A Generalized Macaulay Theorem and Generalized Face Rings

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Abstract

We prove that the f-vector of members in a certain class of meet semi-lattices satisfies Macaulay inequalities $0 \leq \partial^k(f_k) \leq f_{k-1}$ for all $k \geq 0$. We construct a large family of meet semi-lattices belonging to this class, which includes all posets of multicomplexes, as well as meet semi-lattices with the "diamond property", discussed by Wegner [11], as special cases. Specializing the proof to the later family, one obtains the Kruskal-Katona inequalities and their proof as in [11].

For geometric meet semi-lattices we construct an analogue of the exterior face ring, generalizing the classic construction for simplicial complexes. For a more general class, which also includes multicomplexes, we construct an analogue of the Stanley-Reisner ring. These two constructions provide algebraic counterparts (and thus also algebraic proofs) of Kruskal-Katona's and Macaulay's inequalities for these classes, respectively.

1 Introduction

Let us review the characterization of f-vectors of finite simplicial complexes, known as the Schützenberger-Kruskal-Katona theorem (see [4] for a proof and for references). Let C be a (finite, abstract) simplicial complex, $f_i = |\{S \in C : |S| = i+1\}|$. $f = (f_{-1}, f_0, ...)$ is called the f-vector of C (note that $f_{-1} = 1$). For any two integers $k, n > 0$ there exists a unique expansion

$$n = \binom{n_k}{k} + \binom{n_{k-1}}{k-1} + ... + \binom{n_i}{i} \quad (1)$$

such that $n_k > n_{k-1} > ... > n_i \geq i \geq 1$ (details in [4]). Define the function ∂_{k-1} by

$$\partial_{k-1}(n) = \binom{n_k}{k-1} + \binom{n_{k-1}}{k-2} + ... + \binom{n_i}{i-1}, \quad \partial_{k-1}(0) = 0.$$

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Theorem 1.1 (Schützenberger-Kruskal-Katona) *f is the f-vector of some simplicial complex iff f ultimately vanishes and*

$$\forall k \geq 0 \;\; 0 \leq \partial_k(f_k) \leq f_{k-1}. \tag{2}$$

For a ranked meet semi-lattice P, finite at every rank, let f_i be the number of elements with rank $i+1$ in P, and set $\mathrm{rank}(\hat{0}) = 0$ where $\hat{0}$ is the minimum of P. the f-vector of P is $(f_{-1}, f_0, f_1, ...)$.

P has the *diamond property* if for every $x, y \in P$ such that $x < y$ and $\mathrm{rank}(y) - \mathrm{rank}(x) = 2$ there exist at least two elements in the open interval (x, y). The closed interval is denoted by $[x, y] = \{z \in P : x \leq z \leq y\}$.

We identify a simplicial complex with the poset of its faces ordered by inclusion. The following generalization of Theorem 1.1 is due to Wegner [11].

Theorem 1.2 (Wegner) *Let P be a finite ranked meet semi-lattice with the diamond property. Then its f-vector ultimately vanishes and satisfies (2).*

For $\hat{x} \in P$ define $P(\hat{x}) = \{x \in P : \hat{x} \leq x\}$ and let $y' \prec y$ denote y covers y'.

Lemma 1.3 *For a ranked meet semi-lattice P, the diamond property is equivalent to satisfying the following condition:*

(*) For every $\hat{x} \in P$, x which covers $\hat{x}$ and y such that $y \in P(\hat{x})$ and $y \neq \hat{x}$, there exists $y' \in P(\hat{x})$ such that $y' \prec y$ and $x \nleq y'$.

A multicomplex (on a finite ground set) can be considered as an order ideal of monomials I (i.e. if $m|n \in I$ then also $m \in I$) on a finite set of variables. Its f-vector is defined by $f_i = |\{m \in I : \deg(m) = i+1\}|$ (again $f_{-1} = 1$). Define the function ∂^{k-1} by

$$\partial^{k-1}(n) = \binom{n_k - 1}{k-1} + \binom{n_{k-1} - 1}{k-2} + ... + \binom{n_i - 1}{i-1}, \;\; \partial^{k-1}(0) = 0,$$

w.r.t the expansion (1).

Theorem 1.4 (Macaulay,[8]) *(More proofs in [5, 9]) f is the f-vector of some multicomplex iff $f_{-1} = 1$ and*

$$\forall k \geq 0 \;\; 0 \leq \partial^k(f_k) \leq f_{k-1}. \tag{3}$$

Definition 1.5 *(Parallelogram property) A ranked poset P is said to have the parallelogram property if the following condition holds:*

(**) For every $\hat{x} \in P$ and $y \in P(\hat{x})$ such that $y \neq \hat{x}$, if the chain $\{\hat{x} = x_0 \prec x_1 \prec ... \prec x_r\}$ equals the closed interval $[\hat{x}, x_r]$ $(r > 0)$ and is

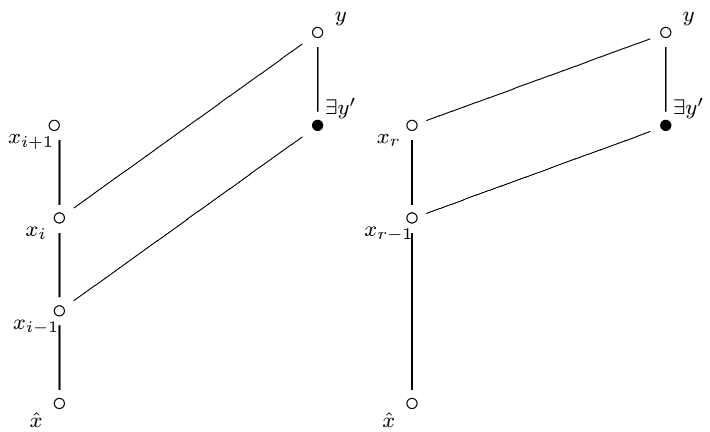

Figure 1: The parallelogram property for $i < r$ (left) and for $i = r$ (right).

maximal w.r.t. inclusion such that $r < \text{rank}(y)$ *(the rank of* y *in the poset* $P(\hat{x})$*), and if* $x_i < y$ *and* $x_{i+1} \nleq y$ *for some* $0 < i \leq r$*, then there exists* $y' \in P(\hat{x})$ *such that* $y' \prec y$*,* $x_{i-1} < y'$ *and* $x_i \nleq y'$*. For* $i = r$ *interpret* $x_{r+1} \nleq y$ *as:* $[\hat{x}, y]$ *is not a chain.*

See Figure 1 for an illustration of the parallelogram property. Note that condition (*) of Lemma 1.3 implies condition (**) of Definition 1.5 (with 1 being the only possible value of r), and that posets of multicomplexes satisfy the parallelogram property.
We identify a multicomplex with the poset of its monomials ordered by division. We now generalize Theorem 1.4.

Theorem 1.6 *Let* P *be a ranked meet semi-lattice, finite at every rank, with the parallelogram property. Then its* f*-vector satisfies (3) and* $f_{-1}(P) = 1$*.*

For generalizations of Macaulay's theorem in a different direction ('compression'), see e.g. [5, 10].

In Section 2 we prove Theorem 1.6 and construct a large family of meet semi-lattices satisfying its hypothesis.

Theorems 1.1 and 1.4 have algebraic counterparts in terms of face rings and algebraic shifting. No such interpretation is known for Theorems 1.2 and 1.6. In Section 3 we extend Theorems 1.1 and 1.4 by constructing analogues of the exterior and symmetric face rings, respectively. More specifically, we define an exterior algebraic shifting operation for geometric meet semi-lattices, and a symmetric algebraic shifting operation for a common generalization of geometric meet semi-lattices and multicomplexes.

2 Macaulay inequalities

We provide proofs of Theorem 1.6 and Lemma 1.3, and construct a large class of examples for which Theorem 1.6 applies.

Proof of Theorem 1.6: Clearly $f_{-1}(P) = 1$. Let us show that P satisfies the inequalities (3). Let X be the set of rank $k+1$ ($\leq \text{rank}(P)$) elements in P, and denote its shadow by ∂X, i.e. $\partial X = \{p \in P : \exists x \in X, p \prec x\}$. We will show that $|\partial X| \geq \partial^k(|X|)$, which clearly proves Theorem 1.6.

The proof is by induction on k and on $|f_k|$. The case $k = 0$ is trivial, as well as the case $|f_k| = 1$ for any k. So assume $k > 0$.

Let us introduce some notation: Let $\hat{0} \neq x_r \in P$ be such that the interval $[\hat{0}, x_r]$ is maximal w.r.t. inclusion such that it is a chain $\{\hat{0} = x_0 \prec x_1 \prec \ldots \prec x_r\}$ and $x_r < x$ for some $x \in X$ (hence $r \leq k$). For $0 \leq i \leq r$, denote $P_i = \{p \in P : x_{i+1} \nleq p, x_i < p\}$ and $X_i = X \cap P_i$. Thus $P = \uplus_{0\leq i\leq r} P_i \uplus [\hat{0}, x_r]$. In addition, $\partial X = \uplus_{0\leq i\leq r}(\partial X \cap P_i)$, unless $r = k$, in which case $\{x_k\}$ should be added to that union. Let $\hat{X}_i$ denote the elements of X_i considered as elements of the induced meet semi-lattice $P(x_i)$. Thus, $\partial(\hat{X}_i) \subseteq \partial X \cap P_i$ unless $x_k \in \partial(\hat{X}_i)$, a case in which $i = k$ and $\partial(\hat{X}_k) = \{x_k\}$. Hence

$$|\partial \hat{X}_i| \leq |\partial X \cap P_i| \quad 0 \leq i \leq \min\{r, k-1\}, \tag{4}$$

and for $r = k$ $|\partial \hat{X}_k| = 1$. By the parallelogram property, for any $0 \leq i \leq \min\{r, k-1\}$ and $y \in X_{i+1}$, there exists $y' \in \partial\{y\} \cap P_i$ (for $i = r$ $X_{r+1} = \emptyset$). Note that y' 's arising from different y 's are distinct: suppose $y' \in \partial X \cap P_i$ arises from two different $y \in X_{i+1}$, then as P is a meet semi-lattice $x_{i+1} \leq y'$, a contradiction. We deduce that

$$|X_{i+1}| \leq |\partial X \cap P_i| \quad 0 \leq i \leq \min\{r, k-1\}. \tag{5}$$

Combining (4) and (5) we get that

$$|\partial X| = \begin{cases} 1 + \sum_{0\leq i\leq k-1} |\partial X \cap P_i| \geq 1 + \sum_{0\leq i\leq k-1} \max\{|X_{i+1}|, |\partial \hat{X}_i|\} & \text{if } r = k \\ \\ \sum_{0\leq i\leq r} |\partial X \cap P_i| \geq \sum_{0\leq i\leq r} \max\{|X_{i+1}|, |\partial \hat{X}_i|\} & \text{if } r < k. \end{cases} \tag{6}$$

By induction hypothesis, $|\partial \hat{X}_i| \geq \partial^{k-i}(|\hat{X}_i|) = \partial^{k-i}(|X_i|)$ for $0 \leq i \leq \min\{r, k-1\}$ (the induction on k implies it for $i \neq 0$, and the induction on $|f_k|$ implies it for $i = 0$). We need the following simple Lemma due to Björner and Vrećica: (One uses Theorem 1.4 to prove it.)

Lemma 2.1 *(Lemma 3.2 of [3]) For $k > 0$, the function ∂^k satisfies for all non-negative integers n_i and $r < k$:*

$$\partial^k(\sum_{0\leq i\leq r} n_i) \leq \sum_{0\leq i\leq r} \max\{n_{i+1}, \partial^{k-i}(n_i)\},$$

$$\partial^k(1+\sum_{0\le i\le k} n_i) \le 1+\sum_{0\le i\le k-1} \max\{n_{i+1}, \partial^{k-i}(n_i)\}.$$

By Lemma 2.1 we see that both right hand sides of (6) are $\ge \partial^k(\sum_{0\le j\le r}|X_j| + \delta_{r,k})$, where $\delta_{r,k}$ is Kronecker's delta. Using the fact that ∂^k is nondecreasing, the right hand side of (6) is $\ge \partial^k(|X|)$ (as $X = \uplus_{0\le i\le r} X_i$ for $r < k$, and $|X| \le 1+\sum_{0\le i\le r}|X_i|$ for $r = k$). Hence $|\partial X| \ge \partial^k(|X|)$ as desired. $\square$

Remark: If P satisfies the diamond property, then $X = X_0 \uplus X_1$ and $|\partial X| \ge \sum_{0\le i\le 1}\max\{|X_{i+1}|, |\partial\hat{X}_i|\}$ (here $X_2 = \emptyset$), an inequality which implies the Kruskal-Katona inequalities for P, via an inequality for the function ∂_k, analogous to the one in Lemma 2.1, which is given in [6]. This is how the proof given in [11] argues.

Proof of Lemma 1.3: Condition (*) clearly implies the diamond property. Conversely, we argue by induction on $r = \text{rank(y)} - \text{rank}(\hat{\text{x}})$. For $r = 1$, take $y' = \hat{x}$. For $r = 2$, this is the diamond property. For $r > 2$, assume $x < y$ (otherwise the assertion is trivial). There exists z such that $x < z \prec y$. By the induction hypothesis, there exists z' such that $z' \prec z$ and $x \not\le z'$. By the diamond property applied to the pair (z', y), there exists y' such that $z' \prec y' \prec y$ and $y' \ne z$. Now, $x \not\le y'$ as otherwise we get $x \le y' \wedge z = z'$, a contradiction. $\square$

Example F: Let $(L, <, r)$ be a finite ranked meet semi-lattice with partial order relation $<$ and rank function r. Denote its minimum by $\hat{0}$. Associate with each $\hat{0} \ne l \in L$ a collection $F(l)$ of multichains in the interval $(\hat{0}, l]$ which is closed w.r.t. the following partial order on multichains in $L \setminus \{\hat{0}\}$: Let $\underline{a} = (a_m \le ... \le a_1 \le a_0)$ and $\underline{b} = (b_k \le ... \le b_1 \le b_0)$ be multichains in $L \setminus \{\hat{0}\}$ and define $\underline{a} <' \underline{b}$ iff $m \le k$, $a_i \le b_i$ for all $0 \le i \le m$ and $\underline{a} \ne \underline{b}$. $F(l)$ is closed if $\underline{a} <' \underline{b} \in F(l)$ implies $\underline{a} \in F(l)$.

We define a new ranked meet semi-lattice $(L', <', r')$ as follows: $L' = \cup_{l\in L\setminus\{\hat{0}\}} F(l)$ where the empty multichain is the minimum $\hat{0}_{L'}$. In addition, $r'(\underline{a}) = \sum_{0\le i\le m} r(a_i)$ for $\underline{a} \in L'$ as above, where the empty multichain has rank 0. We denote it in short by L'. See Figure 2 for an illustration.

It is straightforward to verify that L' is indeed a ranked meet semi-lattice; we merely remark that for $\underline{a}, \underline{b} \in L'$ as above $\underline{a} \wedge \underline{b} = (a_{\min(m,k)} \wedge b_{\min(m,k)} \le ... \le a_0 \wedge b_0)$, which is an element of L' as for $l \in L$ such that $\underline{a} \in F(l)$ indeed $\underline{a} \wedge \underline{b} \in F(l)$.

Lemma 2.2 *Let L be a ranked meet semi-lattice. If L has the diamond property then L' has the parallelogram property.*

Corollary 2.3 *Let L be a ranked meet semi-lattice, finite at every rank. If L has the diamond property then L' satisfies Macaulay inequalities (3).*

Proof: This is immediate from Lemma 2.2 and Theorem 1.6. $\square$

Before proving Lemma 2.2, let us mention that the L' arising in this way include all posets of multicomplexes and all meet semi-lattices with the

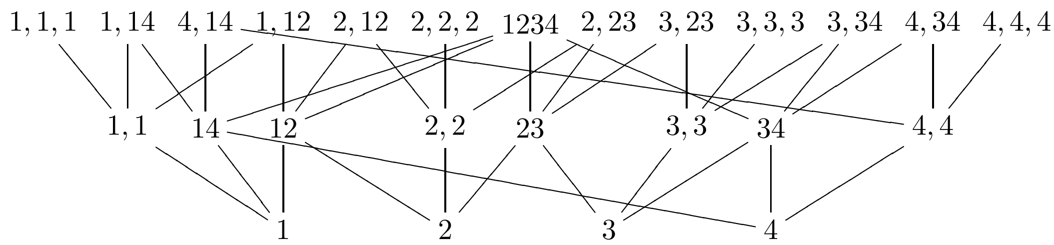

Figure 2: Constructing L': L is a regular CW-complex consisting of a 2-cell, the square 1234. $F(l)$ consists of all multichains of rank $\leq r(l)$ in $(\hat{0}, l]$ for all $l \in L$. $L' \setminus \hat{0}$ is shown.

diamond property. For the later, if L satisfies the diamond property, define $F(l) = \{(l') : \hat{0} < l' \leq l\} \cup \{\emptyset\}$ for all $l \in L$ to obtain $L' \cong L$. For a monomial m in a multicomplex M on the variables $x_1, ..., x_n$, define $f(m)$ to be the unique multichain of simplices obtained by dividing at each step by the largest possible square free monomial, e.g. $m = x_1^5 x_2 x_4^3 \mapsto f(m) = (\{1\} \leq \{1\} \leq \{1,4\} \leq \{1,4\} \leq \{1,2,4\})$. Denote by $\sigma(m)$ the largest simplex in the multichain $f(m)$. $\sigma(m) = \{1,2,4\}$ in the example above. Let $L = (\{\sigma(m) : m \in M\}, \subset)$. It is (the face poset of) a simplicial complex. For $\sigma \in L$ let $F(\sigma) =< f(m) : m \in M, \sigma(m) = \sigma >$ where $<>$ denotes the closure in the set of multichains w.r.t. $\subset'$. Then $L' \cong M$ as ranked posets.
Remarks: (1) If L is a regular CW-complex, L' already gives us new examples for which the inequalities (3) hold, see Figure 2.

(2) The construction $L \mapsto L'$ is a generalization of the barycentric subdivision. If L is a simplicial complex and $F(l)$ is the set of all chains (i.e. multichains *without* repetitions) in $(\hat{0}, l]$ then L' is the barycentric subdivision of L.

Proof of Lemma 2.2: For every $\underline{l'} \in L'$ consider the induced poset $L'(\underline{l'}) = \{\underline{y} \in L' : \underline{l'} \leq' \underline{y}\}$. An interval $[\underline{l'}, \underline{l}]$, $\underline{l} \neq \underline{l'}$, which is a chain in $L'(\underline{l'})$ is of one of the following (intersecting) two types: ($\prec'$ stands for the cover relation in L'.)

(1) $(\underline{l'} = \underline{l_0} \prec' \underline{l_1} \prec' ... \prec' \underline{l_m} = \underline{l})$ where there exists an atom $u \in L$ such that for every i where $1 \leq i \leq m$ $\underline{l_i}$ is obtained from $\underline{l_{i-1}}$ by adding u to its lower end, denoted by $\underline{l_i} = (u, \underline{l_{i-1}})$. In other words, $\overline{[\underline{l'}, \underline{l}]} = \{\underline{l'} \prec' (u, \underline{l'}) \prec' (u, u, \underline{l'}) \prec' ... \prec' (u, u, ..., u, \underline{l'})\}$.

(2) $(\underline{l'} \prec' \underline{l})$.

It follows from the fact that L satisfies the diamond property that indeed every interval not of type (1) nor of type (2) is not a chain: let $\underline{a} \prec' \underline{b} \prec' \underline{c}$ be a chain in such an interval, and assume by contradiction that it equals the interval $[\underline{a}, \underline{c}]$. Combining this with the definition of L', we conclude that the multichains $\underline{a}, \underline{b}, \underline{c}$ must have the same length, i.e. same last index m in the notation $\underline{a} = (a_m \leq ... \leq a_1 \leq a_0)$. If $\underline{a}$ and $\underline{c}$ differ in at least two different indices, denoted by i and j, then clearly there are at least two elements in the open interval $(\underline{a}, \underline{c})$ - just replace in $\underline{a}$ either a_i with c_i or a_j with c_j. We are left to deal with the case where $\underline{a}$ and $\underline{c}$ differ only in a single index, i. As $\underline{a} \prec' \underline{b} \prec' \underline{c}$, we conclude that $a_i \prec b_i \prec c_i$. By the diamond property of L, there exists $d \in L$ such that $d \neq b_i$ and $a_i \prec d \prec c_i$. Replacing b_i with d in $\underline{b}$ results in a multichain $\underline{d}$ such that $\underline{a} \prec' \underline{d} \prec' \underline{c}$; a contradiction. Thus indeed an interval not of type (1) nor of type (2) is not a chain.

We now verify that L' satisfies the parallelogram property.

Let $[\underline{l'}, \underline{l}]$ be of type (1), and let $\underline{x} \in (\underline{l'}, \underline{l}]$, $\underline{x} <' \underline{y}$, $(u, \underline{x}) \not\leq' \underline{y}$. Then $\underline{x} = (u, \underline{x'})$ for some multichain $\underline{x'}$. Let d be the element in the multichain $\underline{y} = (y_m \leq .. \leq y_1 \leq y_0)$ with the same index as the index of u at the lower end of $\underline{x}$ and let c be the next indexed element in $\underline{y}$; put $c = \hat{0}$ if $\underline{y}$ has the same last index as $\underline{x}$. Then $u \not\leq c$. We will show now that there exists $d' \in L$ such that $d' \prec d$, $c \leq d'$ and $u \not\leq d'$. Replacing d with d' in $\underline{y}$ we obtain a multichain $\underline{y'} \in L'$ such that $\underline{y'} \prec' \underline{y}$, $\underline{x'} <' \underline{y'}$ but $\underline{x} \not\leq' \underline{y'}$, as desired.

Let $\underline{\gamma} = (c < ... < d)$ be a maximal chain in $[c, d]$ such that its element of minimal rank in its intersection with the induced poset $L(c \vee u)$, denoted by z, is of maximal possible rank. We need to show that $z = d$ (taking d' as the element covered by d in $\underline{\gamma}$, we are done). Assume $z \neq d$. Clearly $z \neq c$ (as $u \not\leq c$). Let $t \in \underline{\gamma}$, $t \prec z$. By condition (*) of Lemma 1.3, there exists $t' \in L$ such that $t \prec t' < d$ and $t' \neq z$. By the maximality of z, $t' \in L(c \vee u)$. As L is a meet semi-lattice, $c \vee u \leq t$, contradicting the definition of z.

Let $[\underline{l'}, \underline{l}]$ be of type (2), and not of type (1). Let $\underline{l} <' \underline{y}$. By induction on the rank $r(\underline{y})$ we will show the existence of $\underline{y'} \in L'$ such that $\underline{y'} \prec' \underline{y}$, $\underline{l'} <' \underline{y'}$ and $\underline{l} \not\leq' \underline{y'}$. For $r(\underline{y}) = r(\underline{l}) + 1$, nonexistence of such $\underline{y'}$ means that the chain $\underline{l'} \prec' \underline{l} \prec' \underline{y}$ is an interval, thus $\underline{l} = (u, \underline{l'})$ for some atom $u \in L$, hence $[\underline{l'}, \underline{l}]$ is of type (1), a contradiction. Thus, the case $r(\underline{y}) = 2$ is verified. Let $\underline{t} \in [\underline{l}, \underline{y}]$, $\underline{t} \prec' \underline{y}$. By induction hypothesis there exists $\underline{z}$ such that $\underline{l'} <' \underline{z} \prec' \underline{t}$ and $\underline{l} \not\leq' \underline{z}$. If the chain $\underline{z} \prec' \underline{t} \prec' \underline{y}$ in L' is not an interval, let $\underline{y'} \in (\underline{z}, \underline{y})$, $\underline{y'} \neq \underline{t}$. As L' is a meet semi-lattice $\underline{l} \not\leq' \underline{y'}$. We are left to deal with the case $\underline{t} = (u, \underline{z})$ and $\underline{y} = (u, \underline{t})$ for some atom $u \in L$. As $\underline{l} \not\leq' \underline{z}$, the multichains $\underline{l}, \underline{t}$ have equal length, hence $\underline{l} = (u, \underline{\tilde{l}})$ for some multichain $\underline{\tilde{l}}$. As $[\underline{l'}, \underline{l}]$ is not of type (1), also $\underline{l'} = (u, \underline{\tilde{l'}})$ for some multichain $\underline{\tilde{l'}}$. Let us denote by $\underline{\tilde{w}}$ the multichain obtained from $\underline{w}$ by deleting its lower end u, where $\underline{w} \in \{\underline{y}, \underline{t}, \underline{l}, \underline{l'}\}$. Looking at $L'(\underline{\tilde{l'}})$, by induction hypothesis there exists $\underline{\tilde{y'}} \in L'(\underline{\tilde{l'}})$ such that $\underline{\tilde{l'}} <' \underline{\tilde{y'}} \prec' \underline{\tilde{y}}$ and $\underline{\tilde{l}} \not\leq' \underline{\tilde{y'}}$. Then $\underline{y'} = (u, \underline{\tilde{y'}})$ is as desired. $\square$

Example T: Let T be a rooted tree such that all its leaves have the same distance r from the root. Let $P(T)$ be the graded poset with T as its Hesse diagram where the root is its maximal element. Add a minimum to $P(T)$ to obtain the ranked lattice $L(T)$. The parallelogram property trivially holds for $L(T)$, hence by Theorem 1.6 $L(T)$ satisfies Macaulay inequalities. (In this case, of course $f_0 \geq f_1 \geq ... \geq f_r$, yet this family was not "trapped" by the previously known generalizations of Theorems 1.1 and 1.4.)

3 Face rings and algebraic shifting

3.1 Shifting geometric meet semi-lattices

We will associate an analogue of the exterior face ring to geometric ranked meet semi-lattices, which coincides with the usual construction for the case of simplicial complexes. Applying an algebraic shifting operation, à la Kalai [7], we construct a canonically defined shifted simplicial complex, having the same f-vector as its geometric meet semi-lattice.

Let $(L, <, r)$ be a ranked atomic meet semi-lattice with L the set of its elements, $<$ the partial order relation and $r : L \to \mathbb{N}$ its rank function. We denote it in short by L. L is called *geometric* if

$$r(x \wedge y) + r(x \vee y) \leq r(x) + r(y) \tag{7}$$

for every $x, y \in L$ such that $x \vee y$ exists. For example, the intersections of a finite collection of hyperplanes in a vector space form a geometric meet semi-lattice w.r.t. the reverse inclusion order and the codimension rank. Face posets of simplicial complexes are important examples of geometric meet semi-lattices, where (7) holds with equality.

Adding a maximum to a ranked meet semi-lattice makes it a lattice, denoted by $\hat{L}$, but the maximum may not have a rank. Denote by $\hat{0}, \hat{1}$ the minimum and maximum of $\hat{L}$, respectively, and by L_i the set of rank i elements in L. $r(\hat{0}) = 0$.

We now define the algebra $\bigwedge L$ over a field k with characteristic 2. Let V be a vector space over k with basis $\{e_u : u \in L_1\}$. Let $I_L = I_1 + I_2 + I_3$ be the ideal in the exterior algebra $\bigwedge V$ defined as follows. Choose a total ordering of L_1, and denote by e_S the wedge product $e_{s_1} \wedge ... \wedge e_{s_{|S|}}$ where $S = \{s_1 < ... < s_{|S|}\}$. Define:

$$I_1 = (e_S : S \subseteq L_1, \vee S = \hat{1} \in \hat{L}), \tag{8}$$

$$I_2 = (e_S : S \subseteq L_1, \vee S \in L, r(\vee S) \neq |S|), \tag{9}$$

$$I_3 = (e_S - e_T : T, S \subseteq L_1, \vee T = \vee S \in L, r(\vee S) = |S| = |T|, S \neq T). \tag{10}$$

(As $\mathrm{char}(k) = 2$, $e_S - e_T$ is independent of the ordering of the elements in S and in T.) Let $\bigwedge L = \bigwedge V / I_L$. As I_L is generated by homogeneous elements, $\bigwedge L$ inherits a grading from $\bigwedge V$. Let $f(\bigwedge L) = (f_{-1}, f_0, ..)$ be its graded dimensions vector, i.e. f_{i-1} is the dimension of the degree i component of $\bigwedge L$.

Remark: If L is the poset of a simplicial complex, then $I_L = I_1$ and $\bigwedge L$ is the classic exterior face ring of L, as in [7].

The following proposition will be used for showing that $\bigwedge L$ and L have the same f-vector. Its easy proof by induction on the rank is omitted.

Proposition 3.1 *Let L be a geometric ranked meet semi-lattice. Let $l \in L$ and let S be a minimal set of atoms such that $\vee S = l$, i.e. if $T \subsetneq S$ then $\vee T < l$. Then $r(l) = |S|$.* $\square$

Remark: The converse of Proposition 3.1 is also true: Let L be a ranked atomic meet semi-lattice such that every $l \in L$ and every minimal set of atoms S such that $\vee S = l$ satisfy $r(l) = |S|$. Then L is geometric.

Proposition 3.2 $f(\bigwedge L) = f(L)$.

Proof: Denote by $\tilde{w}$ the projection of $w \in \bigwedge V$ on $\bigwedge L$. We will show that picking $S(l)$ such that $S(l) \subseteq L_1, \vee S(l) = l, |S(l)| = r(l)$ for each $l \in L$ gives a basis over k of $\bigwedge L$, $E = \{\tilde{e}_{S(l)} : l \in L\}$.

As $\{\tilde{e}_S : S \subseteq L_1\}$ is a basis of $\bigwedge V$, it is clear from the definition of I_L that E spans $\bigwedge L$. To show that E is independent, we will prove first that the generators of I_L as an ideal, that are specified in (9), (8) and (10), actually span it as a vector space over k.

As $x \vee \hat{1} = \hat{1}$ for all $x \in L$, the generators of I_1 that are specified in (8) span it as a k-vector space. Next, we show that the generators of I_2 and I_1 that are specified in (9) and in (8) respectively, span $I_1 + I_2$ as a k-vector space: if e_S is such a generator of I_2 and $U \subseteq L_1$ then either $e_U \wedge e_S \in I_1$ (if $U \cap S \neq \emptyset$ or if $\vee(U \cup S) = \hat{1}$) or else, by Proposition 3.1, $r(\vee(U \cup S)) < |U \cup S|$ and hence $e_U \wedge e_S$ is also such a generator of I_2.

Let $e_S - e_T$ be a generator of I_3 as specified in (10) and let $U \subseteq L_1$. If $U \cap T \neq \emptyset$ then $e_T \wedge e_U = 0$ and $e_S \wedge e_U$ is either zero (if $U \cap S \neq \emptyset$) or else a generator of $I_1 + I_2$, by Proposition 3.1; and similarly when $U \cap S \neq \emptyset$. If $U \cap T = \emptyset = U \cap S$ then $\vee(S \cup U) = \vee(T \cup U)$ and $|S \cup U| = |T \cup U|$. Hence, if $e_S \wedge e_U - e_T \wedge e_U$ is not the obvious difference of two generators of I_1 or of I_2 as specified in (8) and (9), then it is a generator of I_3 as specified in (10). We conclude that these generators of I_L as an ideal span it as a vector space over k.

Assume that $\sum_{l \in L} a_l \tilde{e}_{S(l)} = 0$, i.e. $\sum_{l \in L} a_l e_{S(l)} \in I_L$ where $a_l \in k$ for all $l \in L$. By the discussion above, $\sum_{l \in L} a_l e_{S(l)}$ is in the span (over k) of the generators of I_3 that are specified in (10). But for every $l \in L$ and every such generator g of I_3, if $g = \sum\{b_S e_S : \vee S \in L, r(\vee S) = |S|\}$ ($b_S \in k$ for all S) then $\sum\{b_S : \vee S = l\} = 0$. Hence $a_l = 0$ for every $l \in L$. Thus E is a basis of $\bigwedge L$, hence $f(\bigwedge L) = f(L)$. $\square$

Now let us shift. Note that Kalai's algebraic shifting [7], which was defined for the exterior face ring, can be applied to any graded exterior algebra finitely generated by degree 1 elements. It results in a simplicial complex with an f-vector that is equal to the vector of graded dimensions of the algebra. This shows that any such graded algebra satisfies Kruskal-Katona inequalities! We apply this construction to $\bigwedge L$:

Let $B = \{b_u : u \in L_1\}$ be a basis of V. Then $\{\tilde{b}_S : S \subseteq L_1\}$ spans $\bigwedge L$. Choosing a basis from this set in the greedy way w.r.t. the lexicographic order $<_L$ on equal sized sets ($S < T$ iff $\min(S \triangle T) \in S$), defines a collection of sets:

$$\Delta_B(L) = \{S : \tilde{b}_S \notin \mathrm{span}_k\{\tilde{b}_T : |T| = |S|, T <_L S\}\}.$$

$\Delta_B(L)$ is a simplicial complex, and by Proposition 3.2 $f(\Delta_B(L)) = f(L)$. For a generic B, $\Delta_B(L)$ is shifted. (B is *generic* if the entries of the transition matrix form the standard basis to B are algebraically independent over a subfield of k. Alternatively, we can extend k by n^2 intermediates and consider the exterior algebra over this bigger field, letting the transition matrix consist of those intermediates. A collection of finite subsets of $\mathbb{N}$, A, is *shifted* if $S \in A$ and T that is componentwise not greater than S as ordered sets of equal size implies $T \in A$.) Moreover, the construction is canonical, i.e. is independent both of the chosen ordering of L_1 and of the generically chosen basis B. It is also independent of the characteristic 2 field that we picked. We denote $\Delta(L) = \Delta_B(L)$ for a generic B. For proofs of the above statements we refer to Björner and Kalai [2] (they proved for the case where L is a simplicial complex, but the proofs remain valid for any graded exterior algebra finitely generated by degree 1 elements).

We summarize the above discussion in the following theorem:

Theorem 3.3 *Let L be a geometric meet semi-lattice, and let k be a field of characteristic* 2*. There exists a canonically defined shifted simplicial complex $\Delta(L)$ associated with L, with $f(\Delta(L)) = f(L)$.* $\square$

Remarks: (1) The fact that L satisfies Kruskal-Katona inequalities follows also without using our algebraic construction, from the fact that it satisfies the diamond property and applying Theorem 1.2. The diamond property easily seen to hold for all ranked atomic meet semi-lattices.

(2) A different operation, which does depend on the ordering of L_1 and results in a simplicial complex with the same f-vector, was described by Björner [1], Chapter 7, Problem 7.25: totally order L_1. For each $x \in L$ choose the lexicographically least subset $S_x \subseteq L_1$ such that $\vee S_x = x$ ($S_{\hat{0}} = \emptyset$). Define $\Delta_<(L) = \{S_x : x \in L\}$. Then $\Delta_<(L)$ is a simplicial complex with the same f-vector as L. An advantage in our operation is that it is canonical (and results in a shifted simplicial complex). To see that these two operations are indeed different, let L be the face poset of a simplicial complex. Then for any total ordering of L_1, $\Delta_<(L) = L$. But if the simplicial complex is not shifted (e.g. a 4-cycle), then $\Delta(L) \neq L$.

3.2 Shifting generalized multicomplexes

We will associate an analogue of the symmetric (Stanley-Reisner) face ring with a common generalization of multicomplexes and geometric meet semi-lattices. Applying an algebraic shifting operation, we construct a multicomplex having the same f-vector as the original poset.

Let $\mathbb{P}$ be the following family of posets: to construct $P \in \mathbb{P}$ start with a geometric meet semi-lattice L. Associate with each $l \in L$ the (square free) monomial $m(l) = \prod_{a<l, a\in L_1} x_a$, and equip it with rank $r(m(l)) = r(l)$. Denote this collection of monomials by M_0. Now repeat the following procedure finitely or countably many times to construct $(M_0 \subseteq M_1 \subseteq ...)$: Choose $m \in M_i$ and $a \in L$ such that $x_a|m$, $\frac{x_a}{x_b}m \in M_i$ for all $b \in L_1$ such that $x_b|m$, and $x_a m \notin M_i$. M_{i+1} is obtained from M_i by adding $x_a m$, setting its rank to be $r(x_a m) = r(m)+1$ and let it cover all the elements $\frac{x_a}{x_b}m$ where $b \in L_1$ such that $x_b|m$. Define $P = \cup M_i$.

Note that the posets in $\mathbb{P}$ are ranked (not necessarily atomic) meet semi-lattices with the parallelogram property, and that $\mathbb{P}$ includes all multicomplexes (start with L, a simplicial complex) and geometric meet semi-lattices ($P = M_0$).

For $P \in \mathbb{P}$ define the following analogue of the Stanley-Reisner ring: Assume for a moment that P is finite. Fix a field k, and denote $P_1 = \{1, .., n\}$. Let $A = k[x_1, .., x_n]$ be a polynomial ring. For j such that $1 \leq j \leq n$ let r_j be the minimal integer number such that $x_j^{r_j+1}$ does not divide any of the monomials $p \in P$. Note that each $i \in P$ of rank 1 belongs to a unique maximal interval which is a chain; whose top element is $x_i^{r_i}$. By abuse of notation, we identify the elements in such intervals with their corresponding monomials in A.

We add a maximum $\hat{1}$ to P to obtain $\hat{P}$ and define the following ideals in A:

$I_0 = (\prod_{i=1}^n x_i^{a_i} : \exists j\ 1 \leq j \leq n,\ a_j > r_j)$,

$I_1 = (\prod_{i=1}^n x_i^{a_i} : \forall j\ a_j \leq r_j,\ \vee_{i=1}^n x_i^{a_i} = \hat{1} \in \hat{P})$,

$I_2 = (\prod_{i=1}^n x_i^{a_i} : \vee_{i=1}^n x_i^{a_i} \in P, r(\vee_{i=1}^n x_i^{a_i}) \neq \sum_i a_i)$,

$I_3 = (\prod_{i=1}^n x_i^{a_i} - \prod_{i=1}^n x_i^{b_i} : \vee_{i=1}^n x_i^{a_i} = \vee_{i=1}^n x_i^{b_i} \in P, r(\vee_{i=1}^n x_i^{a_i}) = \sum_i a_i = \sum_i b_i)$,

$I_P = I_0 + I_1 + I_2 + I_3$.

Define $k[P] := A/I_P$. As I_P is homogeneous, $k[P]$ inherits a grading from A. Let $f(k[P]) = (f_{-1}, f_0, ..)$ where $f_i = \dim_k\{m \in k[P] : r(m) = i+1\}$ $(f_{-1} = 1)$.

The proof of the following proposition is similar to the proof of Proposition 3.2, and is omitted.

Proposition 3.4 $f(k[P]) = f(P)$. $\square$

Denote by $\tilde{w}$ the projection of $w \in A$ on $k[P]$. Let $B = \{y_1, .., y_n\}$ be a basis of A_1. Then

$$\Delta_B(P) := \{\prod_{i=1}^{n} y_i^{a_i} : \prod_{i=1}^{n} \tilde{y_i}^{a_i} \notin \text{span}_k\{\prod_{i=1}^{n} \tilde{y_i}^{b_i} : \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i, \prod_{i=1}^{n} y_i^{b_i} <_L \prod_{i=1}^{n} y_i^{a_i}\}\}$$

is an order ideal of monomials with an f-vector $f(P)$. (The lexicographic order on monomials of equal degree is defined by $\prod_{i=1}^{n} y_i^{b_i} <_L \prod_{i=1}^{n} y_i^{a_i}$ iff there exists j such that for all $1 \leq t < j$ $a_t = b_t$ and $b_j > a_j$.) To prove this, we reproduce the argument of Stanley for proving Macaulay's theorem ([9], Theorem 2.1): as the projections of the elements in $\Delta_B(P)$ form a k-basis of $k[P]$, then by Proposition 3.4 $f(\Delta_B(P)) = f(P)$. If $m \notin \Delta_B(P)$ then $m = \sum\{a_n n : \deg(n) = \deg(m), n <_L m\}$, hence for any monomial m' $m'm = \sum\{a_n m'n : \deg(n) = \deg(m), n <_L m\}$. But $\deg(m'm) = \deg(m'n)$ and $m'n <_L m'm$ for these n's, hence $m'm \notin \Delta_B(P)$, thus $\Delta_B(P)$ is an order ideal of monomials.

Remark: For B a generic basis the construction is canonical in the same sense as defined for the exterior case.

Combining Proposition 3.4 with Theorem 1.4 we obtain

Corollary 3.5 *Every $P \in \mathbb{P}$ satisfies Macaulay inequalities (3).* $\square$

If P is infinite, let $P_{\leq r} := \{p \in P : r(p) \leq r\}$ and construct $\Delta(P_{\leq r})$ for each r. Then $\Delta(P_{\leq r}) \subseteq \Delta(P_{\leq r+1})$ for every r, and $\Delta(P) := \cup_r \Delta(P_{\leq r})$ is an order ideal of monomials with f-vector $f(P)$. Hence, Corollary 3.5 holds in this case too.

To conclude, I wish to address the following open question to the readers:

Problem 3.6 *Find algebraic objects (such as standard graded rings) and notions of algebraic shifting that support Kruskal-Katona's and Macaulay's inequalities for the general combinatorial objects covered by Theorems 1.2 and 1.6, respectively.*

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