THE FLAG f-VECTORS OF GORENSTEIN* ORDER COMPLEXES OF DIMENSION 3

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ABSTRACT. We characterize the **cd**-indices of Gorenstein* posets of rank 5, equivalently the flag f-vectors of Gorenstein* order complexes of dimension 3. As a corollary, we characterize the f-vectors of Gorenstein* order complexes in dimensions 3 and 4. This characterization rise a speculated intimate connection between the f-vectors of flag homology spheres and the f-vectors of Gorenstein* order complexes.

1. INTRODUCTION

The flag f-vector is an important invariant of graded posets. For a Gorenstein* poset P, its flag f-vector is efficiently encoded by its **cd**-index [1] whose integer coefficients turn out to be non-negative [3, 8]. Further restrictions on the **cd**-index of Gorenstein* posets were obtained recently in [4], however a full characterization is not even conjectured yet. In this paper we make a first step in this direction by characterizing the **cd**-indices of Gorenstein* posets of rank 5, which is the first nontrivial case.

We first recall the definition of the **cd**-index. Let P be a graded poset of rank $n+1$ with the minimal element $\hat{0}$ and the maximal element $\hat{1}$. The *order complex* $\mathcal{O}(P)$ of P (or of $P-\{\hat{0},\hat{1}\}$), is the (abstract) simplicial complex whose faces are the chains of $P-\{\hat{0},\hat{1}\}$. Thus

$$\mathcal{O}(P)=\{\{\sigma_1,\sigma_2,\dots,\sigma_k\}\subseteq P-\{\hat{0},\hat{1}\}:\sigma_1<\sigma_2<\cdots<\sigma_k\}.$$

Let $r:P\to\mathbb{Z}_{\geq 0}$ denote the rank function of P. For $S\subseteq[n]=\{1,2,\dots,n\}$, an element $\{\sigma_1,\dots,\sigma_k\}\in\mathcal{O}(P)$ with $\{r(\sigma_1),\dots,r(\sigma_k)\}=S$ is called an S-*flag* of P. Let $f_S(P)$ be the number of S-flags of P. Define $h_S(P)$ by

$$h_S(P)=\sum_{T\subseteq S}(-1)^{|S|-|T|}f_T(P),$$

where $|X|$ denotes the cardinality of a finite set X. The vectors $(f_S(P):S\subseteq[n])$ and $(h_S(P):S\subseteq[n])$ are called the *flag f-vector* and *flag h-vector* of P (or $\mathcal{O}(P)$) respectively. It is convenient to represent flag h-vectors as coefficients of non-commutative polynomials. For $S\subseteq[n]$, we define a non-commutative monomial $u_S=u_1u_2\cdots u_n$ in variables $\mathbf{a}$ and $\mathbf{b}$ by $u_i=\mathbf{a}$ if $i\notin S$ and $u_i=\mathbf{b}$ if $i\in S$, and

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define

$$\Psi_P(\mathbf{a},\mathbf{b}) = \sum_{S\subseteq[n]} h_S(P)u_S.$$

We say that P is *Gorenstein** if the simplicial complex $\mathcal{O}(P)$ is Gorenstein* [9, p. 67]. A typical example of a Gorenstein* poset comes from CW-spheres, namely, a regular CW-complex which is homeomorphic to a sphere. Indeed, if P is the face poset of a CW-sphere then $P \cup \{\hat{0},\hat{1}\}$ is a Gorenstein* poset. It is known that if P is Gorenstein* then $\Psi_P(\mathbf{a},\mathbf{b})$ can be written as a polynomial $\Phi_P(\mathbf{c},\mathbf{d})$ in $\mathbf{c}=\mathbf{a}+\mathbf{b}$ and $\mathbf{d}=\mathbf{ab}+\mathbf{ba}$ [1], and this non-commutative polynomial $\Phi_P(\mathbf{c},\mathbf{d})$ is called the **cd**-*index* of P.

It is known that the coefficients of $\Phi_P(\mathbf{c},\mathbf{d})$ are non-negative integers [3, 8] and the coefficient of $\mathbf{c}^n$ in $\Phi_P(\mathbf{c},\mathbf{d})$ is 1. The main result of this paper is the next result, which characterizes all possible **cd**-indices of Gorenstein* posets of rank 5.

Theorem 1.1. *The* **cd**-*polynomial* $\mathbf{c}^4+\alpha_1\mathbf{dc}^2+\alpha_2\mathbf{cdc}+\alpha_3\mathbf{c}^2\mathbf{d}+\alpha_{13}\mathbf{d}^2 \in \mathbb{Z}_{\geq 0}\langle \mathbf{c},\mathbf{d}\rangle$ *is the* **cd**-*index of a Gorenstein* poset of rank* 5 *if and only if one of the following conditions holds:*

(i) $\alpha_2 = 0$ *and* $\alpha_{13}=\alpha_1\alpha_3$.
(ii) $\alpha_2 = 1$ *and there are non-negative integers* b_1,b_2,b_3,c_1,c_2,c_3 *such that* $\alpha_1 = b_1+b_2+b_3$, $\alpha_3 = c_1+c_2+c_3$ *and* $\alpha_{13}=\alpha_1\alpha_3-(b_1c_1+b_2c_2+b_3c_3)$.
(iii) $\alpha_2 \geq 2$ *and* $\alpha_{13}\leq\alpha_1\alpha_3$.

Note that, since knowing the **cd**-index of P is equivalent to knowing the flag f-vector of $\mathcal{O}(P)$, Theorem 1.1 characterizes the flag f-vectors of Gorenstein* order complexes of dimension 3.

Recall that, for a Gorenstein* poset P of rank $n+1$, the vector $(h_0,h_1,\dots,h_n)$, where $h_i=\sum_{S\subseteq[n],|S|=i}h_S(P)$, is the usual h-vector of $\mathcal{O}(P)$. Thus, as an immediate corollary of Theorem 1.1, we obtain a characterization of the f-vectors of Gorenstein* order complexes of dimension 3. We extend this later result to dimension 4 as well.

The numerical conditions are conveniently given in terms of **d**-vectors of Gorenstein* posets. For a Gorenstein* poset P of rank $n+1$, we define its **d**-*vector* $\mathbf{d}(P)=(\delta_0,\delta_1,\dots,\delta_{\lfloor\frac{n}{2}\rfloor})$ by

$$\Phi_P(1,\mathbf{d}) = \delta_0+\delta_1\mathbf{d}+\cdots+\delta_{\lfloor\frac{n}{2}\rfloor}\mathbf{d}^{\lfloor\frac{n}{2}\rfloor},$$

where $\lfloor x\rfloor$ is the integer part of x. Thus $\delta_0,\delta_1,\dots,\delta_{\lfloor\frac{n}{2}\rfloor}$ are the coefficients of the polynomial which is obtained from the **cd**-index of P by substituting $\mathbf{c}=1$. Note that, by the definition, $\mathbf{d}(P)$ is a non-negative vector with $\delta_0=1$. Since the **d**-vector of P and the h-vector $h=(h_0,h_1,\dots,h_n)$ of $\mathcal{O}(P)$ are related by $\sum_{i=0}^n h_ix^i = \sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}2^i\delta_ix^i(1+x)^{n-2i}$ (cf. [4, Section 4]), knowing the **d**-vector of P is equivalent to knowing the h-vector of $\mathcal{O}(P)$, equivalently the f-vector of $\mathcal{O}(P)$.

Theorem 1.2. *Let* $(1,x,y)\in\mathbb{Z}^3_{\geq 0}$.

(a) *The vector* $(1,x,y)$ *is the* **d**-*vector of a Gorenstein* poset of rank* 5 *if and only if it satisfies* $y\leq\frac{(x-1)^2}{4}$ *or there are non-negative integers* a *and* b *such that* $x=a+b$ *and* $y=ab$.

(b) *The vector* $(1, x, y)$ *is the* $\mathbf{d}$*-vector of a Gorenstein* poset of rank* 6 *if and only if it satisfies* $y \leq \frac{x^2}{4}$.

We notice that the vectors in part (b) are exactly the γ-vectors of homology 4-spheres (details will be given in Section 5), and that the vectors in part (a) are conjectured in [2] to be exactly the γ-vectors of homology 3-spheres. This gives rise to Question 5.3, about an interesting relation between f-vectors of flag homology spheres and those of Gorenstein* order complexes.

Outline of the paper: in Section 2 we describe a poset construction, which we call *unzipping*. Unzipping is in a certain sense an inverse of zipping [7] and will be used to prove sufficiency in Theorems 1.1 and 1.2; Theorem 1.1 is proved in Section 3; $\mathbf{d}$-vectors are discussed in Section 4 where Theorem 1.2 is proved; in Section 5 we discuss questions relating f-vectors of flag homology spheres with those of Gorenstein* order complexes.

2. Unzipping

We introduce a construction on graded posets which we call *unzipping*. Post composition with the corresponding *zipping*, as defined by Reading [7], gives back the original poset.

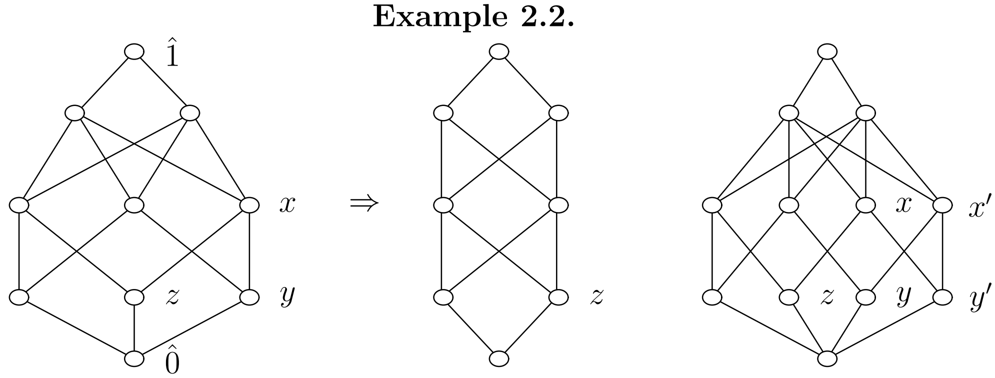
Definition 2.1. Let P be a graded poset with $\hat{0}$ and $\hat{1}$, and let $r : P \to \mathbb{Z}_{\geq 0}$ be the rank function of P.

(1) (Reading [7]) Let $x, y, z \in P - \{\hat{0}, \hat{1}\}$ be such that (i) x covers exactly y and z, (ii) x is the unique minimal upper bound of y and z, and (iii) y and z cover exactly the same elements. Let $\mathcal{Z}(P; x, y, z)$ be the poset obtained from P by deleting x, y and adding the relations $w > z$ for all relations $w > y$.

(2) Let $x, y \in P - \{\hat{0}, \hat{1}\}$ be such that x covers y. We define the graded poset $\mathcal{U}(P; x, y)$ as follows: delete the cover relation $y < x$, add elements x', y' with ranks $r(x') = r(x), r(y') = r(y)$ and add cover relations (i) $x' < w$ for all covers $x < w$, (ii) $w < y'$ for all covers $w < y$ and (iii) $y' < x', y < x'$ and $y' < x$.

The operations $P \to \mathcal{Z}(P; x, y, z)$ and $P \to \mathcal{U}(P; x, y)$ are called *zipping* and *unzipping* respectively.

Example 2.2.



P $\mathcal{Z}(P; x, y, z)$ $\mathcal{U}(P; x, y)$

Remark 2.3. In general $\mathcal{Z}(P;x,y,z)$ may not be graded. However, $\mathcal{Z}(P;x,y,z)$ is graded if P is *thin*, namely, if for every $x \geq y$ in P with $r(x)-r(y)=2$ the closed interval $[x,y]$ is a Boolean algebra of rank 2. See [7, Proposition 4.4].

In the rest of this section, we study basic properties of zipping and unzipping. Let Δ be a simplicial complex on the vertex set V. The *link of* $F \in \Delta$ *in* Δ is the simplicial complex

$$\mathrm{lk}_\Delta(F) = \{G \subseteq V \setminus F : G \cup F \in \Delta\}.$$

Definition 2.4. Let Δ be a simplicial complex and let $\{i,j\}$ be an edge of Δ. The *(stellar) edge subdivision of* Δ *with respect to* $\{i,j\}$ is the simplicial complex

$$\{F \in \Delta : F \not\supset \{i,j\}\} \cup \{F \cup \{v\},\ F \cup \{i,v\},\ F \cup \{j,v\} : F \in \mathrm{lk}_\Delta(\{i,j\})\},$$

where v is a new vertex. The *edge contraction of* i *to* j *in* Δ is the simplicial complex Δ' which is obtained from Δ by identifying the vertices i and j, in other words,

$$\Delta' = \{F \in \Delta : i \notin F\} \cup \{(F \setminus \{i\}) \cup \{j\} : i \in F \in \Delta\}.$$

Proposition 2.5. *With the same notation as in Definition 2.1,*

1. $\mathcal{Z}(\mathcal{U}(P;x,y);x',y',y) = P$.
2. $\mathcal{O}(\mathcal{Z}(P;x,y,z))$ *is obtained from* $\mathcal{O}(P)$ *by two successive edge contractions: first contract* y *to* x*, then contract* x *to* z*.*
3. $\mathcal{O}(\mathcal{U}(P;x,y))$ *is obtained from* $\mathcal{O}(P)$ *by two successive edge subdivisions: first subdivide* $\{x,y\}$ *by* x'*, then subdivide* $\{x,x'\}$ *by* y'*.*

Proof. Part (1) follows directly from Definition 2.1.

By the definition of the edge contraction, contracting y to x in $\mathcal{O}(P)$ and then contracting x to z results in the simplicial complex Δ obtained from $\mathcal{O}(P)$ by replacing x and y by z in all simplices (and deleting repetitions, of vertices in a simplex, and of simplices). As (i) $z \in \mathcal{Z}(P;x,y,z)$ covers the same elements as y and z in P, (ii) $w > z$ in $\mathcal{Z}(P;x,y,z)$ if and only if $w \neq x$ and either $w > y$ or $w > z$ in P, and (iii) x covers only $y, z \in P$, we conclude that the simplices of Δ are exactly the chains in $\mathcal{Z}(P;x,y,z) - \{\hat{0},\hat{1}\}$, proving (2).

To prove (3), let Γ be the simplicial complex obtained from $\mathcal{O}(P)$ by two successive edge subdivisions: first subdivide $\{x,y\}$ by x', then subdivide $\{x,x'\}$ by y'. The maximal simplices of Γ are obtained from those of $\mathcal{O}(P)$ by replacing each $x,y \in F \in \mathcal{O}(P)$ in three ways, by either x,y', or y',x', or x',y; thus a maximal simplex $F \in \mathcal{O}(P)$ corresponds to 3 simplices in Γ. Thus, by construction, these are exactly the maximal chains in $\mathcal{U}(P;x,y) - \{\hat{0},\hat{1}\}$. □

We say that a simplicial complex Δ satisfies the *Link condition with respect to an edge* $\{i,j\} \in \Delta$ if $\mathrm{lk}_\Delta(\{i\}) \cap \mathrm{lk}_\Delta(\{j\}) = \mathrm{lk}_\Delta(\{i,j\})$. As the edge contractions in Proposition 2.5 satisfy the Link Condition w.r.t. $\{x,y\}$ and $\{x,z\}$ in the corresponding simplicial complexes, they preserve the PL-type for simplicial spheres [5, Theorem 1.4], and preserve being a homology sphere [6, Proposition 2.3]. Hence we conclude the following result, obtained by Reading [7, Theorem 4.7] for the case of zipping in Gorenstein* posets [7].

Corollary 2.6. *If P is a Gorenstein* poset (or a CW-sphere) then so are $\mathcal{U}(P;x,y)$ and $\mathcal{Z}(P;x,y,z)$.*

We use the following formula, observed by Reading [7, Theorem 4.6], later on.

Lemma 2.7 (Reading)**.** *The* **cd***-index changes under zipping as follows:*

$$\Phi_P(\mathbf{c},\mathbf{d}) = \Phi_{\mathcal{Z}(P;x,y,z)}(\mathbf{c},\mathbf{d}) + \Phi_{[\hat{0},y]}(\mathbf{c},\mathbf{d}) \cdot \mathbf{d} \cdot \Phi_{[x,\hat{1}]}(\mathbf{c},\mathbf{d}).$$

Thus, for unzipping we get

$$\Phi_{\mathcal{U}(P;x,y)}(\mathbf{c},\mathbf{d}) = \Phi_P(\mathbf{c},\mathbf{d}) + \Phi_{[\hat{0},y]}(\mathbf{c},\mathbf{d}) \cdot \mathbf{d} \cdot \Phi_{[x,\hat{1}]}(\mathbf{c},\mathbf{d}).$$

3. cd-indices of Gorenstein* posets of rank 5

In this section, we prove our first main result, Theorem 1.1.

We first recall the join of two posets. Let P and Q be posets with $\hat{0}$ and $\hat{1}$. The *join* $P * Q$ is the poset on the set $(P - \{\hat{1}\}) \cup (Q - \{\hat{0}\})$ with $x \leq y$ if either (i) $x \leq y$ in P, (ii) $x \leq y$ in Q, or (iii) $x \in P$ and $y \in Q$. It is not hard to see that $\mathcal{O}(P * Q)$ is the join of $\mathcal{O}(P)$ and $\mathcal{O}(Q)$ (as simplicial complexes). Thus, if P and Q are Gorenstein* then so is $P * Q$. The following formula was given in [8, Lemma 1.1].

Lemma 3.1. *If P and Q are Gorenstein* posets then $\Phi_{P*Q}(\mathbf{c},\mathbf{d}) = \Phi_P(\mathbf{c},\mathbf{d})\Phi_Q(\mathbf{c},\mathbf{d})$.*

In the rest of this section, we focus on Gorenstein* posets of rank 5. For a Gorenstein* poset of rank 5, we write its **cd**-index in the form

$$\Phi_P(\mathbf{c},\mathbf{d}) = \mathbf{c}^4 + \alpha_1(P)\mathbf{d}\mathbf{c}^2 + \alpha_2(P)\mathbf{c}\mathbf{d}\mathbf{c} + \alpha_3(P)\mathbf{c}^2\mathbf{d} + \alpha_{13}(P)\mathbf{d}^2.$$

We often use the following formula. (We refer the readers for verification of the formula).

$$f_{\{2,3\}}(P) = \alpha_{13}(P) + 2(\alpha_1(P) + \alpha_2(P) + \alpha_3(P)) + 4. \tag{1}$$

We first study necessary conditions of **cd**-indices of Gorenstein* posets of rank 5. The following results were shown in [4, Propositions 4.4 and 4.6].

Lemma 3.2. *If P is a Gorenstein* poset of rank 5 then $\alpha_{13}(P) \leq \alpha_1(P)\alpha_3(P)$.*

Lemma 3.3. *Let P be a Gorenstein* poset of rank 5 with $\alpha_2(P) = 0$. Then there are Gorenstein* posets Q_1 and Q_2 of rank 3 such that $P = Q_1 * Q_2$. In particular, $\alpha_{13}(P) = \alpha_1(P)\alpha_3(P)$.*

The next result gives a new restriction on **cd**-indices.

Lemma 3.4. *Let P be a Gorenstein* poset of rank 5. If $\alpha_2(P) = 1$ then there are non-negative integers $b_1, b_2, b_3, c_1, c_2, c_3$ such that $\alpha_1(P) = b_1 + b_2 + b_3$, $\alpha_3(P) = c_1 + c_2 + c_3$ and $\alpha_{13}(P) = \alpha_1(P)\alpha_3(P) - (b_1c_1 + b_2c_2 + b_3c_3)$.*

Proof. Consider the subposet $Q = \{\sigma \in P : r(\sigma) \in \{1,2\}\}$, where $r : P \to \mathbb{Z}_{\geq 0}$ is the rank function of P. Since $(\hat{0}, x)$ is the face poset of a cycle for any rank 3 element $x \in P$, the poset Q is the face poset of a CW-complex Γ which is a union of cycles. Also, Q is a Cohen-Macaulay poset since Q is a rank selected subposet of P (cf. [9, III, Theorem 4.5]), which implies that the CW-complex Γ is connected. Similarly, let Q' be the dual poset of $\{\sigma \in P : r(\sigma) \in \{3,4\}\}$. Then, by considering the dual of the above argument, it follows that Q' is also the face poset of a connected CW-complex Γ' which is a union of cycles.

Since $f_{\{1\}}(P) = 2 + \alpha_1(P)$ and $f_{\{2\}}(P) = 2 + \alpha_1(P) + \alpha_2(P)$, $\alpha_2(P) = 1$ implies that the number of edges in Γ is equal to the number of vertices in Γ plus 1. Since Γ is a union of cycles, this fact shows that Γ is a union of two simple cycles C and C' such that the intersection of C and C' is either a point or a nontrivial simple path (namely, a simple path with at least one edge).

Suppose $C \cap C'$ is a point. Then C and C' are the only simple cycles in Γ. Thus, for each rank 3 element $x \in P$, $(\hat{0}, x)$ is equal to the face poset of C or that of C'. Then, for each rank 4 element $y \in P$, y must cover exactly two elements x and x' with $(\hat{0}, x) = (\hat{0}, x')$ since P is Gorenstein*. However, this fact shows that Γ' has two connected components, which contradicts the connectedness of Γ'. Hence $C \cap C'$ is a nontrivial simple path.

Let C'' be the cycle in Γ obtained from $C \cup C'$ by removing the interior of $C \cap C'$. Then, C, C' and C'' are the only simple cycles in Γ, and for each rank 3 element $x \in P$, $(\hat{0}, x)$ must coincide with C, C' or C''. Let c_1, c_2, c_3 be the number of rank 3 elements $x \in P$ such that $(\hat{0}, x)$ coincides with C, C', C'' respectively. Let b_1, b_2, b_3 be the *lengths* of the simple paths $C' \cap C'', C \cap C'', C \cap C'$ respectively, by means of number of edges. Clearly $b_i \geq 1$. Also, since Γ' is connected, by using the same argument as when we concluded that $C \cap C'$ is a nontrivial simple path, we have $c_i \geq 1$.

Then, we have

$$\begin{aligned} f_{\{2,3\}}(P) &= c_1(b_2 + b_3) + c_2(b_1 + b_3) + c_3(b_1 + b_2) \\ &= (b_1 + b_2 + b_3)(c_1 + c_2 + c_3) - (b_1c_1 + b_2c_2 + b_3c_3). \end{aligned}$$

Set $b_i' = b_i - 1$ and $c_i' = c_i - 1$ for $i = 1, 2, 3$. Then, since $\alpha_2(P) = 1$, $\alpha_1(P) = f_{\{2\}}(P) - 3 = b_1' + b_2' + b_3'$ and $\alpha_3(P) = f_{\{3\}}(P) - 3 = c_1' + c_2' + c_3'$. By using (1) and the above equation, a routine computation shows $\alpha_{13}(P) = \alpha_1(P)\alpha_3(P) - (b_1'c_1' + b_2'c_2' + b_3'c_3')$, as desired. □

Now we prove Theorem 1.1. In the proof, we use the following notation. Let P be a Gorenstein* poset and let σ and τ be elements of $P \setminus \{\hat{0}, \hat{1}\}$ such that σ covers τ. We say that Q is obtained from P by unzipping (σ, τ) k times if Q is obtained by the following successive process: First, unzip (σ, τ) and consider $P' = \mathcal{U}(P; \sigma, \tau)$. This unzipping creates new elements σ' and τ' such that σ' covers τ' in P'. Next, unzip (σ', τ') in P', and consider $P'' = \mathcal{U}(P'; \sigma', \tau')$. Again, we obtain new elements σ'' and τ'' such that σ'' covers τ'' in P'', and continue this procedure k times. Note that, by Lemma 2.7, we have $\Phi_Q(\mathbf{c}, \mathbf{d}) = \Phi_P(\mathbf{c}, \mathbf{d}) + k \cdot \Phi_{[\hat{0}, \tau]}(\mathbf{c}, \mathbf{d}) \cdot \mathbf{d} \cdot \Phi_{[\sigma, \hat{1}]}(\mathbf{c}, \mathbf{d})$.

Proof of Theorem 1.1. The necessity follows from Lemmas 3.2, 3.3 and 3.4. We prove the sufficiency. Let C_k be the Gorenstein* poset of rank 3 corresponding to a cycle of length k (i.e. $C_k - \{\hat{0}, \hat{1}\}$ is the face poset of a cycle of length k).

(i) Observe that $\Phi_{C_k}(\mathbf{c},\mathbf{d}) = \mathbf{c}^2 + (k-2)\mathbf{d}$. Then, for all non-negative integers α_1 and α_3, the join $C_{\alpha_1+2} * C_{\alpha_3+2}$ is a Gorenstein* poset of rank 5 with the desired **cd**-index

$$\Phi_{C_{\alpha_1+2}*C_{\alpha_3+2}}(\mathbf{c},\mathbf{d}) = \Phi_{C_{\alpha_1+2}}(\mathbf{c},\mathbf{d}) \cdot \Phi_{C_{\alpha_3+2}}(\mathbf{c},\mathbf{d}) = \mathbf{c}^4 + \alpha_1\mathbf{dc}^2 + \alpha_3\mathbf{c}^2\mathbf{d} + \alpha_1\alpha_3\mathbf{d}^2.$$

(ii) Let $Q = \hat{B}_2 * C_3 * \hat{B}_2$ described in figure (a), where $\hat{B}_2$ is the Boolean algebra of rank 2. Note that $\Phi_Q(\mathbf{c},\mathbf{d}) = \mathbf{c}^4 + \mathbf{cdc}$ by Lemma 3.1.

Let R be the Gorenstein* poset obtained from Q by unzipping (τ_i, ρ) b_i times for $i = 1,2,3$ and by unzipping (π, σ_i) c_i times for $i = 1,2,3$. We claim that R has the desired **cd**-index. By Lemma 2.7, $\alpha_1(R) = b_1 + b_2 + b_3$, $\alpha_2(R) = 1$ and $\alpha_3(R) = c_1 + c_2 + c_3$. It remains to prove $\alpha_{13}(R) = \alpha_1(R)\alpha_3(R) - (b_1c_1 + b_2c_2 + b_3c_3)$.

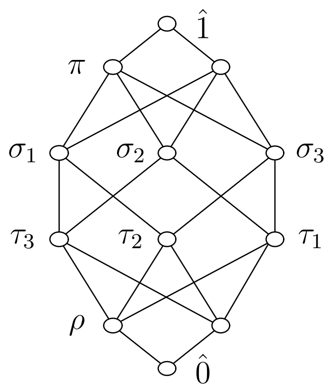


figure (a)

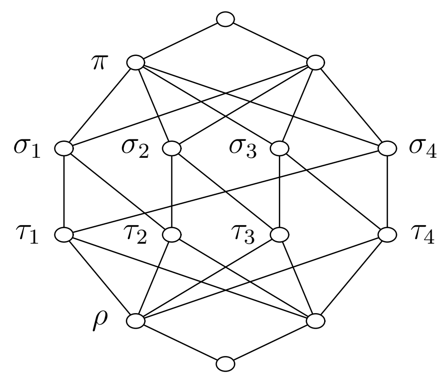


figure (b)

For $i = 1,2,3$, let B_i be the set of rank 2 elements of R that consists of τ_i and elements which are added when we unzip (τ_i, ρ) b_i times, and define the set C_i of rank 3 elements of R with $\sigma_i \in C_i$ similarly. Then, since each element of C_i exactly covers all rank 2 elements which are not in B_i,

$$\begin{aligned} f_{\{2,3\}}(R) &= |C_1|(|B_2| + |B_3|) + |C_2|(|B_1| + |B_3|) + |C_3|(|B_1| + |B_2|) \\ &= (|B_1|+|B_2|+|B_3|)(|C_1|+|C_2|+|C_3|) - (|B_1||C_1|+|B_2||C_2|+|B_3||C_3|). \end{aligned}$$

Observe $b_i = |B_i| - 1$ and $c_i = |C_i| - 1$ for $i = 1,2,3$. Then, in the same way as in the proof of Lemma 3.4, (1) and routine computations guarantee $\alpha_{13}(R) = \alpha_1(R)\alpha_3(R) - (b_1c_1 + b_2c_2 + b_3c_3)$.

(iii) Let $\alpha_1, \alpha_2, \alpha_3$ and α_{13} be non-negative integers with $\alpha_2 \geq 2$ and $\alpha_{13} \leq \alpha_1\alpha_3$. Let $Q' = \hat{B}_2 * C_4 * \hat{B}_2$ described in figure (b). Note that $\Phi_Q(\mathbf{c},\mathbf{d}) = \mathbf{c}^4 + 2\mathbf{cdc}$.

Case 1. Suppose $\alpha_1 = 0$. Then $\alpha_{13} = 0$ by the assumption. Let P be the Gorenstein* poset obtained from Q' by unzipping (π, σ_1) α_3 times and by unzipping (σ_1, τ_1) $(\alpha_2 - 2)$ times. Lemma 2.7 shows that P has the desired **cd**-index $\mathbf{c}^4 + \alpha_2\mathbf{cdc} + \alpha_3\mathbf{c}^2\mathbf{d}$.

Case 2. Suppose $\alpha_1 > 0$ and $\alpha_{13} \leq \alpha_1$. Let R' be the Gorenstein* poset obtained from Q' by applying the following unzipping.

(U1) unzip (τ_1, ρ) α_{13} times;
(U2) unzip (τ_4, ρ) $(\alpha_1 - \alpha_{13})$ times;
(U3) unzip (π, σ_1);

(U4) unzip (π, σ_2) $(\alpha_3 - 1)$ times.

By Lemma 2.7, $\alpha_1(R') = \alpha_{13} + (\alpha_1 - \alpha_{13}) = \alpha_1$, $\alpha_2(R') = 2$ and $\alpha_3(R') = 1 + (\alpha_3 - 1) = \alpha_3$. We claim $\alpha_{13}(R') = \alpha_{13}$.

Let B_1 be the set of rank 2 elements of R' consisting of τ_1 and elements which are added by unzipping (U1), and let B_4 be that consisting of τ_4 and elements added by (U2). Similarly, let C_1 be the set of rank 3 elements of R' consisting of σ_1 and the element added by (U3), and let C_2 be that consisting of τ_2 and elements added by (U4). Finally, let $B_2 = \{\tau_2\}$, $B_3 = \{\tau_3\}$, $C_3 = \{\sigma_3\}$ and $C_4 = \{\sigma_4\}$.

Then $B_1 \cup \cdots \cup B_4$ and $C_1 \cup \cdots \cup C_4$ are partitions of the sets of elements of R' of ranks 2 and 3 respectively. Also, elements in C_i exactly cover elements in B_i and B_{i+1}, where we consider $B_5 = B_1$. This implies

$$\begin{aligned} f_{\{2,3\}}(R) &= |C_1|(|B_1|+|B_2|)+|C_2|(|B_2|+|B_3|)+|C_3|(|B_3|+|B_4|)+|C_4|(|B_4|+|B_1|) \\ &= 2 \cdot (\alpha_{13} + 2) + \alpha_3 \cdot 2 + 1 \cdot (\alpha_1 - \alpha_{13} + 2) + 1 \cdot (\alpha_1 + 2) \\ &= \alpha_{13} + 2(\alpha_1 + 2 + \alpha_3) + 4. \end{aligned}$$

Hence $\alpha_{13}(R) = \alpha_{13}$ by (1).

Finally, consider the Gorenstein* poset P obtained from R' by unzipping (σ_1, τ_1) $(\alpha_2 - 2)$ times. Then Lemma 2.7 show that P has the desired **cd**-index

$$\Phi_P(\mathbf{c}, \mathbf{d}) = \Phi_{R'}(\mathbf{c}, \mathbf{d}) + (\alpha_2 - 2)\mathbf{cdc} = \mathbf{c}^4 + \alpha_1\mathbf{dc}^2 + \alpha_2\mathbf{cdc} + \alpha_3\mathbf{c}^2\mathbf{d} + \alpha_{13}\mathbf{d}^2.$$

Case 3. Suppose $\alpha_1 > 0$ and $\alpha_{13} > \alpha_1$. Recall $\alpha_{13} \leq \alpha_1\alpha_3$. Let $\beta \leq \alpha_3$ be the integer satisfying

$$\alpha_1(\beta - 1) < \alpha_{13} \leq \alpha_1\beta$$

and let

$$p = \alpha_1\beta - \alpha_{13}.$$

Note that $\beta \geq 2$ and $0 \leq p < \alpha_1$. Let R' be the Gorenstein* poset obtained from Q' by applying the following unzipping.

(U1) unzip (τ_1, ρ) $(\alpha_1 - p)$ times and unzip (τ_2, ρ) p times;

(U2) unzip (π, σ_1) $(\beta - 1)$ times, unzip (π, σ_3) $(\alpha_3 - \beta)$ times, and unzip (π, σ_4).

Then, Lemma 2.7 shows $\alpha_1(R') = \alpha_1$, $\alpha_2(R') = 2$ and $\alpha_3(R') = \alpha_3$. Also, a computation similar to Case 2 shows

$$\begin{aligned} f_{\{2,3\}}(R) &= \beta \cdot (\alpha_1 + 2) + 1 \cdot (p + 2) + (\alpha_3 - \beta + 1) \cdot 2 + 2 \cdot (\alpha_1 - p + 2) \\ &= (\beta\alpha_1 - p) + 2(\alpha_3 + 2 + \alpha_1) + 4. \end{aligned}$$

Since $\alpha_{13} = \beta\alpha_1 - p$, the above equation and (1) show $\alpha_{13}(R') = \alpha_{13}$.

Then the Gorenstein* poset P obtained from R' by unzipping (σ_1, τ_1) $(\alpha_2 - 2)$ times has the desired **cd**-index. □

4. d-vectors of Gorenstein* posets of rank 5 and 6

In this section, we study **d**-vectors of Gorenstein* posets of rank 5 and 6. We often use the following obvious fact.

Lemma 4.1. *Let x and y be non-negative integers. Then $y \leq \frac{x^2}{4}$ if and only if there are non-negative integers a and b such that $y \leq ab$ and $a + b \leq x$.*

We first classify $\mathbf{d}$-vectors of Gorenstein* posets of rank 5.

Theorem 4.2. *The vector* $(1,x,y) \in \mathbb{Z}^3_{\geq 0}$ *is the* $\mathbf{d}$*-vector of a Gorenstein* poset of rank* 5 *if and only if it satisfies* $y \leq \frac{(x-1)^2}{4}$ *or there are non-negative integers* a *and* b *such that* $x = a+b$ *and* $y = ab$.

Proof. (Necessity). Let P be a Gorenstein* poset and $\mathbf{d}(P) = (1,x,y)$. Suppose $y > \frac{(x-1)^2}{4}$. We show that there are non-negative integers a and b such that $x = a+b$ and $y = ab$.

Observe that $x = \alpha_1(P) + \alpha_2(P) + \alpha_3(P)$ and $y = \alpha_{13}(P)$. Then

$$\frac{(x-1)^2}{4} < y = \alpha_{13}(P) \leq \alpha_1(P)\alpha_3(P) \leq \frac{(\alpha_1(P)+\alpha_3(P))^2}{4} \leq \frac{x^2}{4}.$$

This says that $\alpha_1(P) + \alpha_3(P) = x$ and therefore $\alpha_2(P) = 0$. Then Theorem 1.1(i) shows $y = \alpha_1(P)\alpha_3(P)$.

(Sufficiency). For all non-negative integers a and b, $\mathbf{d}(C_{a+2} * C_{b+2}) = (1, a+b, ab)$ by Lemma 3.1. Let $(1,x,y) \in \mathbb{Z}^3_{\geq 0}$ with $y \leq \frac{(x-1)^2}{4}$. What we must prove is that there is a Gorenstein* poset of rank 5 such that $\mathbf{d}(P) = (1,x,y)$.

Since $y \leq \frac{(x-1)^2}{4}$, there are non-negative integers a and b such that $y \leq ab$ and $a+b \leq x-1$. We may choose these integers so that $a(b-1) < y \leq ab$. Let $\alpha_1 = a$, $\alpha_2 = x-a-b$, $\alpha_3 = b$ and $\alpha_{13} = y$. It is enough to show that there is a Gorenstein* poset of rank 5 whose $\mathbf{cd}$-index is $\mathbf{c}^4 + \alpha_1\mathbf{dc}^2 + \alpha_2\mathbf{cdc} + \alpha_3\mathbf{c}^2\mathbf{d} + \alpha_{13}\mathbf{d}^2$. If $\alpha_2 \geq 2$ then the existence of such a poset follows from Theorem 1.1(iii). If $\alpha_2 < 2$ then $\alpha_2 = 1$. We claim that $\alpha_1, \alpha_3, \alpha_{13}$ satisfy the conditions in Theorem 1.1(ii). If $b = 0$ then the statement is obvious. If $b > 0$ then, since $0 \leq (ab-y) < a$, the partition of integers $a = (ab-y) + 0 + (a-(ab-y))$ and $b = 1 + (b-1) + 0$ shows that $\alpha_1, \alpha_3, \alpha_{13}$ satisfy the desired conditions. □

Next, we consider Gorenstein* posets of rank 6.

Theorem 4.3. *The vector* $(1,x,y) \in \mathbb{Z}^3_{\geq 0}$ *is the* $\mathbf{d}$*-vector of a Gorenstein* poset of rank* 6 *if and only if it satisfies* $y \leq \frac{x^2}{4}$.

Proof. (Necessity). Let P be a Gorenstein* poset of rank 6 and $\mathbf{d}(P) = (1,x,y)$. We write the $\mathbf{cd}$-index of P in the form

$$\Phi(P) = c^5 + \alpha_1\mathbf{dc}^3 + \alpha_2\mathbf{cdc}^2 + \alpha_3\mathbf{c}^2\mathbf{dc} + \alpha_4\mathbf{c}^3\mathbf{d} + \alpha_{13}\mathbf{d}^2\mathbf{c} + \alpha_{14}\mathbf{dcd} + \alpha_{24}\mathbf{cd}^2.$$

It was proved in [4, Proposition 4.4] that $\alpha_{13} \leq \alpha_1\alpha_3$, $\alpha_{14} \leq \alpha_1\alpha_4$ and $\alpha_{24} \leq \alpha_2\alpha_4$. Then, since $x = \alpha_1+\alpha_2+\alpha_3+\alpha_4$ and $y = \alpha_{13}+\alpha_{14}+\alpha_{24}$, we have

$$y \leq \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_4 \leq (\alpha_1+\alpha_2)(\alpha_3+\alpha_4) \leq \frac{x^2}{4},$$

as desired.

(Sufficiency). Let $(1,x,y) \in \mathbb{Z}^3_{\geq 0}$ with $y \leq \frac{x^2}{4}$. We show that there is a Gorenstein* poset of rank 6 whose $\mathbf{d}$-vector is $(1,x,y)$. Since $\mathbf{d}(P) = \mathbf{d}(P * \hat{B}_2)$ for any Gorenstein* poset P of rank 5, by Theorem 4.2, we may assume $\frac{(x-1)^2}{4} < y \leq \frac{x^2}{4}$.

Then there are positive integers a and b such that $a(b-1) < y \leq ab$ and $a+b = x$. Let $r = ab - y$. Then $0 \leq r < a$ and

$$(b-1)(a-r) + r(b-1) + (a-r) = y.$$

Let $Q = C_{a-r+2} * \hat{B}_2 * C_{b+1}$. By Lemma 3.1, $\mathbf{d}(Q) = (1, a-r+b-1, (a-r)(b-1))$. Let ρ be a rank 2 element of P, π a rank 4 element of P, and let σ, τ be the rank 3 elements of P. Note that $[\hat{0}, \rho] = [\pi, \hat{1}] = \hat{B}_2$, $[\hat{0}, \tau] = C_{a-r+2}$ and $[\sigma, \hat{1}] = C_{b+1}$. Let P be the Gorenstein* poset obtained from Q by unzipping (σ, ρ) r times and by unzipping (π, τ). Then, by Lemma 2.7, we have

$$\begin{aligned}\mathbf{d}(P) &= \mathbf{d}(Q) + r(0,1,b-1) + (0,1,a-r)\\ &= (1, a+b, (a-r)(b-1) + r(b-1) + (a-r))\\ &= (1, x, y),\end{aligned}$$

as desired. □

Remark 4.4. Gal [2] proved that $\gamma_2(\Delta) \leq \frac{\gamma_1(\Delta)^2}{4}$ for any flag homology 4-sphere Δ (see Section 5 for details). This result and the relation between $\mathbf{d}$-vectors and γ-vectors give an alternative proof of the necessity of Theorem 4.3.

Remark 4.5. For the posets P constructed to show sufficiency in Theorems 1.1, 4.2 and 4.3, their order complexes $\mathcal{O}(P)$ are *polytopal*, namely, can be realized as the boundary of a polytope.

Indeed, it is not hard to see that P is obtained from the join $\hat{B}_2 * \hat{B}_2 * \cdots * \hat{B}_2$ of Boolean algebras of rank 2 by applying unzipping repeatedly. The order complex of the join of Boolean algebras of rank 2 is the boundary of a cross polytope. As edge subdivisions preserve polytopality (just place the new vertex *beyond* the edge, cf. [10, p. 78]), by Proposition 2.5(3) we conclude that $\mathcal{O}(P)$ is polytopal.

5. Questions

Let Δ be an $(n-1)$-dimensional simplicial complex. Recall that the *f-vector* $f(\Delta) = (1, f_0, f_1, \dots, f_{n-1})$ of Δ is defined by $f_i = |\{F \in \Delta : |F| = i+1\}|$, and the *$h$-vector* $h(\Delta) = (h_0, h_1, \dots, h_n)$ of Δ is defined by the relation

$$\sum_{i=0}^{n} h_i x^{n-i} = \sum_{i=0}^{n} f_{i-1}(x-1)^{n-i}.$$

If Δ is Gorenstein* then $h_i = h_{n-i}$ for all i by the Dehn-Sommerville equations, and in this case the γ-vector $\gamma(\Delta) = (\gamma_0, \gamma_1, \dots, \gamma_{\lfloor \frac{n}{2} \rfloor})$ of Δ is defined by the relation

$$\sum_{i=0}^{n} h_i x^i = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_i x^i (1+x)^{n-2i}.$$

Thus, if Δ is the order complex $\mathcal{O}(P)$ of a Gorenstein* poset P, $\gamma(\Delta)$ and $\mathbf{d}(P) = (1, \delta_1, \dots, \delta_{\lfloor \frac{n}{2} \rfloor})$ are related by $\gamma_i = 2^i \delta_i$ for all i.

A simplicial complex is said to be *flag* if every minimal non-face has at most two elements. The study of γ-vectors of flag homology spheres (namely, flag Gorenstein* complexes) is one of current trends in face enumeration. On γ-vectors of flag homology spheres, one of the most important open problems is the conjecture of Gal [2, Conjecture 2.1.7] which states that the γ-vector of a flag homology sphere is non-negative. Gal's conjecture is known to be true in dimension ≤ 4. Moreover, in low dimension Gal essentially proved the following result.

Theorem 5.1 (Gal). *Let Λ_k be the set of γ-vectors of flag homology k-spheres.*

(i) $\Lambda_3 \supset \{(1,x,y) \in \mathbb{Z}^3_{\geq 0} : y \leq \frac{(x-1)^2}{4}$ *or* $(x,y) = (a+b, ab)$ *for some* $a, b \in \mathbb{Z}_{\geq 0}\}$.
(ii) $\Lambda_4 = \{(1,x,y) \in \mathbb{Z}^3_{\geq 0} : y \leq \frac{x^2}{4}\}$,

Proof. (i) is proved in [2, Theorem 3.2.1] and $\Lambda_4 \subseteq \{(1,x,y) \in \mathbb{Z}^3_{\geq 0} : y \leq \frac{x^2}{4}\}$ is proved in [2, Theorem 3.1.3]. It remains to show the reverse containment in (ii).

We sketch the proof since it is similar to that of Theorem 4.3. Let $(1,x,y) \in \mathbb{Z}^3_{\geq 0}$ with $y \leq \frac{x^2}{4}$. As taking suspension does not change the γ-vector, we may assume $\frac{(x-1)^2}{4} < y \leq \frac{x^2}{4}$; in particular $x \geq 2$ and $y \geq 1$. Then there are $a, b, r \in \mathbb{Z}_{\geq 0}$ with $a, b \geq 1$ and $r < a$ such that $a+b = x$ and $ab-r = y$. Consider the simplicial complex $K = \tilde{C}_{a-r+4} * \tilde{B}_2 * \tilde{C}_{b+3}$, where $\tilde{C}_k$ is the cycle (1-dimensional simplicial sphere) of length k and where $\tilde{B}_2$ is the 0-sphere with vertices $\{x,y\}$. (Here we consider the join as simplicial complexes.) Then, by multiplicativity of γ w.r.t. joins (cf. [2, Remark 2.1.9]), $\gamma(K) = (1, a-r+b-1, (a-r)(b-1))$. Let $\{s\} \in C_{a-r+4}$ and $\{t\} \in C_{b+3}$. If we subdivide the edge $\{x,s\}$ r-times and subdivide the edge $\{y,t\}$, by [2, Proposition 2.4.3], we obtain a flag 4-sphere Δ with $\gamma(\Delta) = (1,x,y)$. $\square$

Moreover, Gal [2, Conjecture 3.2.2] conjectured the following, which, if true, gives a complete characterization of the f-vectors of flag homology 3-spheres.

Conjecture 5.2 (Gal)**.** Let Δ be a flag homology 3-sphere and let $\gamma(\Delta) = (1, \gamma_1, \gamma_2)$. If $\gamma_2 > \frac{(\gamma_1-1)^2}{2}$ then Δ is the join of two cycles.

Note that Lemma 3.3 and the proof of Theorem 4.2 show that the above conjecture is true for order complexes. If the above conjecture is true then the inclusion in Theorem 5.1(i) becomes an equality. These facts and Theorem 1.2 suggest the following question.

Question 5.3. Let $\mathcal{D}_k$ be the set of $\mathbf{d}$-vectors of Gorenstein* posets of rank $k+2$. Is there any relation between Λ_k and $\mathcal{D}_k$? Is it true that $\Lambda_k = \mathcal{D}_k$ for all k? Or at least does Λ_k contain $\mathcal{D}_k$?

Note that equality in Question 5.3 would imply Gal's conjecture on the non-negativity of the γ-vector of flag spheres, by Karu's result on the non-negativity of the $\mathbf{cd}$-index of Gorenstein* posets [3].

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References

[1] Margaret M. Bayer and Andrew Klapper. A new index for polytopes. *Discrete Comput. Geom.*, 6(1):33–47, 1991.

[2] Światosław R. Gal. Real root conjecture fails for five- and higher-dimensional spheres. *Discrete Comput. Geom.*, 34(2):269–284, 2005.

[3] Kalle Karu. The *cd*-index of fans and posets. *Compos. Math.*, 142(3):701–718, 2006.

[4] Satoshi Murai and Eran Nevo. On the **cd**-index and γ-vector of s*-shellable cw-spheres. *Math. Z.*, to appear.

[5] Eran Nevo. Higher minors and Van Kampen's obstruction. *Math. Scand.*, 101(2):161–176, 2007.

[6] Eran Nevo and Eyal Novinsky. A characterization of simplicial polytopes with $g_2 = 1$. *J. Combin. Theory Ser. A*, 118(2):387–395, 2011.

[7] Nathan Reading. The cd-index of Bruhat intervals. *Electron. J. Combin.*, 11(1):Research Paper 74, 25, 2004.

[8] Richard P. Stanley. Flag f-vectors and the *cd*-index. *Math. Z.*, 216(3):483–499, 1994.

[9] Richard P. Stanley. *Combinatorics and commutative algebra*, volume 41 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, second edition, 1996.

[10] Günter M. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

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