Regularity of edge ideals of C_4-free graphs via the topology of the lcm-lattice

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Abstract

We study the topology of the lcm-lattice of edge ideals and derive upper bounds on the Castelnuovo-Mumford regularity of the ideals. In this context it is natural to restrict to the family of graphs with no induced 4-cycle in their complement. Using the above method we obtain sharp upper bounds on the regularity when the complement is a chordal graph, or a cycle, or when the original graph is claw free with no induced 4-cycle in its complement. For the last family we show that the second power of the edge ideal has a linear resolution.

1 Preliminaries

Fix a field k and let I be a monomial ideal in the polynomial ring $S = k[x_1, ..., x_n]$ with a minimal set of generators $G(I) = \{m_1, ..., m_d\}$. Let $L(I)$ be the lcm-lattice of I, i.e. the poset whose elements are labeled by the least common multiples of subsets of monomials in $G(I)$ ordered by divisibility. Indeed $L(I)$ is a lattice, its minimum is 1 (corresponding to the empty set) and its maximum is $m_I = \text{lcm}(m : m \in G(I))$.

The minimal free resolution of I over S is $\mathbb{N}^n$-graded. Denote the corresponding multi-graded Betti numbers by $\beta_{i,m}$ for a monomial m and $i \geq 0$ an integer. The main result in [11] shows how to compute the Betti numbers from the reduced homology of the order complex of open intervals in $L(I)$.

Theorem 1.1. *[11, Theorem 2.1] Let $\Delta((1,m))$ denote the order complex of the open interval $(1,m)$ in $L(I)$. For every $i \geq 0$ and $m \in L(I)$ one has*

$$\beta_{i,m} = \dim_k \tilde{H}_{i-1}(\Delta((1,m)); k).$$

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If $m \notin L(I)$ then $\beta_{i,m} = 0$ for every i.

(In [11] S/I, rather than I, was resolved, hence the shift in the index.) It follows that the Castelnuovo-Mumford regularity of I is

$$\operatorname{reg}(I) = \sup_{i \geq 0}(\max\{j : \exists m \in L(I),\ \deg(m) = i+j,\ \tilde{H}_{i-1}(\Delta((1,m));k) \neq 0\}). \tag{1}$$

Further work on $L(I)$ appeared in [16]. For unexplained terminology on posets, simplicial complexes and topology we refer to Björner [1].

For a graph $G = (V, E)$ let $I(G)$ be its *edge ideal*, namely $I(G) = (x_i x_j : \{i, j\} \in E(G))$. This is the case where $G(I)$ consists of squarefree monomials of degree 2. Denote $m_G = m_{I(G)}$ in this case. In this paper we consider edge ideals. These have recieved much attention in recent years, from both algebraists and combinatorialists. For example, in the recent papers [6, 8, 19] algebraic properties of certain edge ideals are derived from the topology of the clique complex of the complementary graph. We study the topology of the lcm-lattice of the edge ideal (of the original graph) and its powers, which in turn implies upper bounds on their regularity.

Let G^c be the complement of G, namely $G^c = (V, \binom{V}{2} - E)$. When considering $L(I(G))$ it is natural to assume that G^c has no induced 4-cycles, as is explained in Section 2, so we restrict our attention to this class of graphs and some subclasses of it.

In Section 3 we consider chordal graphs and in Section 4 we consider cycles. From our results on the lcm-lattice of their complement we derive a new proof of Fröberg's theorem, that $I(G)$ has a linear resolution iff G^c is chordal. Moreover, the main result in [8] also easily follows. Further, the relation between the homology of the lcm-lattice and the homology of the clique complex of the complementary graph is explained.

In [9] Francisco, Hà and Van Tuyl suspected that if G^c has no induced 4-cycles then for any $k \geq 2$, $I(G)^k$ has a linear resolution. While this is not true in general (see [15] for examples), it may be true for the subfamily where in addition G is *claw free*, i.e. has no induced bipartite subgraph with one vertex on one side and 3 vertices on the other. Note that this family contains all graphs G such that G^c has no induced $3-$ nor $4-$ cycles.

Theorem 1.2. *Let G be claw free such that G^c has no induced 4-cycle. Then:*

(1) $I(G)^2$ has a linear resolution.

(2) If G^c is not chordal then $\operatorname{reg}(I(G)) = 3$.

This seems to be the first infinite family of graphs G with the property that although each edge ideal $I(G)$ does not have a linear resolution, a higher power ($I(G)^2$ in this case) does. Theorem 1.2 is proved in Section 5.

2 Why not C_4?

Let C_4 denote the 4-cycle. Recall that a poset P is *pure* if all its maximal chains have the same finite length. It is shown in [15, Theorem 2.2] that

Proposition 2.1. *If G^c has no induced C_4 then for any $k \geq 1$ the lcm-lattice $L(I(G)^k)$ is* pure*, and except for the minimum, the rank function is given by* $\mathrm{rank}(m) = \deg(m) - 2k + 1$.

This makes tools from graded poset topology applicable. In this situation any interval $[x, y]$ in $L(I(G)^k)$, where $x \neq 1$, is a semimodular lattice, and hence shellable [2, Theorem 3.1], a fact which we will use in the sequel to derive information on the regularity of $I(G)^k$. (A lattice is *semimodular* iff for any two elements in the lattice if one of them covers their meet then the other is covered by their join.)

If G^c has an induced C_4, equivalently if G has two disjoint edges as an induced subgraph, then $L(I(G)^k)$ is not graded by degree of monomials (up to a shift). Moreover,

Lemma 2.2. *If G is a connected graph and G^c has an induced C_4 then $L(I(G))$ is not pure.*

Proof. As G is connected there is a maximal chain in $[1, m_G]$ of length $|V(G)|$: look on a sequence of edges which form a spanning tree in G and such that every initial segment forms a connected graph. The joins corresponding to initial segments form a maximal chain of length $|V(G)|$.

As G has induced two disjoint edges $\{a, b\}, \{c, d\}$ there is a maximal chain in $[1, m_G]$ of length smaller than $|V(G)|$: look on a maximal chain $(1, x_a x_b, x_a x_b x_c x_d, \ldots)$. □

For the path of length $n \geq 5$, P_n, the lcm-lattice of P_n is therefore not pure. It can be shown that e.g. the conclusion of Theorem 5.1 on regularity fails for these graphs (which are claw free but contain an induced C_4 in the complement). Actually $\mathrm{reg}(I(P_n)) \to \infty$ as $n \to \infty$. See [13] for a detailed analysis.

3 Chordal graphs

A graph is *chordal* if it has no induced cycles of length > 3. In particular, chordal graphs have no induced C_4. Dirac characterization of chordal graphs [5] implies that if G^c is chordal then the vertices of G can be totally ordered such that if $i, j, k \in V(G)$, $k > i, j$ and $\{i, j\} \in E(G)$ then either $\{i, k\} \in E(G)$ or $\{j, k\} \in E(G)$. Such an order is called a *Dirac order* on $V(G)$.

A pure simplicial complex Δ is *constructible* if it is a simplex or empty, or inductively, if $\Delta = \Delta_1 \cup \Delta_2$ such that Δ_1, Δ_2 and $\Delta_1 \cap \Delta_2$ are constructible and $\dim(\Delta_1) = \dim(\Delta_2) = \dim(\Delta_1 \cap \Delta_2) + 1$. If Δ is constructible of dimension d then Δ is $(d-1)$-connected; in particular, a nonzero reduced homology $\tilde{H}_i(\Delta)$ may appear only in dimension $i = d$. We remark that shellable complexes are constructible, and the reverse implication is false (see [1] and the references therein).

Theorem 3.1. *If G^c is chordal then $\Delta((1, m_G))$ is constructible.*

Proof. If $|V(G)| \leq 3$ or $E(G) = \emptyset$ then the assertion is trivial. For $|V(G)| > 3$ and $E(G) \neq \emptyset$, let $v_1 < v_2 < ... < v_t$ be a Dirac order on $V(G)$. Note that the induced graph on a subset of the vertices of a chordal graph is chordal. By induction the assertion holds for the induced subgraphs $G_l = G[v_1, ..., v_l]$ for $2 \leq l \leq t-1$. Let $\Delta_l = \Delta_l(G)$ be the subcomplex of $\Delta((1, m_G))$ spanned by the maximal chains in $(1, m_G)$ whose bottom is an edge contained in $\{v_1, ..., v_l\}$. Then $\Delta((1, m_G)) = \Delta_t$. Let l_0 be the minimal i such that $\Delta_i \neq \emptyset$.

We now show that for any $l \geq l_0$ Δ_l is constructible. For $l = l_0$ $\Delta_{l_0} = \Delta((x_{v_{l_0}}, m_G))$ where $[x_{v_l}, m_G]$ is the restriction of $[1, m_G]$ to monomials divisible by x_{v_l}, and adding x_{v_l} as a minimum. Note that $[x_{v_{l_0}}, m_G]$ is a semimodular lattice, hence by [2] $\Delta((x_{v_{l_0}}, m_G))$ is shellable and in particular constructible. Note that $\dim(\Delta_{l_0}) = \deg(m_G) - 3$ by Proposition 2.1. For $l > l_0$,

$$\Delta_l = \Delta_{l-1} \cup \Delta((x_{v_l}, m_G)).$$

Again, as $[x_{v_l}, m_G]$ is a semimodular lattice then $\Delta((x_{v_l}, m_G))$ is constructible. Further, Δ_{l-1} is constructible by the induction hypothesis and $\dim(\Delta_{l-1}) = \dim(\Delta((x_{v_l}, m_G))) = \deg(m_G) - 3$. The intersection $\Delta_{l-1} \cap \Delta((x_{v_l}, m_G))$ is pure of dimension $\deg(m_G) - 4$, and its (nonempty collection of) facets are the maximal chains in $(1, m_G)$ with bottom $x_{v_i} x_{v_j} x_{v_l}$ where $\{v_i, v_j\} \in G$ and $i, j < l$. This follows from the definition of Dirac order. Moreover, $\Delta_{l-1} \cap \Delta((x_{v_l}, m_G))$ is combinatorially isomorphic, i.e. as an abstract simplicial complex, to $\Delta_{l-1}(G[V - \{v_l\}])$, which is constructible by the induction hypothesis. We conclude that Δ_t is constructible. $\square$

Corollary 3.2. *If G^c is chordal then $\Delta((1,m_G))$ is Cohen-Macaulay.*

Proof. Using Reisner theorem, e.g. [18, Corollary 4.2], we need to show that for any $F \in \Delta = \Delta((1,m_G))$ and $i < \deg(m_G) - 3 - |F|$, the link $\mathrm{lk}_\Delta F$ satisfies $\tilde{H}_i(\mathrm{lk}_\Delta F) = 0$. For $F = \emptyset$ this holds by Theorem 3.1. For $F = \{a_1 < ... < a_f\}$, $\mathrm{lk}_\Delta F = \Delta((1,a_1)) * \Delta((a_1,a_2)) * ... * \Delta((a_{f-1},a_f)) * \Delta((a_f,m_G))$ where $*$ denotes join. By Theorem 3.1 $\Delta((1,a_1))$ is constructible and by semimodularity $\Delta((a_i,a_{i+1}))$ and $\Delta((a_f,m_G))$ are shellable, hence in each of these pure complexes only the top dimensional homology group may not vanish. By Künneth formula only the top dimensional homology group of their join may not vanish. $\square$

Corollary 3.3. *[10] If G^c is chordal then $I(G)$ has a linear resolution.*

Proof. By Theorem 3.1 and equation (1), $\mathrm{reg}(I(G)) = 2$ hence $I(G)$ has a linear resolution. $\square$

The converse of Corollary 3.3, also proved by Fröberg, will follow from Theorem 4.1 in the next section.

4 Induced cycles

If H is an *induced* subgraph of G, then $[1,m_H]$ is an interval in $[1,m_G]$ and hence $\mathrm{reg}(I(H)) \leq \mathrm{reg}(I(G))$. If G^c is not chordal then it contains an induced cycle $H^c = C_n$, of length $n > 3$. If $H^c = C_4$ then $\Delta((1,m_H))$ is the zero dimensional sphere while $\deg(m_H) = 4$ hence $3 \leq \mathrm{reg}(I(G))$, thus $I(G)$ does not have a linear resolution. What happens if $n \geq 5$?:

Theorem 4.1. *Let $n \geq 3$ and $G^c = C_n$. Then $H_*(\Delta((1,m_G))) \cong H_*(\mathbb{S}^{n-4})$, where $\mathbb{S}^d$ is the d-dimensional sphere.*

Proof. For $n = 3$ the assertion is trivial. For $n \geq 4$ let Δ be the barycentric subdivision of the boundary of the simplex on n vertices. Thus, the vertices of Δ are labeled by the proper nonempty subsets of $[n]$ and its faces correspond to chains of subsets ordered by inclusion. Let Γ be the induced subcomplex of Δ with vertex set V consisting of all singletons, all consecutive pairs $\{i,i+1\}$ and all consecutive triples $\{i-1,i,i+1\}(\mod n)$ in $[n]$.

One easily checks that Γ deformation retracts on C_n (retract the triangles with vertex $\{i-1,i,i+1\}$ on the length 2 path $(i-1,i,i+1)$). As Γ is induced, $\Delta - \Gamma$ deformation retracts onto the induced subcomplex on the complementary set of vertices $\Delta[V(\Delta)-V(\Gamma)]$. As Δ is a $(n-2)$-sphere, it follows from Alexander duality [14, Chapter 8, §71] that for every i

$$\tilde{H}_i(\Delta[V(\Delta)-V(\Gamma)]) \cong \tilde{H}_i(\Delta-\Gamma) \cong \tilde{H}^{n-3-i}(\Gamma) \cong \tilde{H}^{n-3-i}(\mathbb{S}^1).$$

By the obvious bijection between subsets of $[n]$ and square free monomials with variables in $\{x_1,...,x_n\}$, we get a combinatorial isomorphism $\Delta((1,m_G)) \cong \Delta[V(\Delta)-V(\Gamma)]$, and hence $H_i(\Delta((1,m_G))) \cong H_i(\mathbb{S}^{n-4})$ for all i. □

Corollary 4.2. *[10] If G^c is not chordal then $I(G)$ does not have a linear resolution.*

Proof. By Theorem 4.1 and (1), there is some $n \geq 4$ such that $\mathrm{reg}(I(G)) \geq \mathrm{reg}(C_n) > 2$ hence $I(G)$ does not have a linear resolution. □

In the recent papers [19, 6, 8] Hochster's formula (Theorem 5.2), applied to the clique complex of G^c, was used to derive Fröberg's theorem, i.e. Corollaries 3.3 and 4.2. The main result in the latter reference, namely [8, Theorem 1.1], easily follows from Theorems 3.1 and 4.1.

We show now that as far as homology is concerned, Hochster's formula and the lcm method are equivalent, yielding yet another proof of Theorem 4.1. More precisely:

Proposition 4.3. *Let $E(G) \neq \emptyset$, $|V(G)| = n$ and denote by $\Delta(G^c)$ the clique complex on G^c. Then for any i the reduced homology and cohomology groups satisfy*

$$\tilde{H}^i(\Delta(G^c)) \cong \tilde{H}_{n-3-i}(\Delta((1,m_G))).$$

In particular, over any field k, $\tilde{H}_i(\Delta(G^c);k) \cong \tilde{H}_{n-3-i}(\Delta((1,m_G));k)$.

Proof. Let C be the set of minimal non faces of $\Delta(G^c)$. Then $C = E(G)$. Let Γ be the simplicial complex on the vertex set C with faces F such that $\cup_{u\in F} u \neq V(G)$. As $V(G^c) \notin \Delta(G^c)$, by [3, Theorem 2]

$$\tilde{H}_i(\Delta(G^c)) \cong \tilde{H}^{n-3-i}(\Gamma) \text{ and } \tilde{H}^i(\Delta(G^c)) \cong \tilde{H}_{n-3-i}(\Gamma) \qquad (2)$$

for all i, where $\tilde{H}^j$ denotes the j-th cohomology group (the second isomorphism is not stated explicitly in [3] but can be proved similarly to the first).

To show that Γ is homotopy equivalent to $\Delta((1,m_G))$ consider $\Gamma' := \Gamma - \{\emptyset\}$ as a poset where faces are ordered by inclusion, and the poset map

$$\pi : \Gamma' \longrightarrow (1,m_G),\ \pi(F) = \prod\{x_i :\ i \in \cup_{u\in F} u\}.$$

Note that π is onto. For $W \subsetneq V(G)$ such that $x_W := \prod_{i\in W} x_i \in (1, m_G)$ the fiber $\pi^{-1}(\{y : y \leq x_W\})$ has a unique maximal element $\{c \in C : \ c \subseteq W\}$, hence its order complex is contractible. By Quillen's fiber theorem [17, Proposition 1.6] the barycentric subdivision of Γ is homotopy equivalent to $\Delta((1, m_G))$, and hence so is Γ. Combining with (2) and the isomorphism between homology and cohomology when working with field coefficients, the result follows. $\square$

We now strengthen Theorem 4.1 by specifying the homotopy type of $\Delta((1, m_G))$.

Theorem 4.4. *Let $G^c = C_n$ for $n \geq 4$. Then $\Delta((1, m_G))$ is homotopy equivalent to $\mathbb{S}^{n-4}$.*

Proof. For $n = 4$, $\Delta((1, m_G))$ consists of two points, and for $n = 5$, $\Delta((1, m_G))$ is easily seen to deformation retract to the 10-cycle formed by the monomials of degree 2 and 3, and the assertion holds. Assume $n \geq 6$. By Theorem 4.1 and the Hurewicz theorem, the assertion would follow by showing that the fundamental group of $\Delta((1, m_G))$ is trivial (see [12, Theorem 4.32 and Example 4.34]). Assume by contradiction that γ is a nontrivial loop in $\Delta((1, m_G))$. Then γ is homotopic to a loop that consists of a sequence of edges in the 1-skeleton of $\Delta((1, m_G))$, so without loss of generality we can assume that $\gamma = (v_1, v_2, ..., v_s, v_1)$ where the $\{v_i, v_{i+1}\}$ and $\{v_s, v_1\}$ are edges in $\Delta((1, m_G))$, and s is the smallest possible for a nontrivial loop. In particular γ is simple, in other words $v_i \neq v_j$ for all $1 \leq i < j \leq s$.

By the minimality of s, for any $1 \leq i < j \leq s$, if v_i and v_j are comparable in $[1, m_G]$ then $j \in \{i-1, i+1\} \mod s$. Indeed, otherwise one of the shorter loops $(v_1, ...v_{i-1}, v_i, v_j, v_{j+1}, ..., v_s, v_1)$ and $(v_i, v_{i+1}, ..., v_j, v_i)$ would be nontrivial. Thus, working modulo s, for each v_i either both v_{i-1} and v_{i+1} are above v_i in $[1, m_G]$ or both of them are below v_i. In the former case, replacing v_i by an atom of $[1, m_G]$ which is covered by it yields a homotopic loop with same length s. Thus, we may assume that the vertices in γ alternate between atoms and non atoms of $[1, m_G)$.

If there are $1 \leq i < j < k \leq s$ such that v_i, v_j, v_k are atoms then one of the loops $(v_1, \cdots, v_i, v_i \vee v_k, v_k, \cdots, v_s, v_1)$ and $(v_i, v_{i+1}, \cdots, v_k, v_i \vee v_k, v_i)$ must be nontrivial (otherwise γ would be trivial), and similarly for the other two pairs of atoms from the above triple. If the length of γ, namely s, is > 6 then at least for one of the three pairs both of these loops are shorter than γ, contradicting the minimality of s. If $\gamma = (v_1, v_2, v_3, v_4, v_1)$ and the two atoms are v_1 and v_3 then $v_1 \vee v_3$ is comparable to all elements in γ. By coning the 4-cycle γ over $v_1 \vee v_3$, γ is trivial.

Thus, we can assume $\gamma = (v_1, v_2, v_3, v_4, v_5, v_6, v_1)$ with atoms v_1, v_3, v_5. Replace v_{2i} by $v_{2i-1} \vee v_{2i+1} \mod 6$ for $i = 1, 2, 3$ to obtain a homotopic loop γ' of the same length. If $n > 6$, as the join $v = v_1 \vee v_3 \vee v_5$ is a monomial of degree ≤ 6 then $v \neq m_G$, and coning the 6-cycle γ' over v shows that γ' is trivial, a contradiction.

Thus, the above argument shows that we can assume also that $n = 6$ and that v_1, v_3, v_5 are pairwise disjoint edges in G (otherwise $v \neq m_G$ and we are done as before). It is not difficult to find discs with boundary γ' in this case: let G^c be the 6-cycle $(1, 2, 3, 4, 5, 6, 1)$, and denote faces of $\Delta((1, m_G))$ by the shorthand notation $[ij, ijkl]$ to mean $\{x_i x_j, x_i x_j x_k x_l\}$ etc. By rotational symmetry of the 6-cycle and permuting the 3 atoms determining γ', it is enough to consider the following two cases:

Case 1: $v_1 = 14, v_3 = 25, v_5 = 36$. Then γ' is the boundary of the embedded disc in $\Delta((1, m_G))$ with the 18 facets

$$[[14, 1245, 12456], [25, 1245, 12456], [25, 2356, 12356], [36, 2356, 12356],$$

$$[36, 1346, 13456], [14, 1346, 13456], [14, 145, 12456], [15, 145, 12456], [15, 145, 13456],$$

$$[14, 145, 13456], [25, 125, 12356], [15, 125, 12356], [15, 125, 12456], [25, 125, 12456],$$

$$[36, 1356, 12356], [15, 1356, 12356], [15, 1356, 13456], [36, 1356, 13456]].$$

Denote this disk by D.

Case 2: $v_1 = 26, v_3 = 35, v_5 = 14$. The involution $\nu : [6] \longrightarrow [6]$ fixing $1, 4, 5$ and 6 and exchanging 2 and 3 shows that the disc $\nu(D)$ has boundary γ'.

Thus, γ is trivial, a contradiction completing the proof. □

5 Claw free graphs

A graph G is *claw free* if it contains no 4 vertices on which the induced graph is a *star*, i.e. a connected graph where all vertices but one have exactly one neighbor, which is common to all of them. Claw free graphs are of great interest in combinatorics. The connectivity of the independence complex of claw free graphs was studied in [7]; in particular it follows that a nonzero homology in the independence complex of G, which is the clique complex of G^c, can occur in arbitrarily high dimension. Using Hochster's formula it means that $\sup\{\text{reg}(I(G)) : G \text{ is claw free}\} = \infty$.

If we restrict to claw free graphs with no induced C_4 in their complement, denote this family by $\mathcal{CF}$, the situation is drastically different, as Theorem 5.1 below shows.

Theorem 5.1. *If $G \in \mathcal{CF}$ then* $\mathrm{reg}(I(G)) \leq 3$.

As we have seen, both values of the regularity permitted by this theorem are possible: if G^c is a tree then $\mathrm{reg}(I(G)) = 2$ and if $G^c = C_n$ for $n \geq 5$ then $\mathrm{reg}(I(G)) = 3$.

Proof of Theorem 5.1. We shall make use of Hochster's formula:

Theorem 5.2. *(Hochster's formula)[4, Theorem 5.5.1] For a simplicial complex Δ on vertex set V, the Betti numbers of its Stanley-Reisner ideal I_Δ over a field k satisfy for every i*

$$\beta_{i,i+j}(I_\Delta) = \sum_{W \subseteq V,\ |W|=i+j} \dim_k(\tilde{H}_{j-2}(\Delta[W]; k)).$$

(Note that we resolve the ideal, not the face ring, hence the shift in the index w.r.t. [4, Theorem 5.5.1], which states a multi-graded version.) Thus, to prove the theorem we need to show that for every $l > 1$ and every $W \subseteq V(G)$, $\tilde{H}_l(\Delta[W]; k)) = 0$ where $\Delta[W]$ is the clique complex on the induced graph $G^c[W]$.

If $\Delta[W]$ has no 2-dimensional faces, this is obvious. Assume that $F = \{a, b, c\}$ is a 2-face of $\Delta[W]$. Decompose the geometric realization $|\Delta[W]|$ as a union of two open spaces $\Delta[W] = (|\Delta[W]| - |F|) \cup (\cup_{v \in F} \mathrm{st}(v))$ where $\mathrm{st}(v)$ is the open star of v in $\Delta[W]$. Let $L = (|\Delta[W]| - |F|) \cap (\cup_{v \in F} \mathrm{st}(v))$. Then $|\Delta[W]| - |F|$ retracts on $\Delta[W - F]$ and by induction on the number of vertices all of its homology groups in dimension > 1 vanish. Note that $\cup_{v \in F} \mathrm{st}(v)$ is contractible. The intersection L is homotopy equivalent to $\cup_{v \in F} \mathrm{lk}(v) - \partial F$, where ∂F is the boundary of the simplex with vertex set F, and in turn is homotopy equivalent to the complex M generated by the faces $T \subseteq (W - F)$ such that one of the sets $T \cup \{v\}$ where $v \in F$ is in $\Delta[W]$.

We now show that if $|T| \geq 3$ then $T \in M$ iff $T \in \Delta[W - F]$, and hence, again by induction, the homology groups of M and hence of L in dimension > 1 vanish. Clearly $M \subseteq \Delta[W - F]$. Let $T \in \Delta[W - F]$, $|T| \geq 3$ and assume by contradiction that $T \notin M$, i.e. for any $v \in F$ the number of its neighbors in G^c among T is smaller than $|T|$; without loss of generality let a maximize this number among the elements of F, and denote this number by t and the neighbors of v in G^c among T by $T(v)$. Let $u \in T - T(a)$. By claw freeness u has a neighbor in F, and w.l.o.g. let b be such neighbor. We will show now that the disjoint union $T(a) \uplus \{u\} \subseteq T(b)$, a contradiction to the choice of a: for each $w \in T(a)$, look at the 4-cycle (a, b, u, w) in G^c and conclude that $\{w, b\} \in G^c$, hence $w \in T(b)$.

By Mayer-Vietoris long exact sequence over $\mathbb{Z}$ we get for $i > 1$

$$0 = H_i(|\Delta[W]| - |F|) \oplus H_i(\bigcup_{v\in F} \mathrm{st}(v)) \to H_i(\Delta[W]) \to H_{i-1}(L) \stackrel{j_*}{\to} H_{i-1}(|\Delta[W]| - |F|),$$

thus we will be done if we show that j_* is injective. This will follow from showing that the diagram

$$\begin{array}{ccc} H_l(L) & \stackrel{j_*}{\to} & H_l(|\Delta[W]| - |F|) \\ \cong\downarrow & & \downarrow\cong \\ H_l(M) & \stackrel{i_*}{\to} & H_l(\Delta[W - F]) \end{array}$$

commutes for any l, where i_* is induced by inclusion. Indeed, i_* is injective for $l \geq 1$: we already showed that M and $\Delta[W - F]$ have the same faces in dimension ≥ 2. Thus, for $l > 1$, $i_* : H_l(M) \longrightarrow H_l(\Delta[W - F])$ is an isomorphism, and as $M \subseteq \Delta[W - F]$, we obtain also that $i_* : H_1(M) \longrightarrow H_1(\Delta[W - F])$ is injective. Commutativity follows from taking a retract $|\Delta[W]| - |F| \to \Delta[W - F]$ whose restriction to L is a retract onto M; this is easy to do, we omit the details. □

Denote by L_i the restriction of the lcm lattice $L = L(M)$ to monomials of degree at least i (not to be confused with the rank of them as elements in the poset). For a simplicial complex Γ let $\alpha(\Gamma)$ be the maximal number such that $\tilde{H}_{\dim(\Gamma)-\alpha(\Gamma)}(\Gamma) \neq 0$, and set $\alpha(\Gamma) = 0$ if Γ is acyclic. For a monomial m in L let $\alpha(m) = \alpha((1, m)) := \alpha(\Delta((1, m)))$. Let $\alpha(M) := \max_{1\neq m\in L(M)}\{\alpha(m)\}$. If M is generated by monomials of degree r then $\mathrm{reg}(M) = r + \alpha(M)$ (use (1), or see [15, Proposition 2.3]). Also, denote $\mathrm{supp}(m) = \{v : x_v | m\}$.

The following proposition was suggested to me by Irena Peeva, generalizing a result of Phan [16] who proved the case where $s = 2$ and M has a linear resolution. It will be used in the proof of Theorem 1.2(1).

Proposition 5.3. *Let M be a monomial ideal minimally generated by monomials of a fixed degree $s \geq 2$. Suppose that its lcm-lattice $L(M)$ is graded and except for the minimum, the rank function is given by rank$(m) = \deg(m) - s + 1$ (m is a monomial). Suppose that there exist monomials of degree $s+1$ in $L(M)$, and let Q be the monomial ideal generated by all such monomials, that is, Q is generated by the multidegrees of the first minimal syzygies of M. Then for any $m \in Q$, $\alpha((1, m)_{L(Q)}) \leq \max(0, \alpha((1, m)_{L(M)} - 1)$. In particular,*

$$\mathrm{reg}(Q) \leq \max(s + 1, \mathrm{reg}(M)).$$

Proof. Fix a monomial $m \in L(M)$. Let A be the set of atoms in $(1,m)_{L(M)}$, $\Delta = \Delta((1,m)_{L(M)})$, $\Gamma = \Delta - A$, and Λ the induced subcomplex of Δ on the complement of A, i.e. $\Lambda = \Delta((1,m)_{L(Q)})$. Then Γ deformation retracts on Λ and $\dim(\Lambda) = \dim(\Gamma) - 1$.

The Mayer-Vietoris sequence gives

$$\tilde{H}_i\big(\uplus_{a\in A}\,\mathrm{lk}(a,\Delta)\big) \longrightarrow \tilde{H}_i(\Gamma) \oplus \tilde{H}_i\big(\uplus_{a\in A}\,\mathrm{st}(a,\Delta)\big) \longrightarrow \tilde{H}_i(\Delta)\,.$$

For any $a \in A$, the link $\mathrm{lk}(a,\Delta) = \Delta((a,m))$ is shellable (by [2, Theorem 3.1] again). Therefore, we get that $\alpha(\mathrm{lk}(a,\Delta)) = 0$, hence $\alpha(\Gamma) \leq \max(1, \alpha(\Delta))$. Now, the assertion follows as $\dim(\Lambda) = \dim(\Gamma) - 1$. □

Corollary 5.4. *If G^c has no induced C_4, then for every $m \in L = L(I_G)$ and $i \geq 2$*

$$\alpha((1,m)_{L_i}) \leq \max(0, \alpha((1,m)_L) - i + 2).$$

Proof. Combine Propositions 2.1 and 5.3. □

The following lemma is needed in the proof of Theorem 1.2(1).

Lemma 5.5. *Let $G \in \mathcal{CF}$ and $m \in L^2 := L(I(G)^2)$ be squarefree. Let $L := L(I(G))$. Then $(1,m]_{L^2} = (1,m]_{L_4}$.*

Proof. As in both posets the elements are all the joins of monomial of degree 4, it is enough to show that a monomial of degree 4 is in $(1,m]_{L^2}$ iff it is in $(1,m]_{L_4}$. Let m' be a monomial of degree 4. If $m' \in (1,m]_{L^2}$ clearly $m' \in (1,m]_{L_4}$. Conversely, if $m' \in (1,m]_{L_4}$ then $G[\mathrm{supp}(m')]$ contains two (not induced!) disjoint edges as G is claw free, and their product shows $m' \in (1,m]_{L^2}$. □

The following theorem is a restatement of Theorem 1.2(1).

Theorem 5.6. *If $G \in \mathcal{CF}$ then* $\mathrm{reg}(I(G)^2) = 4$.

Proof. By (1) and Proposition 2.1 we need to show that $\alpha(m) = 0$ holds for any $m \in L^2 := L(I(G)^2)$.

If $|\,\mathrm{supp}(m)| \leq 3$ then one easily checks that $\alpha(m) = 0$ (note that any variable appears in degree at most 2 in m). So assume that $|\,\mathrm{supp}(m)| \geq 4$. As $G[\mathrm{supp}(m)]$ is claw free, it contains two disjoint edges, and their product divides m. Let m_{sf} be the (nonempty) join of squarefree atoms in $(1,m]$. We distinguish two cases.

Case 1: $m_{sf} = m$. Combining Lemma 5.5, Corollary 5.4 and Theorem 5.1 gives $\alpha((1,m)_{L^2}) \leq \max(0,\, \alpha((1,m)_L) - 2) = 0$ as desired.

Case 2: $m_{sf} \neq m$. For an induced subposet L of an lcm lattice generated by monomials of degree 4, denote by $L_{\neg 2}$ the restriction of L to the joins of atoms which are not squares, i.e. not of the form $(ab)^2$. First we show that:

Lemma 5.7. *For any $m \in L(I(G)^2)$, $\alpha((1,m)) \leq \alpha((1,m)_{\neg 2})$.*

We postpone the proof for later. To conclude in Case 2, it is enough to show that $\alpha((1,m)_{\neg 2}) = 0$.

Let $P_0 = (1, m_{sf}]$ and for $i > 0$ let P_i be the restriction of $(1,m)_{\neg 2}$ to P_0 union with the elements of degree at least $\deg(m) - i$ in $(1,m)_{\neg 2}$. Then $P_0 \subseteq P_1 \subseteq ... \subseteq P_{\deg(m)-4} = (1,m)_{\neg 2}$. Note that $\Delta(P_0)$ as acyclic as it is a cone.

We will show first that $\Delta(P_i)$ is acyclic for $0 \leq i \leq \deg(m) - 7$. Let $i > 0$ and $x \in P_i - P_{i-1}$. Then

$$\mathrm{lk}(x, \Delta(P_i)) = \Delta((x,m)_{\neg 2}) * \Delta((1, x_{sf}])$$

where $\Delta((1, x_{sf}]) = \emptyset$ if x_{sf} does not exist. However, recall that claw freeness guarantees that x_{sf} exists if $|\operatorname{supp}(x)| \geq 4$ which is the case if $\deg(x) > 6$. If x_{sf} exists then $\mathrm{lk}(x, \Delta(P_i))$ is acyclic.

Let $1 \leq i \leq \deg(m) - 7$. Order the vertices in $P_i - P_{i-1}$, say $x_1, x_2, ..., x_j$. Let P_{x_l} be the induced poset of $(1,m)$ on $P_{i-1} \cup \{x_1, ..., x_l\}$ and $\Delta(P_{x_l})$ be its order complex. Define $P_{x_0} := P_{i-1}$. Let $1 \leq l \leq j$ and by induction we assume that $\Delta(P_{x_{l-1}})$ is acyclic. Consider the Mayer-Vietoris long exact sequence for the union $\Delta(P_{x_l}) = (\Delta(P_{x_l}) - \{x_l\}) \cup \mathrm{st}(x_l, \Delta(P_{x_l}))$. Note that $\Delta(P_{x_l}) - \{x_l\}$ is homotopy equivalent to $\Delta(P_{x_{l-1}})$, $\mathrm{st}(x_l, \Delta(P_{x_l}))$ is acyclic, and their intersection is homotopy equivalent to $\mathrm{lk}(x_l, \Delta(P_i))$ which is a cone. We conclude that $\Delta(P_{x_l})$ is acyclic too.

Thus, $\Delta(P_{deg(m)-7})$ is acyclic. For $x_l \in P_{deg(m)-6} - P_{deg(m)-7}$, if $(x_l)_{sf}$ exists then as we showed before, adding it to the poset $P_{x_{l-1}}$ will not affect the homology. If $(x_l)_{sf}$ does not exist then $\mathrm{lk}(x_l, \Delta(P_{x_{l-1}})) = \Delta((x_l, m)_{\neg 2})$ which is shellable (as $[x_l, m]_{\neg 2}$ is semimodular and see Section 2), hence adding x_l to $P_{x_{l-1}}$ may create nontrivial homology in dimension $\dim(\Delta(1,m)) - 3$ only. Thus, the Mayer-Vietoris sequence shows that $\Delta(P_{x_l})$ may have nonzero homology only in dimension $\dim(\Delta(1,m)) - 2 = \deg(m) - 7$. Moreover, it shows that $\tilde{H}_{\deg(m)-7}(\Delta(P_{\deg(m)-6})) \cong \mathbb{Z}^k$ where k is the number of monomials $x \in P_{\deg(m)-6} \setminus P_{\deg(m)-7}$ such that $\Delta((x,m))$ has nonvanishing top dimensional homology.

Note that for such x $\Delta((x,m))$ is a pseudomanifold (indeed every chain $x < c_1 < ... < c_{\deg(m)-\deg(x)-2} < m$ is contained in at most two maximal chains in $[x,m]_{\neg 2}$). It follows that for x as above $\Delta((x,m)_{\neg 2})$ is a sphere. As a representative of the homology induced by x we need to find a cycle in $\Delta(P_{\deg(m)-6})$ (actually we will find a sphere) whose support contains the ball $\Delta([x,m)_{\neg 2})$. For this, we need the following lemma.

Lemma 5.8. *Let* $x \in \{a^2b^2c, a^2b^2c^2\}$ *and* $y \in (x,m)_{\neg 2}$ *where* a,b,c *are different variables. Then* $y/a \in (x/a,m)_{\neg 2}$.

We postpone its proof for later. Back to the proof of Theorem 5.6, we need to consider $x \in P_{deg(m)-6} - P_{deg(m)-7}$ with $\Delta((x,m))$ not acyclic and such that $|\operatorname{supp}(x)| \leq 3$, hence $x = a^2b^2c^2$. By Lemma 5.8, $\{y/a : y \in (x,m)_{\neg 2}\} \subseteq P_{\deg(m)-6}$. The join of these y/a is $m/a \in P_{\deg(m)-6}$. For each facet $\{c_1 < ... < c_l\}$ of $\Delta((x,m)_{\neg 2})$, triangulate the prism with top $\{c_1 < ... < c_l\}$ and bottom $\{c_1/a < ... < c_l/a\}$ in the standard way using the facets $\{c_1/a < ... < c_i/a < c_i < ... < c_l\}$. The union of all these prisms and $\Delta([x,m)_{\neg 2})$ and $\Delta((x/a,m/a]_{\neg 2})$ is a sphere of codimension 2 in $\Delta(P_{\deg(m)-6})$ representing the nontrivial homology induced by x.

On the other hand, the cone over this sphere with apex x/a shows that the map

$$\tilde{H}_{\deg(m)-7}(\Delta(P_{\deg(m)-6}) \longrightarrow \tilde{H}_{\deg(m)-7}(\Delta(P_{\deg(m)-5}),$$

induced by the Mayer-Vietoris sequence for the union

$$\Delta(P_{\deg(m)-5}) = (\Delta(P_{\deg(m)-5}) - (P_{\deg(m)-5} - P_{\deg(m)-6}))\cup$$
$$(\bigcup_{x\in P_{\deg(m)-5}-P_{\deg(m)-6}} \operatorname{st}(x, \Delta(P_{\deg(m)-5})),$$

is zero. Arguing as before with the Mayer-Vietoris sequence, $\Delta(P_{\deg(m)-5})$ may have nonvanishing homology only in dimension $\deg(m)-6$ (i.e. codimension 1), and by Lemma 5.8 applied to $x = a^2b^2c$ and the above argument, this homology maps to zero in $\Delta(P_{\deg(m)-4})$. Thus, applying again the Mayer-Vietoris sequence, $\tilde{H}_i(\Delta(P_{\deg(m)-4}))$ may be nonzero only if $i = \deg(m)-5$ (depending on whether there are atoms $x \in (1,m)_{\neg 2}$ such that $\Delta(x,m)_{\neg 2}$ is a sphere), i.e. $\alpha((1,m)_{\neg 2}) = 0$. □

Proof of Lemma 5.7. If m has degree 4 there is nothing to prove, as both posets are empty. Otherwise, $|\operatorname{supp}(m)| > 2$ and hence there is an atom below m which is not a square. Let m_{ns} be the join of all such atoms. Let $P_0 = (1,m)_{\neg 2}$ (it is not empty!) and for $i > 0$ let P_i be the restriction of $(1,m)$ to P_0 union with the elements of degree at least $\deg(m)-i$ in $(1,m)$. Then $P_0 \subseteq P_1 \subseteq ... \subseteq P_{\deg(m)-4} = (1,m)$.

Let $i > 0$ and $x \in P_i \setminus P_{i-1}$. Then

$$\mathrm{lk}(x, \Delta(P_i)) = \Delta((x, m)) * \Delta((1, x_{ns}]_{\neg 2})$$

where $\Delta((1, x_{ns}]_{\neg 2}) = \emptyset$ iff x_{ns} does not exist. Note that as long as $\deg(x) > 4$ then x_{ns} exists, hence $\mathrm{lk}(x, \Delta(P_i))$ is acyclic.

Similar Mayer-Vietoris sequences to the ones using x_{sf} in the proof of Theorem 5.6(Case 2) show that $\Delta(P_i)$ is homologous to $\Delta(P_0)$ for $0 \leq i \leq \deg(m) - 5$.

Let $x \in P_{\deg(m)-4} \setminus P_{\deg(m)-5}$. Then $\mathrm{lk}(x, \Delta(P_{\deg(m)-4})) = \Delta((x, m))$, which is shellable, hence $\alpha((x, m)) = 0$.

Now add the vertices $\{x_1, ..., x_j\} = P_{\deg(m)-4} - P_{\deg(m)-5}$ to $P_{\deg(m)-5}$ one by one, denoting by P_{x_l} the induced poset in $(1, m)$ on $P_{\deg(m)-5} \cup \{x_1, ..., x_l\}$, where $P_{x_0} := P_{\deg(m)-5}$. The Mayer-Vietoris sequence shows that the homology of $\Delta(P_{x_l})$ may differ from the homology of $\Delta(P_{x_{l-1}})$ only in the top dimension and in codimension 1, where a difference in codimension 1 is possible only if the codimension 1 homology group of $\Delta(P_{x_{l-1}})$ is nonzero. Inductively, this implies that $\alpha(P_{\deg(m)-4}) \leq \alpha(P_0)$, i.e. that $\alpha((1, m)) \leq \alpha((1, m)_{\neg 2})$. □

We remark that if $m_{ns} < m$ then the proof above gives that $\alpha((1, m)) = 0$ as $(1, m)_{\neg 2}$ is a cone (with apex m_{ns}).

Proof of Lemma 5.8. Note that $G[\mathrm{supp}(x)]$ is a triangle, hence $x/a \in (1, m)_{\neg 2}$. If $|\mathrm{supp}(y)| = 3$ then $y = a^2 b^2 c^2$ and the claim is clear. So assume $|\mathrm{supp}(y)| > 3$, and as we argued before, claw freeness guarantees the existence of y_{sf}. If a variable $v \neq a$ appears in degree 2 in y, then there are two different edges containing the vertex v in $G[\mathrm{supp}(y)]$, and their product, denoted by $e(v)$ is in $(1, m)_{\neg 2}$. Thus, the join of y_{sf} with all the $e(v)$ for v as above equals y/a and is in $(x/a, m)_{\neg 2}$. □

Theorem 1.2 readily follows from Theorems 5.1 and 5.6.

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