NONPOLYTOPAL NONSIMPLICIAL LATTICE SPHERES WITH NONNEGATIVE TORIC g-VECTOR

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Abstract. We construct many nonpolytopal nonsimplicial Gorenstein* meet semi-lattices with nonnegative toric g-vector, supporting a conjecture of Stanley. These are formed as Bier spheres over the face posets of multiplexes, polytopes constructed by Bisztriczky as generalizations of simplices.

1. Introduction

A poset P with a minimum $\hat{0}$ is called *Gorenstein** if the order complex of $P - \hat{0}$ is Gorenstein* (that is, if we add a maximum to P, then $\hat{P} = P \cup \{\hat{1}\}$ is both Eulerian and Cohen-Macaulay). For P a Gorenstein* poset, denote its toric g-vector by $g(P)$. Our starting point is the following conjecture of Stanley.

Conjecture 1.1. ([18], Conjecture 4.2.(c,d)) Let P be a Gorenstein* meet semi-lattice. Then $g(P)$ is an M-sequence. In particular, $g(P)$ is nonnegative.

Conjecture 1.1 is known to hold in the following cases: face posets of the boundary complexes of simplicial polytopes [17], strongly-edge-decomposible spheres [2, 16] (these include Kalai's squeezed spheres [12, 16]), Bier spheres [7] - all these cases are simplicial spheres, and the nonnegativity of $g(P)$ is known for the boundary poset of arbitrary polytopes [14].

In this work we construct many nonpolytopal nonsimplicial Gorenstein* meet semi-lattices with nonnegative toric g-vector, supporting Conjecture 1.1. Specifically,

Theorem 1.2. *Let $n \geq d \geq 4$ be integers and M be the boundary poset of the d-dimensional multiplex with n vertices. Let I be an ideal in M which contains all the elements in M of rank at most $\lceil \frac{d}{2} \rceil + 1$. Let*

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P be the Bier poset $P = \mathrm{Bier}(M, I)$. Then P is a Gorenstein meet semi-lattice and $g(P)$ is nonnegative.*

In Section 2 we give the needed background on g-vectors, Bier posets and multiplexes; in Section 3 we prove Theorem 1.2 by analyzing the effect on the toric g-vector of a local move on Gorenstien* posets; in Section 4 we remark on nonpolytopality and nonsimpliciality of $\mathrm{Bier}(M, I)$.

2. Preliminaries

The g-vector. We follow Stanley [18]. Let P be a Gorenstein* poset of rank r, or more generally, a lower Eulerian poset (i.e., a graded poset such that all of its intervals are Eulerian posets, that is, for each interval, the number of elements of even degree equals the number of elements of odd degree), and define recursively the following polynomials in x, $g(P, x)$, $h(P, x)$ and $f(P, x)$: $h(P, x) = x^r f(P, \frac{1}{x})$, $f(\emptyset, x) = g(\emptyset, x) = 1$, for $f(P, x) = h_0 + h_1x + h_2x^2 + ...$ let

$$g(P, x) = h_0 + (h_1 - h_0)x + ... + (h_{\lfloor \frac{r}{2} \rfloor} - h_{\lfloor \frac{r}{2} \rfloor - 1})x^{\lfloor \frac{r}{2} \rfloor},$$

and define

$$f(P, x) = \sum_{t \in P} g([\hat{0}, t), x)(x - 1)^{r - r(t)}$$

where $r(t)$ is the rank of t and for $x, y \in P$

$$[x, y) := \{z \in P : \ x \le z < y\}.$$

When adding a maximal element to P results in an Eulerian poset, we have $f(P, x) = h(P, x)$. If P is the face poset of an Eulerian simplicial complex Δ then $g(P)$ coincides with the simplicial g-vector $g(\Delta)$.

For P the face poset of the boundary of a polytope P', $h(P)$ encodes the dimensions of the combinatorial intersection homology associated with P' [3, 4, 10], and a hard-Lefschetz type theorem for this module shows the nonnegativity of $g(P)$ [14].

For a poset P denote by $\widehat{P}$ the poset obtained by adding to it a maximum $\hat{1}$. If P has a maximum $\hat{1}$, let $\partial P := P - \{\hat{1}\}$.

For posets P and Q let $P * Q$ be their product poset, i.e. $(x, y) \le (x', y')$ in $P * Q$ iff $x \le x'$ in P and $y \le y'$ in Q. We need the following facts, stated by Kalai [13] for polytopes.

Observation 2.1. Let P_1 and P_2 be Eulerian posets. Then
(a) $g(\partial(P_1 * P_2), x) = g(\partial P_1, x) \cdot g(\partial P_2, x)$, and
(b) $h(\partial P_1 * \partial P_2, x) = h(\partial P_1, x) \cdot h(\partial P_2, x)$.

For completeness, we include their proofs.

Proof. We prove part (a) by induction on $r' := r(\partial(P_1 * P_2))$. The case $r' \leq 1$ is trivial. For part (b) we prove move generally for lower Eulerian posets Q_1, Q_2 that $f(Q_1 * Q_2, x) = f(Q_1, x) \cdot f(Q_2, x)$, by induction on $r'' := r(Q_1 * Q_2)$. Call this assertion (b$''$). The case $r'' \leq 1$ is trivial.

First we prove (b$''$) for r'' assuming (a) is true for all $r' < r''$:

$$\begin{aligned} f(Q_1 * Q_2, x) = & \\ & \sum_{F_1 \in Q_1, F_2 \in Q_2} g(\partial([\hat{0}, F_1] * [\hat{0}, F_2]))(x-1)^{r(Q_1)+r(Q_2)-r(F_1)-r(F_2)} \\ & = (\sum_{F_1 \in Q_1} g([\hat{0}, F_1])(x-1)^{r(Q_1)-r(F_1)})(\sum_{F_2 \in Q_2} g([\hat{0}, F_2])(x-1)^{r(Q_2)-r(F_2)}) \\ & \qquad = f(Q_1, x) f(Q_2, x), \end{aligned}$$

where the second equality follows by induction from part (a). Thus, (b$''$) follows, hence also (b).

Next we prove (a) for $r' = l$ assuming (a) is true for $r' < l$. Each face of $\partial(P_1 * P_2)$ satisfy exactly one of the following 3 possibilities - it contains either P_1, or P_2, or neither. Thus, using the recursive definition of $f(P, x)$ and the induction hypothesis we get

$$\begin{aligned} f(\partial(P_1 * P_2), x) = g(\partial P_1, x) f(\partial P_2, x) + g(\partial P_2, x) f(\partial P_1, x) \\ + (x-1) f(\partial P_1, x) f(\partial P_2, x). \end{aligned}$$

Let T_r be the operator that truncates a polynomial at degree r. Thus, if P is a Gorestein* poset of rank r, $g(P, x) = T_{\lfloor \frac{r}{2} \rfloor}((1-x) f(P, x))$. Clearly, the polynomial $\bar{g}(P, x) := (1-x) f(P, x) - g(P, x)$ satisfies that all its nonzero terms have degree larger than $\lfloor \frac{r}{2} \rfloor$. Thus,

$$\begin{aligned} (1-x) f(\partial(P_1 * P_2), x) &= g(\partial P_1, x)[g(\partial P_2, x) + \bar{g}(\partial P_2, x)] \\ &\quad + g(\partial P_2, x)[g(\partial P_1, x) + \bar{g}(\partial P_1, x)] \\ &\quad - [g(\partial P_1, x) + \bar{g}(\partial P_1, x)][g(\partial P_2, x) + \bar{g}(\partial P_2, x)] \\ &= g(\partial P_1, x) g(\partial P_2, x) - \bar{g}(\partial P_1, x) \bar{g}(\partial P_2, x), \end{aligned}$$

and we conclude that

$$\begin{aligned} g(P, x) &= T_{\lfloor \frac{l}{2} \rfloor}(g(\partial P_1, x) g(\partial P_2, x) - \bar{g}(\partial P_1, x) \bar{g}(\partial P_2, x)) \\ &= g(\partial P_1, x) g(\partial P_2, x). \end{aligned}$$

□

Bier posets. Bier posets were introduced in [7], generalizing the construction of Bier spheres, see e.g. [15]. Here we slightly modify the definition in order to simplify notation. Let P be a finite poset with a minimum $\hat{0}$ and add to it a new elements $\hat{1}$ greater than all elements of P to obtain the poset $\hat{P} = P \cup \{\hat{1}\}$. Let I be a proper ideal in $\hat{P}$. Then the poset $\mathrm{Bier}(P, I)$ consists of all intervals $[x, y]$ where $x \in I, y \in \hat{P} - I$, ordered by reverse inclusion. If $\hat{P}$ is Eulerian then $\mathrm{Bier}(P, I)$ is Eulerian, and the order complexes of the posets $P - \{\hat{0}\}$ and $\mathrm{Bier}(P, I) - \{[\hat{0}, \hat{1}]\}$ are homeomorphic [7]. In particular,

Theorem 2.2. *([7]) Let I be an ideal in a poset P. If P is Gorenstein* then* $\mathrm{Bier}(P, I)$ *is Gorenstein*.*

In the case where P is the face poset of the boundary of a simplex the posets $\mathrm{Bier}(P, I)$ are exactly the face posets of the Bier spheres, and their g-vector is shown to be a Kruskal-Katona vector [7], and in particular an M-sequence, thus nonnegative. Note that the number of vertices in a d-dimensional Bier sphere is at most $2(d + 2)$. Note that for $I = P$, $\mathrm{Bier}(P, I)$ is isomorphic to P.

Multiplex. Multiplexes were introduced by Bisztriczky [6] as a generalization of simplices. For any integers $n \geq d \geq 2$ there exists a d-dimensional polytope on $n + 1$ vertices $x_0, x_1, ..., x_n$ with facets

$$F_i := \mathrm{conv}(\mathrm{x_{i-d+1}}, \mathrm{x_{i-d+2}}, ..., \mathrm{x_{i-1}}, \mathrm{x_{i+1}}, \mathrm{x_{i+2}}, ..., \mathrm{x_{i+d-1}})$$

for $0 \leq i \leq n$, with the convention that $x_j = x_0$ if $j < 0$ and $x_j = x_n$ if $j > n$. Such polytope was constructed in [6], is called a *multiplex*, and we denote it by $M^{d,n}$. Note that $M^{d,d}$ is a simplex. We need the following known properties of multiplexes, shown by Bisztriczky [6] and Bayer et. al. [5].

Theorem 2.3. *(a) Every multiplex is self dual. ([6])*

(b) If F is a face in a multiplex M then both F and the quotient M/F are multiplexes. ([6])

(c) $M^{d,n}$ has the same flag f-vector as the $(d-2)$-fold pyramid over the $(n - d + 3)$-gon. Thus, for P the boundary poset of $M^{d,n}$ we get $g(P, x) = 1 + (n - d)x$ and $h(P, x) = 1 + (n - d + 1)x + ... + (n - d + 1)x^{d-1} + x^d$. ([5])

3. Computing $g(\mathrm{Bier}(P, I))$

We start by analyzing the effect of a certain local move on a pair (P, I), of a Gorenstein* poset and an ideal in it, on the toric g-vector of $\mathrm{Bier}(P, I)$. This local move can be thought of as a generalization of bistellar moves on simplicial spheres (however, this connection is not necessary for the topic of this paper and will not be stressed here).

Denote by X^* the opposite of a poset X, that is its elements are $\{x^* : x \in X\}$ and $x < y$ in X iff $y^* < x^*$ in X^*.

Lemma 3.1. *Let P be a Gorenstein* poset, let Q be an ideal in P, different from $\{\hat{0}\}$, and let t be a maximal element in Q. Then:*

$$h(\mathrm{Bier}(P, Q-\{t\}), x) - h(\mathrm{Bier}(P,Q), x) = h([\hat{0}, t), x) g((t, \hat{1}]^*, x) - g([\hat{0}, t), x) h((t, \hat{1}]^*, x),$$

where $\hat{1}$ is the maximum of $\hat{P}$.

Proof. By Theorem 2.2, the h-vectors on the left are of Gorenstein* posets, hence are symmetric, thus the left hand side equals

$$f(\mathrm{Bier}(P, Q-\{t\}), x) - f(\mathrm{Bier}(P,Q), x),$$

which, by definition, equals

$$\sum_{z: t>z\in P} g([[\hat{0},\hat{1}],[z,t]),x)(x-1)^{r(t)-r(z)} - \sum_{y: t<y\in P} g([[\hat{0},\hat{1}],[t,y]),x)(x-1)^{r(y)-r(t)}.$$

The last expression equals

$$\sum_{z\in[\hat{0},t)} g(\partial([\hat{0},z]*[t,\hat{1}]^*),x)(x-1)^{r(t)-r(z)} - \sum_{y\in(t,\hat{1}]} g(\partial([\hat{0},t]*[y,\hat{1}]^*),x)(x-1)^{r(y)-r(t)},$$

and by Observation 2.1 it equals

$$\sum_{z\in[\hat{0},t)} g([\hat{0},z),x)g((t,\hat{1}]^*,x)(x-1)^{r(t)-r(z)} - \sum_{y\in(t,\hat{1}]} g([\hat{0},t),x)g((y,\hat{1}]^*,x)(x-1)^{r(y)-r(t)}$$
$$= g((t,\hat{1}]^*,x)f([\hat{0},t),x) - g([\hat{0},t),x)f((t,\hat{1}]^*,x).$$

As the posets on the right hand side are Gorenstein*, the assertion follows. □

We now specialize to the case of a multiplex. Let M be the boundary poset of the multiplex $M^{d,n}$, $I \neq \{\hat{0}\}$ an ideal in M, t a maximal element in I, and denote $\Delta h := h(\mathrm{Bier}(M, I-\{t\}), x) - h(\mathrm{Bier}(M,I), x)$ and

$$\Delta g := g(\mathrm{Bier}(M, I-\{t\}), x) - g(\mathrm{Bier}(M,I), x).$$

Lemma 3.2. *With notation as above, if $d > r(t) > \lceil \frac{d}{2} \rceil + 1$ then*

$$\Delta g = g_1((t,\hat{1}]^*)x^{d-r(t)} + (g_1([\hat{0},t))g_1((t,\hat{1}]^*)+1)x^{d-r(t)+1} + g_1([\hat{0},t))x^{d-r(t)+2},$$

and hence is nonnegative, and if $r(t) = d > \lceil \frac{d}{2} \rceil$ *(thus* t *is a facet) then* $\Delta g = x + g_1([\hat{0}, t))x^2$ *and hence is nonnegative.*

Proof. Denote $A = (t, \hat{1}]^*$ and $B = [\hat{0}, t)$. They are the boundary posets of multiplexes by Theorem 2.3(a,b). By Lemma 3.1 and Theorem 2.3(c) we get for $r(t) < d$:

$$\begin{aligned}\Delta h =& (1 + h_1(B)x + ... + h_1(B)x^{r(t)-2} + x^{r(t)-1})(1 + g_1(A)x) \\ & - (1 + g_1(B)x)(1 + h_1(A)x + ... + h_1(A)x^{d-r(t)-1} + x^{d-r(t)}).\end{aligned}$$

The coefficient of x^j in Δh, denoted by $[x^j]$, equals $h_{j-1}(B)g_1(A) + h_j(B) - h_j(A) - g_1(B)h_{j-1}(A)$, with the convention $h_{-1} = 0$. Thus, assuming $r(t) < d$, for $j \leq \lfloor \frac{d}{2} \rfloor$ we get: $[x^j] = 0$ if $0 \leq j < d - r(t)$, $[x^{d-r(t)}] = g_1(A)$, $[x^{d-r(t)+1}] = 1 + g_1(A)h_1(B)$, and $[x^j] = h_1(A)h_1(B)$ for $d - r(t) + 1 < j \leq \lfloor \frac{d}{2} \rfloor$. Thus, Δg is as claimed for $r(t) < d$, and in particular is nonnegative by Theorem 2.3(c).

In case $r(t) = d$ we get that $\Delta h = x + h_1(B)x^2 + ... + h_1(B)x^{d-2} + x^{d-1}$ and $\Delta g = x + g_1(B)x^2$ is nonnegative. □

The conditions imposed on $r(t)$ in Lemma 3.2 imply that $d \geq 4$, which is the interesting range for Conjecture 1.1.

Proof of Theorem 1.2. By repeated application of Lemma 3.2 we conclude the nonnegativity of $g(P)$ in Theorem 1.2. By Theorem 2.2, P is Gorenstein*, and clearly P is a meet semi-lattice: let $\wedge_M, \wedge_P$ be the meet operations in M, P respectively, and $\vee_M$ the join operation in M. Then $[a, b], [c, d] \in P$ satisfy $[a, b] \wedge_P [c, d] = [a \wedge_M c, b \vee_M d]$. □

In fact, Lemma 3.2 can be simplified, as we will show below. The inequality $g_2(M) \geq g_1([\hat{0}, t))g_1((t, \hat{1}]^*)$ (see [9] for the general statement for polytopes, and e.g. [8] for the validity for arbitrary polytopes) together with $g_2(M) = 0$ shows that at least one of $g_1([\hat{0}, t)), g_1((t, \hat{1}]^*)$ is zero. Here is a simple combinatorial criterion to tell which. Bisztriczky [6, Lemma 3] described the 4-gons among the 2-faces of a multiplex. In particular:

Observation 3.3. For a multiplex M, $g_1(M)$ is the number of 4-gons among the 2-dimensional faces of M.

Thus, combining with Lemma 3.2, we obtain the following.

Proposition 3.4. *Let M and I be as in Theorem 1.2. Denote by $\Box^*$ the set of faces in M whose duals are 4-gons. Then,*

$$\begin{aligned} g(\mathrm{Bier}(M,I),x) = & \sum_{t\in\hat{M}-I} x^{r(\hat{M})-r(t)} + \ g_1(M)x \\ & + \sum_{t\in M-I} g_1([\hat{0},t))x^{r(\hat{M})-r(t)+1} \\ & + \sum_{t\in M-I}\sum_{s\in\Box^*} \delta_{t\subseteq s}x^{r(\hat{M})-r(t)-1} \end{aligned} \tag{1}$$

□

In the case where M is the boundary of a simplex, $\Box^* = \emptyset$, and by Proposition 3.4 one can show that $g(\mathrm{Bier}(M,I)$ is the f-vector of a simplicial complex as follows: by self-duality of polytopes $(\hat{M}-I)^*$ is the face poset of a polyhedral complex Γ, whose f-vector corresponds to the sum in the first line of (1). Denote by t^* the face in Γ corresponding to $t \in \hat{M}-I$. Take the disjoint union of Γ with a totally ordered set V of $g_1(M)$ vertices, and for any face $t^* \in \Gamma$ take its cone over each of the first $g_1([\hat{0},t])$ vertices in V. As $t \subseteq s$ implies $g_1([\hat{0},t]) \leq g_1([\hat{0},s])$, this results in a polyhedral complex, denoted by Δ, whose f-polynomial is given by the right hand side of (1). (In this case the sum in the third row of (1) is zero.)

Any polyhedral complex is a meet semi-lattice with the *diamond property*, namely, any interval of rank 2 has 4 elements. Thus, by a result of Wegner [19], there is a simplicial complex with the same f-vector as Δ, proving that $g(\mathrm{Bier}(M,I),x)$ is a Kruskal-Katona vector in this case.

For a different proof see [7]. Whether all $g(\mathrm{Bier}(M,I))$ are M-sequences is still open. However, for any M,I,P as in Theorem 1.2, $(1,g_1(P),g_2(P))$ is a Kruskal-Katona vector: indeed, to the 1-skeleton of Δ constructed above one can add a diagonal at each 4-gon face t^* where $t \in (M-I)\cap\Box^*$, and by (1) the resulted graph G has f-vector $(1,g_1(P),g_2(P))$.

4. Concluding remarks

We keep the notation of Theorem 1.2.

Note that $M^{d,n}$ has non-simplicial 2-faces, they are of the form $F = \mathrm{conv}(\mathrm{x_i},\mathrm{x_{i+1}},\mathrm{x_{i+d}},\mathrm{x_{i+d+1}})$ [6], and that I contains all degree 3 elements in M. Thus, $\mathrm{Bier}(M,I)$ is non-simplicial - for example $[F,\hat{1}]$ is a 2-face.

We now argue that $\mathrm{Bier}(M,I)$ is not polytopal for many choices of I. M has $\binom{d+1}{i+1}+(n-d)\binom{d-1}{i}$ i-faces (this was computed already in [6], and follows also from Theorem 2.3(c)). Let F be a facet of M. Then F is $(d-1)$-dimensional with at most $2(d-1)$ vertices. A simple computation shows that the number of ideals I_F in $[\hat{0}, F)$ which contain all the elements in $[\hat{0}, F)$ of rank at most $\lceil \frac{d}{2} \rceil + 1$ is at least $2^{\binom{d}{\lceil \frac{d}{2} \rceil + 2}}$, thus the number of non-isomorphic such $\mathrm{Bier}(F, I_F)$ is at least $\frac{2^{\binom{d}{\lceil \frac{d}{2} \rceil + 2}}}{(4(d-1))!} = \Omega(2^{(1-\epsilon_d)2^d/\sqrt{d}})$ where $\epsilon_d \to 0$ as $d \to \infty$ (see e.g. [7]). It is known that the number of non-isomorphic polytopes with at most $4(d-1)$ vertices is less than $2^{64d^3+O(d^2)}$ [11, 1], hence most of the $\mathrm{Bier}(F, I_F)$ as above are nonpolytopal for $d >> 1$. Fix an ideal I_F such that $\mathrm{Bier}(F, I_F)$ is nonpolytopal. For an ideal I in P such that $I \cap [\hat{0}, F) = I_F$ note that $\mathrm{Bier}(F, I_F)$ is an interval in $\mathrm{Bier}(M, I)$, hence $\mathrm{Bier}(M, I)$ is nonpolytopal.

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