]\}, \partial)$, where $\partial(1T) = \sum_{t\in T} sign(t, T) T \setminus \{t\}$ and $sign(t, T) = (-1)^{|\{s\in T: s<t\}|}$. Define *subchain*, $minimal\ d - cycle$ and $minimal\ d - cycle\ complex$ as follows:

$c' = \sum\{b_T T : T \subseteq [n], |T| = d+1\}$ is a *subchain* of a d-chain $c = \sum\{a_T T : T \subseteq [n], |T| = d+1\}$ iff for every such T, $b_T = a_T$ or $b_T = 0$. A d-chain c is a $d-cycle$ if $\partial(c) = 0$, and is a *minimal* $d-cycle$ if its only subchains which are cycles are c and 0. A simplicial complex K which is spanned by the support of a *minimal* $d-cycle$ is called a *minimal* $d-cycle$ *complex* (over k), i.e. $K = \{S : \exists T\ S \subseteq T, a_T \neq 0\}$ for some minimal d-cycle c as above. For example, triangulations of connected manifolds without boundary are minimal cycle complexes - fix $k = \mathbb{Z}_2$ and let the cycle be the sum of all facets.

The following is the main result in Fogelsanger's thesis.

Theorem 2.2 *(Fogelsanger [6]) For $d \geq 3$, every minimal $(d-1)$-cycle complex has a generically d-rigid 1-skeleton.*

We will need the following gluing lemma, due of Asimov and Roth, who introduced the concept of generic rigidity of graphs [1].

Theorem 2.3 *(Asimov and Roth [2]) Let G_1 and G_2 be generically d-rigid graphs. If $G_1 \cap G_2$ contains at least d vertices, then $G_1 \cup G_2$ is generically d-rigid.*

Now we are ready to conclude Theorem 1.3 from the decomposition theorem, Theorem 3.4.

proof of Theorem 1.3: Consider a decomposition sequence of K as guaranteed by Theorem 3.4, $K = \cup_{i=1}^{m} S_i$. By Theorem 2.2 each S_i has a generically d-rigid 1-skeleton. By Theorem 2.3 for all $2 \leq i \leq m$ $\cup_{j=1}^{i} S_j$ has a generically d-rigid 1-skeleton, in particular K has a generically d-rigid 1-skeleton ($i = m$). $\square$

Remark: One can verify that Theorems 2.2 and 2.3, and hence also Theorem 1.3, continue to hold when replacing "generically d-rigid" by the notion "d-hypperconnected", introduced by Kalai [7]. Both of these assertions have an interpretation in terms of algebraic shifting, introduced by Kalai (see e.g. his survey [9]), namely: for both the exterior and symmetric shifting operators over the field $\mathbb{R}$, denoted by Δ, $\{d, n\} \in \Delta(K)$. The existence of this edge in the shifted complex implies the non-negativity of $g_2(K)$.

3 Decomposing a 2-CM complex

Definition 3.1 *A simplicial complex K is $2-CM$ (over a fixed field k) if it is Cohen-Macaulay and for every vertex $v \in K$, $K - v$ is Cohen-Macaulay of the same dimension as K.*

Here $K - v$ is the simplicial complex $\{T \in K : v \notin T\}$. By a theorem of Reisner [11], a simplicial complex L is Cohen-Macaulay iff it is pure and for every face $T \in L$ (including the empty set) and every $i < dim(lk_L(T)$,

$\tilde{H}_i(lk_L(T);k) = 0$ where $lk_L(T) = \{S \in L : T \cap S = \emptyset, T \cup S \in L\}$ and $\tilde{H}_i(M;k)$ is the reduced i-th homology of M over k. The proof of Theorem 3.4 is by induction on $dim(K)$. Let us first consider the case where K is 1-dimensional.

A (simple finite) graph is 2-*connected* if after a deletion of any vertex from it, the remaining graph is connected and non trivial (i.e. is not a single vertex nor empty). Note that a graph is 2-CM iff it is 2-connected.

Lemma 3.2 *A graph G is 2-connected iff there exists a decomposition $G = \cup_{i=1}^{m} C_i$ such that each C_i is a simple cycle and for every $1 < i \leq m$, $C_i \cap (\cup_{j<i} C_j)$ contains an edge.*

Moreover, for each $i_0 \in [m]$ the C_i's can be reordered by a permutation $\sigma : [m] \to [m]$ such that $\sigma^{-1}(1) = i_0$ and for every $i > 1$, $C_{\sigma^{-1}(i)} \cap (\cup_{j<i} C_{\sigma^{-1}(j)})$ contains an edge.

Proof: Whitney [15] showed that a graph G is 2-connected iff it has an open ear decomposition, i.e. there exists a decomposition $G = \cup_{i=0}^{m} P_i$ such that each P_i is a simple open path, P_0 is an edge, $P_0 \cup P_1$ is a simple cycle and for every $1 < i \leq m$ $P_i \cap (\cup_{j<i} P_j)$ equals the 2 end vertices of P_i.

Assume that G is 2-connected and consider an open ear decomposition as above. Let $C_1 = P_0 \cup P_1$. For $i > 1$ choose a simple path $\tilde{P}_i$ in $\cup_{j<i} P_j$ that connects the 2 end vertices of P_i, and let $C_i = P_i \cup \tilde{P}_i$. $(C_1, ..., C_m)$ is the desired decomposition sequence of G.

Let C be the graph whose vertices are the C_i's and two of them are neighbors iff they have an edge in common. Thus, C is connected, and hence the 'Moreover' part of the Lemma is proved.

The other implication, that such a decomposition implies 2-connectivity, will not be used in the sequel, and its proof is omitted. □

For the induction step we need the following cone lemma. For v a vertex not in the support of a $(d-1)$-chain c, let $v * c$ denote the following d-chain: if $c = \sum\{a_T T : v \notin T \subseteq [n], |T| = d\}$ where $a_T \in k$ for all T, then $v * c = \sum\{sign(v,T) a_T T \cup \{v\} : v \notin T \subseteq [n], |T| = d\}$ where $sign(v,T) = (-1)^{|\{t \in T : t < v\}|}$.

Lemma 3.3 *Let s be a minimal $(d-1)$-cycle and let c be a minimal d-chain such that $\partial(c) = s$, i.e. c has no proper subchain c' such that $\partial(c') = s$. For v a vertex not in any face in $supp(c)$,the support of c, define $\tilde{s} = c - v * s$. Then $\tilde{s}$ is a minimal d-cycle.*

Proof: $\partial(\tilde{s}) = \partial(c) - \partial(v * s) = s - (s - v * \partial(s)) = 0$ hence $\tilde{s}$ is a d-cycle. To show that it is minimal, let $\hat{s}$ be a subchain of $\tilde{s}$ such that $\partial(\hat{s}) = 0$. Note that $supp(c) \cap supp(v * s) = \emptyset$.
Case 1: v is contained in a face in $supp(\hat{s})$. By the minimality of s, $supp(v * s) \subseteq supp(\hat{s})$. Thus, by the minimality of c also $supp(c) \subseteq supp(\hat{s})$ and

hence $\hat{s} = \tilde{s}$.
Case 2: v is not contained in any face in $supp(\hat{s})$. Thus, $supp(\hat{s}) \subseteq supp(c)$. As $\partial(\hat{s}) = 0$ then $\partial(c - \hat{s}) = s$. The minimality of c implies $\hat{s} = 0$. $\square$

Theorem 3.4 *Let K be a d-dimensional 2-CM simplicial complex over a field k ($d \geq 1$). Then there exists a decomposition $K = \cup_{i=1}^{m} S_i$ such that each S_i is a minimal d-cycle complex over k and for every $i > 1$, $S_i \cap (\cup_{j<i} S_j)$ contains a d-face.*

Moreover, for each $i_0 \in [m]$ the S_i's can be reordered by a permutation $\sigma : [m] \to [m]$ such that $\sigma^{-1}(1) = i_0$ and for every $i > 1$, $S_{\sigma^{-1}(i)} \cap (\cup_{j<i} S_{\sigma^{-1}(j)})$ contains a d-face.

proof: The proof is by induction on d. For $d = 1$, by Lemma 3.2 $K = \cup_{i=1}^{m(K)} C_i$ such that each C_i is a simple cycle and for every $i > 1$ $C_i \cap (\cup_{j<i} C_j)$ contains an edge. Define $s_i = \sum\{sign_e(i)e : e \in (C_i)_1\}$, then s_i is a minimal 1-cycle (orient the edges properly: $sign_e(i)$ equals 1 or -1 accordingly) whose support spans the simplicial complex C_i. Moreover, by Lemma 3.2 each C_{i_0}, $i_0 \in [m(K)]$, can be chosen to be the first in such a decomposition sequence.

For $d > 1$, note that the link of every vertex in a 2-CM simplicial complex is 2-CM. For a vertex $v \in K$, as $lk_K(v)$ is 2-CM then by the induction hypothesis $lk_K(v) = \cup_{i=1}^{m(v)} C_i$ such that each C_i is a minimal $(d-1)$-cycle complex and for every $i > 1$ $C_i \cap (\cup_{j<i} C_j)$ contains a $(d-1)$-face. Let s_i be a minimal $(d-1)$-cycle whose support spans C_i. As $K - v$ is CM of dimension d, $\tilde{H}_{d-1}(K - v; k) = 0$. Hence there exists a d-chain c such that $\partial(c) = s_i$ and $supp(c) \subseteq K - v$.

Take c_i to be such a chain with a support of minimal cardinality. By Lemma 3.3, $\tilde{s_i} = c_i - v * s_i$ is a minimal d-cycle. Let $S_i(v)$ by the simplicial complex spanned by $supp(\tilde{s_i})$; it is a minimal d-cycle complex. By the induction hypothesis, for every $i > 1$ $S_i(v) \cap (\cup_{j<i} S_j(v))$ contains a d-face (containing v). Thus, $K(v) := \cup_{j=1}^{m(v)} S_j(v)$ has the desired decomposition for every $v \in K$. $K = \cup_{v \in Ver(K)} K(v)$ as $st_K(v) \subseteq K(v)$ for every v, where $st_K(v) = \{T \in K : T \cup \{v\} \in K\}$.

Let v be any vertex of K. Since the 1-skeleton of K is connected, we can order the vertices of K such that $v_1 = v$ and for every $i > 1$ v_i is a neighbor of some v_j where $1 \leq j < i$. Let $v_{l(i)}$ be such a neighbor of v_i. By the induction hypothesis we can order the $S_j(v_i)$'s such that $S_1(v_i)$ will contain $v_{l(i)}$, and hence, as K is pure, will contain a d-face which appears in $K(v_{l(i)})$ (this face contains the edge $\{v_i, v_{l(i)}\}$). The resulting decomposition sequence $(S_1(v_1), .., S_{m(v_1)}(v_1), S_1(v_2), .., S_{m(v_n)}(v_n))$ is as desired.

Moreover, every $S_j(v_{i_0})$ where $i_0 \in [n]$ and $j \in [m(v_{i_0})]$ can be chosen to be the first in such a decomposition sequence. Indeed, by the induction hypothesis $S_j(v_{i_0})$ can be the first in the decomposition sequence of $K(v_{i_0})$, and as mentioned before, the connectivity of the 1-skeleton of K guarantees that each such prefix $(S_1(v_{i_0}),..,S_{m(v_{i_0})}(v_{i_0}))$ can be completed to a decomposition sequence of K on the same $S_j(v_i)$'s. $\square$

Theorem 1.3 follows also from the following corollary combined with Theorem 2.2.

Corollary 3.5 *Let K be a d-dimensional 2-CM simplicial complex over a field k ($d \geq 1$). Then K is a minimal cycle complex over the Abelian group $\tilde{k} = k(x_1, x_2, ...)$ whose elements are finite linear combinations of the (variables) x_i's with coefficients in k.*

Proof: Consider a decomposition $K = \cup_{i=1}^m S_i$ as guaranteed by Theorem 3.4, where $S_i = \overline{supp(c_i)}$ (the closure w.r.t. inclusion of $supp(c_i)$) for some minimal d-cycle c_i over k. Define $\tilde{c_i} = x_i c_i$, thus $\tilde{c_i}$ is a minimal cycle over $\tilde{k}$. Define $\tilde{c} = \sum_{i=1}^m \tilde{c_i}$. Clearly $\tilde{c}$ is a cycle over $\tilde{k}$ whose support spans K. It remains to show that $\tilde{c}$ is minimal. Let $\tilde{c}'$ be a subchain of $\tilde{c}$ which is a cycle, $\tilde{c}' \neq \tilde{c}$. We need to show that $\tilde{c}' = 0$. Denote by $\tilde{\alpha_T}$ ($\tilde{\alpha_T}'$) the coefficient of the set T in $\tilde{c}$ ($\tilde{c}'$) and by $\tilde{\alpha_T}(i)$ the coefficient of the set T in $\tilde{c_i}$. If $\tilde{\alpha_T}' = 0$ then for every i such that $\tilde{\alpha_T}(i) \neq 0$, the minimality of $\tilde{c_i}$ implies that $\tilde{\alpha_F}' = 0$ whenever $\tilde{\alpha_F}(i) \neq 0$. By assumption, there exists a set T_0 such that $\tilde{\alpha_{T_0}}' = 0 \neq \tilde{\alpha_{T_0}}$. In particular, there exists an index i_0 such that $\tilde{\alpha_{T_0}}(i_0) \neq 0$, hence $\tilde{\alpha_F}' = 0$ whenever $\tilde{\alpha_F}(i_0) \neq 0$. As $S_{i_0} \cap (\cup_{j<i_0} S_j)$ contains a d-face in case $i_0 > 1$, repeated application of the above argument implies $\tilde{\alpha_F}' = 0$ whenever $\tilde{\alpha_F}(1) \neq 0$. Repeated application of the fact that $S_i \cap (\cup_{j<i} S_j)$ contains a d-face for $i = 2, 3, ..$ and of the above argument shows that $\tilde{\alpha_F}' = 0$ whenever $\tilde{\alpha_F}(i) \neq 0$ for some $1 \leq i \leq m$, i.e. $\tilde{c}' = 0$. $\square$

A pure simplicial complex has a *nowhere zero flow* if there is an assignment of integer non-zero wights to all of its facets which forms a $\mathbb{Z}$-cycle. This generalizes the definition of a nowhere zero flow for graphs (e.g. [12] for a survey).

Corollary 3.6 *Let K be a d-dimensional 2-CM simplicial complex over $\mathbb{Q}$ ($d \geq 1$). Then K has a nowhere zero flow.*

Proof: Consider a decomposition $K = \cup_{i=1}^m S_i$ as guaranteed by Theorem 3.4. Multiplying by a common denominator, we may assume that each $S_i = \overline{supp(c_i)}$ for some minimal d-cycle c_i over $\mathbb{Z}$ (instead of just over $\mathbb{Q}$). Let N be the maximal $|\alpha|$ over all nonzero coefficients α of the c_i's, $1 \leq i \leq m$. Let $\tilde{c} = \sum_{i=1}^m (N^m)^i c_i$. $\tilde{c}$ is a nowhere zero flow for K; we omit the details. $\square$

Problem 3.7 *Can the S_i's in Theorem 3.4 be taken to be homology spheres?*

Yhonatan Iron and I proved (unpublished) the following lemma:

Lemma 3.8 *Let K, L and $K \cap L$ be simplicial complexes of the same dimension $d-1$. Assume that K and L are weak-Lefschetz, i.e. that multiplication by a generic degree-one element g in $H = H(K), H(L)$, $g : H_{i-1} \longrightarrow H_i$, is injective for all $i \leq \lfloor d/2 \rfloor$. If $K \cap L$ is CM then $K \cup L$ is weak-Lefschetz.*

In view of this lemma, if the intersections $S_i \cap (\cup_{j<i} S_j)$ in Theorem 3.4 can be taken to be CM, and the S_i's can be taken to be homology spheres, then Conjecture 1.1 would be reduced to the long standing g-conjecture for homology spheres. Can the intersections be guaranteed to be CM?

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