ON THE cd-INDEX AND $\gamma$-VECTOR OF S*-SHELLABLE CW-SPHERES

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Abstract. We show that the $\gamma$-vector of the order complex of any polytope is the $f$-vector of a balanced simplicial complex. This is done by proving this statement for a subclass of Stanley’s S-shellable spheres which includes all polytopes. The proof shows that certain parts of the cd-index, when specializing $c = 1$ and considering the resulted polynomial in $d$, are the $f$-polynomials of simplicial complexes that can be colored with “few” colors. We conjecture that the cd-index of a regular CW-sphere is itself the flag $f$-vector of a colored simplicial complex in a certain sense.

1. Introduction

Let $P$ be an $(n-1)$-dimensional regular CW-sphere (that is, a regular CW-complex which is homeomorphic to an $(n-1)$-dimensional sphere). In face enumeration, one of the most important combinatorial invariants of $P$ is the cd-index. The cd-index $\Phi_P(c,d)$ of $P$ is a non-commutative polynomial in the variables $c$ and $d$ that encodes the flag $f$-vector of $P$. By the result of Stanley [St1] and Karu [Ka], it is known that the cd-index $\Phi_P(c,d)$ has non-negative integer coefficients. On the other hand, a characterization of the possible cd-indices for regular CW-spheres, or other related families, e.g Gorenstien* posets, is still beyond reach. In this paper we take a step in this direction and establish some non-trivial upper bounds, as we detail now.

If we substitute 1 for $c$ in $\Phi_P(c,d)$, we obtain a polynomial of the form

$$\Phi_P(1,d) = \delta_0 + \delta_1 d + \cdots + \delta_{\lfloor \frac{n}{2} \rfloor} d^{\lfloor \frac{n}{2} \rfloor},$$

where $\lfloor \frac{n}{2} \rfloor$ is the integer part of $\frac{n}{2}$, such that each $\delta_i$ is a non-negative integer. In other words, $\delta_i$ is the sum of coefficients of monomials in $\Phi_P(c,d)$ for which $d$ appears $i$ times.

Let $\Delta$ be a (finite abstract) simplicial complex on the vertex set $V$. We say that $\Delta$ is $k$-colored if there is a map $c : V \to [k] = \{1, 2, \ldots, k\}$, called a $k$-coloring map of $\Delta$, such that if $\{x, y\}$ is an edge of $\Delta$ then $c(x) \neq c(y)$. Let $f_i(\Delta)$ denote the number of elements $F \in \Delta$ having cardinality $i + 1$, where $f_{-1}(\Delta) = 1$. The main result of this paper is the following.

Theorem 1.1. Let $P$ be an $(n-1)$-dimensional S*-shellable regular CW-sphere, and let $\Phi_P(1,d) = \delta_0 + \delta_1 d + \cdots + \delta_{\lfloor \frac{n}{2} \rfloor} d^{\lfloor \frac{n}{2} \rfloor}$. Then there exists an $\lfloor \frac{n}{2} \rfloor$-colored simplicial complex $\Delta$ such that

$$\delta_i = f_{i-1}(\Delta) \text{ for } i = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor.$$
The precise definition of the $S^*$-shellability is given in Section 2. The most important class of $S^*$-shellable CW-spheres are the boundary complexes of polytopes. By the Kruskal-Katona Theorem (see e.g. [St2, II, Theorem 2.1]), the above theorem gives certain upper bound on $\delta_i$ in terms of $\delta_{i-1}$. Better upper bounds are given by Frankl-Füredi-Kalai theorem which characterizes the $f$-vectors of $k$-colored complexes [FFK].

The numbers $\delta_0, \delta_1, \delta_2, \ldots$ relate to the $\gamma$-vector (see Section 4 for the definition) of the barycentric subdivision (order complex) of $P$, namely the simplicial complex whose elements are the chains of nonempty cells in $P$ ordered by inclusion. Indeed, as an application of Theorem 1.1 we prove the following.

**Theorem 1.2.** Let $P$ be an $(n-1)$-dimensional $S^*$-shellable regular CW-sphere and let $sd(P)$ be the barycentric subdivision of $P$. Then there exists an $\lfloor \frac{n}{2} \rfloor$-colored simplicial complex $\Gamma$ such that

$$\gamma_i(sd(P)) = f_{i-1}(\Gamma) \quad \text{for } i = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor.$$ 

 Recall that an $(n-1)$-dimensional simplicial complex is said to be balanced if it is $n$-colored. If $P$ is the boundary complex of an arbitrary convex $n$-dimensional polytope, then $\delta_{\lfloor \frac{n}{2} \rfloor}(P) > 0$ and we conclude the following.

**Corollary 1.3.** Let $P$ be the boundary complex of an $n$-dimensional polytope. Then the $\gamma$-vector of $sd(P)$ is the $f$-vector of a balanced simplicial complex.

The above corollary supports the conjecture of Nevo and Petersen [NP, Conjecture 6.3] which states that the $\gamma$-vector of a flag homology sphere is the $f$-vector of a balanced simplicial complex. This conjecture was verified for the barycentric subdivision of simplicial homology spheres (in this case all the cells are simplices) in [NPT].

It would be natural to ask if the above theorems hold for all regular CW-spheres (or more generally, Gorenstein* posets). We conjecture a stronger statement on the cd-index, see Conjecture 4.3.

This paper is organized as follows: in Section 2 we recall some known results on the cd-index and define $S^*$-shellability, in Section 3 we prove our main theorem, Theorem 1.1, in Section 4 we derive consequences for $\gamma$-vectors and present a conjecture on the cd-index, Conjecture 4.3.

## 2. cd-index of $S^*$-shellable CW-spheres

In this section we recall some known results on the cd-index.

Let $P$ be a graded poset of rank $n+1$ with the minimal element $\hat{0}$ and the maximal element $\hat{1}$. Let $\rho$ denote the rank function of $P$. For $S \subseteq [n] = \{1, 2, \ldots, n\}$, a chain $\hat{0} = \sigma_0 < \sigma_1 < \sigma_2 < \cdots < \sigma_{k+1} = \hat{1}$ of $P$ is called an $S$-flag if $\{\rho(\sigma_1), \ldots, \rho(\sigma_k)\} = S$. Let $f_S(P)$ be the number of $S$-flags of $P$. Define $h_S(P)$ by

$$h_S(P) = \sum_{T \subseteq S} (-1)^{|S| - |T|} f_T(P),$$

where $|X|$ denotes the cardinality of a finite set $X$. The vectors $(f_S(P) : S \subseteq [n])$ and $(h_S(P) : S \subseteq [n])$ are called the flag $f$-vector and flag $h$-vector of $P$ respectively.
Now we recall the definition of the $cd$-index. For $S \subset [n]$, we define a non-commutative monomial $u_S = u_1u_2 \cdots u_n$ in variables $a$ and $b$ by $u_i = a$ if $i \notin S$ and $u_i = b$ if $i \in S$. Let

$$\Psi_P(a, b) = \sum_{S \subset [n]} h_P(S)u_S.$$ 

For a graded poset $P$, let $sd(P)$ be the order complex of $P - \{\emptyset, \hat{1}\}$. Thus

$$sd(P) = \{\{\sigma_1, \sigma_2, \ldots, \sigma_k\} \subset P - \{\emptyset, \hat{1}\} : \sigma_1 < \sigma_2 < \cdots < \sigma_k\}.$$ 

We say that $P$ is Gorenstein* if the simplicial complex $sd(P)$ is a homology sphere. It is known that if $P$ is Gorenstein* then $\Psi_P(a, b)$ can be written as a polynomial $\Phi_P(c, d)$ in $c = a + b$ and $d = ab + ba$ [BK], and this non-commutative polynomial $\Phi_P(c, d)$ is called the cd-index of $P$. Moreover, by the celebrated results due to Stanley [St1] (for convex polytopes) and Karu [Ka] (for Gorenstein* posets), the coefficients of $\Phi_P(c, d)$ are non-negative integers.

Next, we define S*-shellability of regular CW-spheres by slightly modifying the definition of S-shellability introduced by Stanley [St1, Definition 2.1].

Let $P$ be a regular CW-sphere (a regular CW-complex which is homeomorphic to a sphere) and $F(P)$ its face poset. Then the order complex of $F(P)$ is a triangulation of a sphere, so the poset $F(P) \cup \{0, \hat{1}\}$ is Gorenstein*. We define the cd-index of $P$ by $\Phi_P(c, d) = \Phi_{F(P) \cup \{0, \hat{1}\}}(c, d)$. For any cell $\sigma$ of $P$, we write $\bar{\sigma}$ for the closure of $\sigma$. For an $(n - 1)$-dimensional regular CW-sphere $P$, let $\Sigma P$ be the suspension of $P$, in other words, $\Sigma P$ is the $n$-dimensional regular CW-sphere obtained from $P$ by attaching two $n$-dimensional cells $\tau_1$ and $\tau_2$ such that $\partial \tau_1 = \partial \tau_2 = P$. Also, for an $(n - 1)$-dimensional regular CW-ball $P$ (a regular CW-complex which is homeomorphic to an $(n - 1)$-dimensional ball), let $P'$ be the $(n - 1)$-dimensional regular CW-sphere which is obtained from $P$ by adding an $(n - 1)$-dimensional cell $\tau$ so that $\partial \tau = \partial P$.

**Definition 2.1.** Let $P$ be an $(n - 1)$-dimensional regular CW-sphere. We say that $P$ is S*-shellable if either $P = \{\emptyset\}$ or there is an order $\sigma_1, \sigma_2, \ldots, \sigma_r$ of the facets of $P$ such that the following conditions hold.

(a) $\partial \bar{\sigma}_1$ is S*-shellable.
(b) For $1 \leq i \leq r - 1$, let

$$\Omega_i = \bar{\sigma}_1 \cup \bar{\sigma}_2 \cup \cdots \cup \bar{\sigma}_i$$

and for $2 \leq i \leq r - 1$ let

$$\Gamma_i = [\partial \bar{\sigma}_i \backslash (\partial \bar{\sigma}_i \cap \Omega_{i-1})].$$

Then both $\Omega_i$ and $\Gamma_i$ are regular CW-balls of dimension $(n - 1)$ and $(n - 2)$ respectively, and $\Gamma_i'$ is S*-shellable with the first facet of the shelling being the facet which is not in $\Gamma_i$. 

**Remark 2.2.** The difference between the above definition and Stanley’s S-shellability is that S-shellability only assume that $P$ and $\Gamma_i'$ are Eulerian and assume no conditions on $\Omega_i$. However, S*-shellable regular CW-spheres are S-shellable, and the boundary complex of convex polytopes are S*-shellable by the line shelling [BM]. We leave the verification of this fact to the readers.
The next recursive formula is due to Stanley [St1].

Lemma 2.3 (Stanley). With the same notation as in Definition 2.1 for \( i = 1, 2, \ldots, r - 2 \), one has

\[
\Phi_{i+1}^j(c, d) = \Phi_i^j(c, d) + \left\{ \Phi_{i+1}^j(c, d) - \Phi_{\partial_i^j(\partial_i^j \Pi^j)}(c, d) \right\} c + \Phi_{\partial_i^j}(c, d)d.
\]

Since \( \Pi^j_{i-1} = P \) the above formula gives a way to compute the \( cd \)-index of \( P \) recursively.

Next, we recall a result of Ehrenborg and Karu proving that the \( cd \)-index increases by taking subdivisions. Let \( P \) and \( Q \) be regular CW-complexes, and let \( \phi : F(P) \rightarrow F(Q) \) be a poset map. For a subcomplex \( Q' = \sigma_1 \cup \cdots \cup \sigma_s \subset Q \), where each \( \sigma_i \) is a cell of \( Q \), we write \( \phi^{-1}(Q') = \phi^{-1}(\sigma_1) \cup \cdots \cup \phi^{-1}(\sigma_s) \).

Following [EK, Definition 2.6], for \((n-1)\)-dimensional regular CW-spheres \( P \) and \( \hat{P} \), we say that \( \hat{P} \) is a subdivision of \( P \) if there is an order preserving surjective poset map \( \phi : F(\hat{P}) \rightarrow F(P) \), satisfying that for any cell \( \sigma \) of \( P \), \( \phi^{-1}(\sigma) \) is a homology ball having the same dimension as \( \sigma \) and \( \partial \phi^{-1}(\partial \sigma) = \partial(\phi^{-1}(\partial \sigma)) \).

The following result was proved in [EK, Theorem 1.5].

Lemma 2.4 (Ehrenborg-Karu). Let \( P \) and \( \hat{P} \) be \((n-1)\)-dimensional regular CW-spheres. If \( \hat{P} \) is a subdivision of \( P \) then one has a coefficientwise inequality \( \Phi_{\hat{P}}(c, d) \geq \Phi_{P}(c, d) \)

Back to \( S^* \)-shellable regular CW-spheres, with the same notation as in Definition 2.1 \( \Omega_i^j \) is a subdivision of \( \Sigma(\partial \Omega_i) \) and \( \partial \Omega_i \) is a subdivision of \( \Sigma(\partial \Gamma_{i+1}) \). Indeed, for the first statement, if \( \tau_1 \) and \( \tau_2 \) are the facets of \( \Sigma(\partial \Omega_i) \) then define \( \phi : F(\Omega_i^j) \rightarrow F(\Sigma(\partial \Omega_i)) \) by

\[
\phi(\sigma) = \begin{cases} 
\sigma, & \text{if } \sigma \in \partial \Omega_i, \\
\tau_1, & \text{if } \sigma \text{ is an interior face of } \Omega_i, \\
\tau_2, & \text{if } \sigma \notin \Omega_i.
\end{cases}
\]

Similarly, for the second statement, if \( \tau_1 \) and \( \tau_2 \) are the facets of \( \Sigma(\partial \Gamma_{i+1}) \) then define \( \phi : F(\partial \Omega_i) \rightarrow F(\Sigma(\partial \Gamma_{i+1})) \) by

\[
\phi(\sigma) = \begin{cases} 
\sigma, & \text{if } \sigma \in \partial \Gamma_{i+1}, \\
\tau_1, & \text{if } \sigma \in \partial_i^j \Gamma_{i+1}, \\
\tau_2, & \text{otherwise}.
\end{cases}
\]

Since \( \Phi_{\Sigma P}(c, d) = \Phi_P(c, d)c \) for any regular CW-sphere \( P \) (see [St1, Lemma 1.1]), Lemma 2.4 shows

Lemma 2.5. With the same notation as in Definition 2.1 for \( i = 2, 3, \ldots, r - 2 \), one has \( \Phi_{i+1}(c, d) \geq \Phi_{\partial_i^j}(c, d)c^2 \).

3. PROOF OF THE MAIN THEOREM

In this section, we prove Theorem 1.1. For a homogeneous \( cd \)-polynomial \( \Phi \) (i.e., homogeneous polynomial of \( \mathbb{Z}(c, d) \) with \( \deg c = 1 \) and \( \deg d = 2 \)) of degree \( n \), we define \( \Phi_0, \Phi_2, \ldots, \Phi_n \) by

\[
\Phi = \Phi_0 + \Phi_2 dc^{n-2} + \Phi_3 dc^{n-3} + \cdots + \Phi_{n-1} dc + \Phi_n d.
\]
where $\Phi_0 = \alpha c^n$ for some $\alpha \in \mathbb{Z}$ and each $\Phi_k$ is a $cd$-polynomial of degree $k - 2$ for $k \geq 2$. Also, we write $\Phi_{\leq k} = \Phi_0 + \Phi_2 dc^{n-2} + \cdots + \Phi_k dc^{n-k}$.

**Definition 3.1.**

- A vector $(\delta_0, \delta_1, \ldots, \delta_s) \in \mathbb{Z}^{s+1}$ is said to be $k$-FFK if there is a $k$-colored simplicial complex $\Delta$ such that $\delta_i = f_{i-1}(\Delta)$ for $i = 0, 1, \ldots, s$. \{\emptyset\} is a 0-colored simplicial complex. A homogeneous $cd$-polynomial $\Phi = \Phi(c, d)$ is said to be $k$-FFK if, when we write $\Phi(1, d) = \delta_0 + \delta_1 d + \cdots + \delta_s d^s$, the vector $(\delta_0, \delta_1, \ldots, \delta_s)$ is $k$-FFK.
- A homogeneous $cd$-polynomial $\Phi$ of degree $n$ is said to be primitive if the coefficient of $c^n$ in $\Phi$ is 1.
- Let $\Phi$ be a homogeneous $cd$-polynomial. A primitive homogeneous $cd$-polynomial $\Psi$ is said to be $k$-good for $\Phi$ if $\Psi$ is $k$-FFK and $\Phi(1, d) \geq \Psi(1, d)$. Also, we say that a homogeneous $cd$-polynomial $\Psi$ is $k$-good for $\Phi$ if it is the sum of primitive homogeneous $cd$-polynomials that are $k$-good for $\Phi$.

We will use the following observation, which follows from [NPT, Lemma 3.1]:

**Lemma 3.2.** If $\Phi$ is a $k$-FFK homogeneous $cd$-polynomial of degree $n$, and if $\Psi'$ and $\Psi''$ are homogeneous $cd$-polynomials of degree $n'$ and $n''$ respectively, where $n', n'' \leq n - 2$, which are $k$-good for $\Phi$ then

$$\Phi + \Psi' dc^{n-n'-2} \text{ and } \Phi + \Psi' dc^{n-n'-2} + \Psi'' dc^{n-n''-2}$$

are $(k + 1)$-FFK.

*Proof.* By Frankl-Füredi-Kalai theorem [FFK], for any $k$-colored simplicial complex $\Gamma$, there is the unique $k$-colored simplicial complex $C(\Gamma)$, called a $k$-colored compressed complex, such that $f_i(\Gamma) = f_i(C(\Gamma))$ for all $i$. Moreover, if $\Gamma'$ is a $k$-colored complex satisfying $f_i(\Gamma') \leq f_i(C(\Gamma))$ for all $i$, then one has $C(\Gamma') \subset C(\Gamma)$.

For a simplicial complex $\Gamma$, we write $f(\Gamma, d) = 1 + f_0(\Gamma) d + f_1(\Gamma) d^2 + \cdots$. There are $k$-colored complexes $\Delta, \Delta^{(1)}, \ldots, \Delta^{(m)}, \ldots, \Delta^{(s)}$ such that $f(\Delta, d) = \Phi(1, d), \sum_{1 \leq i \leq m} f(\Delta^{(i)}, d) = \Psi'(1, d), \sum_{m+1 \leq i \leq s} f(\Delta^{(i)}, d) = \Psi''(1, d)$ and each $\Delta^{(i)}$ is a subcomplex of $\Delta$. Let

$$\Gamma^{(i)} = \Delta \bigcup \left\{ \bigcup_{k=1}^{i} \left\{ F \cup \{v_k\} : F \in \Delta^{(k)} \right\} \right\},$$

where $v_1, \ldots, v_s$ are new vertices. Since each $\Delta^{(k)}$ is a subcomplex of $\Delta$, $\Gamma^{(i)}$ is a simplicial complex. Also, $f(\Gamma^{(m)}, d) = (\Phi + \Psi' dc^{n-n'-2})(1, d)$ and $f(\Gamma^{(s)}, d) = (\Phi + \Psi' dc^{n-n'-2} + \Psi'' dc^{n-n''-2})(1, d)$. We claim that each $\Gamma^{(i)}$ is $(k + 1)$-colored. Let $V$ be the vertex set of $\Delta$ and $c : V \to [k]$ a $k$-coloring map of $\Delta$. Then the map $\hat{c} : V \cup \{v_1, \ldots, v_s\} \to [k + 1]$ defined by $\hat{c}(x) = c(x)$ if $x \in V$ and $\hat{c}(x) = k + 1$ if $x \notin V$ is a $(k + 1)$-coloring map of $\Gamma^{(i)}$. \hfill $\square$

Let $P$ be an $(n - 1)$-dimensional $S^*$-shellable regular CW-sphere with the shelling $\sigma_1, \ldots, \sigma_r$. Keeping the notation in Definition 2.1 to simplify notations, we use the
following symbols.

\[
\Phi^{(i)} = \Phi^{(i)}(c, d) = \Phi_{\Omega_1}(c, d) \\
\Phi = \Phi_{\Gamma}(c, d) = \Phi^{(r-1)} \\
\Psi^{(i)} = \Phi_{\Gamma_{i+1}}(c, d) - \Phi_{\Sigma(\partial \Gamma_{i+1})}(c, d) \\
\Psi = \sum_{i=1}^{r-2} \Psi^{(i)} \\
\Pi = \Phi - \Phi^{(1)}.
\]

Thus Stanley’s recursive formula, Lemma 2.3, says

\[
\Phi^{(i+1)} = \Phi^{(i)} + \Psi^{(i)} c + \Phi_{\partial \Gamma_{i+1}} d
\]

and

\[
\Pi = \Psi c + \sum_{i=1}^{r-2} \Phi_{\partial \Gamma_{i+1}}(c, d)d.
\]

The last part of the following proposition is a restatement of Theorem 1.1.

**Proposition 3.3.** With notation as above, the following holds.

1. For \(2 \leq k \leq n\), \(\Psi_k^{(i)}\) is \(\lfloor \frac{k}{2} - 1 \rfloor\)-good for \(\Phi_{\leq k-2}^{(i)} + \Psi_{\leq k-2}^{(i)} c\).
2. For \(2 \leq k \leq n\), \(\Pi_k = \lfloor \frac{k}{2} - 1 \rfloor\)-good for \(\Phi_{\leq k-2}^{(i)} + \Pi_{\leq k-2}\).
3. For \(2 \leq k \leq n\), \(\Phi_{\leq k}\) is \(\lfloor \frac{k}{2} - 1 \rfloor\)-good for \(\Phi_{\leq k-2}\).
4. For \(0 \leq k \leq n\), \(\Phi_{\leq k}\) is \(\lfloor \frac{k}{2} \rfloor\)-FFK. In particular, the \(cd\)-index of \(P\) is \(\lfloor \frac{n}{2} \rfloor\)-FFK.

**Proof.** The proof is by induction on dimension, where all statements clearly hold for \(n = 0, 1\). Suppose that all statements are true up to dimension \(n - 2\). To simplify notations, for a regular CW-sphere \(Q\), we write \(\Phi_Q = \Phi_Q(c, d)\).

**Proof of (1).** By applying the induction hypothesis to \(\Gamma_{i+1}'\) (use statement (2)), each \(\Psi_k^{(i)}\) is \(\lfloor \frac{k}{2} - 1 \rfloor\)-good for \(\Phi_{\Sigma(\partial \Gamma_{i+1})}^{(i)} \leq k-2 + \Psi_{\leq k-2}^{(i)} c\). Since \((\Sigma c)_k = \Sigma c_k\) for any homogeneous \(cd\)-polynomial \(\Sigma\), \(\Psi_k^{(i)}\) is \(\lfloor \frac{k}{2} - 1 \rfloor\)-good for \(\Phi_{\Sigma(\partial \Gamma_{i+1})}^{(i)} \leq k-2 c + \Psi_{\leq k-2}^{(i)} c\).

By Lemma 2.5,

\[
\Phi_{\Sigma(\partial \Gamma_{i+1})} c = \Phi_{\partial \Gamma_{i+1}} c^2 \leq \Phi_{\Omega_1}^{(i)} = \Phi^{(i)},
\]

thus \(\Psi_k^{(i)}\) is \(\lfloor \frac{k}{2} - 1 \rfloor\)-good for \(\Phi_{\leq k-2}^{(i)} + \Psi_{\leq k-2}^{(i)} c\).

**Proof of (2).** By the definition of \(\Pi\),

\[
\Pi_k = \sum_{i=1}^{r-1} \Psi_k^{(i)} \text{ for } k < n
\]

and

\[
\Pi_n = \sum_{i=1}^{r-2} \Phi_{\partial \Gamma_{i+1}}.
\]

By (1), each \(\Psi_k^{(i)}\) is \(\lfloor \frac{k}{2} - 1 \rfloor\)-good for \(\Phi_{\leq k-2}^{(i)} + \Psi_{\leq k-2}^{(i)} c\). Then since

\[
\Phi_{\leq k-2}^{(i)} + \Psi_{\leq k-2}^{(i)} c \leq \Phi_{\leq k-2}^{(1)} + \Pi_{\leq k-2},
\]

...
Π_k is \([k/2 - 1]\)-good for \(\Phi_{\leq k - 2}^{(1)} + \Pi_{\leq k - 2}\) for \(k < n\). Also, each \(\Phi_{\partial i + 1}\) is \([n/2 - 1]\)-FFK by the induction hypothesis (use (4)), and \(\Phi_{\partial i + 1} \leq \Phi^{(i)}\) by Lemma 2.5. The latter condition clearly says
\[
\Phi_{\partial i + 1} c^2 \leq \Phi_{\leq n - 2}^{(i)} \leq \Phi_{\leq n - 2} = \Phi_{\leq n - 2} + \Pi_{\leq n - 2}.
\]
Hence \(\Pi_n\) is \([n/2 - 1]\)-good for \(\Phi_{\leq n - 2} + \Pi_{\leq n - 2}\).

Proof of (3). Observe that since \(\Phi^{(1)} = \Phi_{\partial 1} c\),
\[
\Phi_k = \Phi_k^{(1)} + \Psi_k \quad \text{for} \quad k < n
\]
and
\[
\Phi_n = \Pi_n.
\]
We already proved that \(\Phi_n = \Pi_n\) is \([n/2 - 1]\)-good for \(\Phi_{\leq n - 2}\) in the proof of (2).

Suppose \(k < n\). Since \(\Phi^{(1)} = \Phi_{\partial 1} c\), by the induction hypothesis (use (3)), \(\Phi^{(1)}\) is \([k/2 - 1]\)-good for \(\Phi_{k - 2}^{(1)}\). Since \(\Phi_{\leq k - 2}^{(1)} \leq \Phi_{\leq k - 2}\) and since we already proved that \(\Psi_k = \Pi_k\) is \([k/2 - 1]\)-good for \(\Phi_{\leq k - 2}\) in the proof of (2), \(\Phi_k\) is \([k/2 - 1]\)-good for \(\Phi_{\leq k - 2}\).

Proof of (4). This statement easily follows from (3). For \(k = 0, 1\), the statement is obvious (as \(\Phi_{\leq 0} = \Phi_{\leq 1} = c^n\)). Suppose that \(\Phi_{\leq 2m + 1}\) is \(m\)-FFK, where \(m \in \mathbb{Z}_{\geq 0}\). Then both \(\Phi_{2m + 2}\) and \(\Phi_{2m + 3}\) are \(m\)-good for \(\Phi_{\leq 2m + 1}\) by (3), and therefore \(\Phi_{\leq 2m + 2}\) and \(\Phi_{\leq 2m + 3}\) are \((m + 1)\)-FFK by Lemma 3.2.

4. \(\gamma\)-vectors of polytopes and a conjecture on the \(cd\)-index

\(\gamma\)-vectors and the \(cd\)-index. Let \(\Delta\) be an \((n - 1)\)-dimensional simplicial complex. Then the \(h\)-vector \(h(\Delta) = (h_0, h_1, \ldots, h_n)\) of \(\Delta\) is defined by the relation
\[
\sum_{i=0}^{n} h_i x^{n-i} = \sum_{i=0}^{n} f_{i-1}(\Delta)(x-1)^{n-i}.
\]
If \(\Delta\) is a homology sphere (that is, a triangulation of a sphere), or more generally a \(h\)-vector of \(\Delta\) is defined by the relation
\[
\sum_{i=0}^{n} h_i x^{i} = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^{i}(1 + x)^{n-2i}.
\]
It was conjectured by Gal [Ga] that if \(\Delta\) is a flag homology sphere then its \(\gamma\)-vector is non-negative. Recently Nevo and Peterson [NP] further conjectured that the \(\gamma\)-vector of a flag homology sphere is the \(f\)-vector of a balanced simplicial complex. These conjectures are open in general, the latter conjecture was verified for barycentric subdivisions of simplicial homology spheres [NPT], and Gal’s conjecture is known to be true for barycentric subdivisions of regular CW-spheres by the following fact, combined with Karu’s result on the nonnegativity of the \(cd\)-index for Gorenstein* posets:

Let \(P\) be an \((n - 1)\)-dimensional regular CW-sphere. The \textit{barycentric subdivision} \(sd(P)\) of \(P\) is the order complex of \(F(P)\). Let \((h_0, h_1, \ldots, h_n)\) and \((\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor n/2 \rfloor})\) be the \(h\)-vector and \(\gamma\)-vector of \(sd(P)\), respectively. Then it is easy to see that
Let $w$ be a homogeneous $cd$-polynomial of degree $n$ (where $0 \leq s_i$ for all $i$ and $s_0 + \ldots + s_k + 2k = n$), let $F_w$ be the following subset of $[n - 1]$:

$$F_w = \{s_0 + 1, s_0 + s_1 + 3, s_0 + s_1 + s_2 + 5, \ldots, s_0 + \cdots + s_k - 1 + 2k - 1\}.$$ 

Note that $F_w$ contains no two consecutive numbers. For example, $F_{c^0} = \emptyset$, $F_{d^k} = \{1, 3, \ldots, 2k - 1\}$ and $F_{cd^k} = \{2, 4, \ldots, 2k\}$. Let $\mathcal{A}$ be the set of subsets of $[n - 1]$ that have no two consecutive numbers, and let $\mathcal{B}$ be the set of $cd$-monomials of degree $n$. Then $w \mapsto F_w$ is a bijection from $\mathcal{B}$ to $\mathcal{A}$ (as $k = |F_w|$ and $s_k = n - 2k - s_k - 1 - \cdots - s_0$ we see that the inverse map exists).

Let $\Delta$ be a $k$-colored simplicial complex with the vertex set $V$ and a $k$-coloring map $c : V \to [k]$. For any subset $S \subset [k]$, let $f_S(\Delta) = |\{F \in \Delta : c(F) = S\}|$. The vector $(f_S(\Delta) : S \subset [k])$ is called the flag $f$-vector of $\Delta$. Note that the flag $f$-vector of a Gorenstein* poset $P$ is equal to the flag $f$-vector of $sd(P)$ by the coloring map defined by the rank function.

**Definition 4.2.** Let $\Phi = \sum_w a_w w$ be a homogeneous $cd$-polynomial of degree $n$ with $w$ the $cd$-monomials and $a_w \in \mathbb{Z}$. For $S \subset [n - 1]$, we define

$$\alpha_S(\Phi) = \begin{cases} a_w, & \text{if } S = F_w \text{ for some } w \in \mathcal{B} \\ 0, & \text{if } S \notin \mathcal{A}. \end{cases}$$
Conjecture 4.3. Let $P$ be an $(n - 1)$-dimensional regular CW-sphere (or more generally, Gorenstein* poset of rank $n + 1$). Then there exists an $(n - 1)$-colored simplicial complex $\Delta$ such that $f_S(\Delta) = \alpha_S(\Phi_P)$ for all $S \subset [n - 1]$.

Thus the above conjecture states that the cd-index is itself the flag $f$-vector of a colored complex. If the above conjecture is true then $\Phi_P(1, d) = 1 + f_0(\Delta)d + \cdots + f_{\left\lfloor \frac{n}{2} \right\rfloor - 1}(\Delta)d^{\left\lfloor \frac{n}{2} \right\rfloor}$. Although $\Delta$ is $(n - 1)$-colored, this fact implies Theorem 1.1.

Indeed, since $f_S(\Delta) = \alpha_S(\Phi_P) = 0$ if $S$ has consecutive numbers, if $c : V \to [n - 1]$ is an $(n - 1)$-coloring map of $\Delta$ then the map $\hat{c} : V \to [\left\lfloor \frac{n}{2} \right\rfloor]$ defined by $\hat{c}(v) = \left\lfloor \frac{c(v) + 1}{2} \right\rfloor$ is an $\left\lfloor \frac{n}{2} \right\rfloor$-coloring map of $\Delta$.

The next result supports the conjecture in low dimension.

Proposition 4.4. Let $P$ be a Gorenstein* poset of rank $n + 1$. For all $i, j \in [n - 1]$, 
$$\alpha_{\{i\}}(\Phi_P)\alpha_{\{j\}}(\Phi_P) \geq \alpha_{\{i, j\}}(\Phi_P).$$

Proof. Let $(h_S(P) : S \subset [n])$ be the flag $h$-vector of $P$. Let $\{i, i + j\} \subset [n - 1]$ with $j \geq 2$. What we must prove is $\alpha_{\{i\}}(\Phi_P)\alpha_{\{i, j\}}(\Phi_P) \geq \alpha_{\{i, i + j\}}(\Phi_P)$.

Observe that
$$h_{[i, i+j+1, \ldots, n]}(P) = \alpha_{\{i, i+j\}}(\Phi_P) + \alpha_{\{i\}}(\Phi_P) + \alpha_{\{i, j\}}(\Phi_P) + \alpha_{\emptyset}(\Phi_P),$$
$$h_{[i]}(P) = \alpha_{\{i\}}(\Phi_P) + \alpha_{\emptyset}(\Phi_P),$$
$$h_{[i, i+j+1, \ldots, n]}(P) = \alpha_{\{i, j\}}(\Phi_P) + \alpha_{\emptyset}(\Phi_P),$$
(as $h_{[i]}(P)$ is the coefficient of $b_i^a$ in $\Psi_P(a, b)$, etc.). Since $\alpha_{\emptyset} = 1$, it is enough to prove that 
$$h_{[i]}(P)h_{[i, i+j+1, \ldots, n]}(P) \geq h_{[n-i-j], [n-i+1, \ldots, n]}(P).$$

It follows from [St2, III, Theorem 4.6] that there is an $n$-colored simplicial complex $\Delta$ with a coloring map $c : V \to [n]$ such that $f_S(\Delta) = h_S(P)$ for all $S \subset [n]$. Let 
$$\Delta_S = \{F \in \Delta : c(F) = S\}$$
for $S \subset [n]$. Then it is clear that 
$$\Delta_{[i]} \cup \Delta_{[i+j+1, \ldots, n]} \subset \{F \cup G : F \in \Delta_{[i]}, G \in \Delta_{[i+j+1, \ldots, n]}\},$$
which implies the desired inequality. 

It is straightforward that the above proposition proves the next statement.

Corollary 4.5. Conjecture 4.3 holds for $n \leq 5$.

Non-existence of d-polynomials. For a Gorenstein* poset $P$, we call $\Phi_P(1, d)$ the $d$-polynomial of $P$. It is a challenging problem to classify all possible $d$-polynomials of Gorenstein* posets, which give a complete characterization of all possible face vectors of Gorenstein* order complexes since knowing the d-polynomials is equivalent to knowing the $\gamma$-vectors. The problem is open even for the 3-dimensional case. To study this problem, by virtue of Theorem 1.1 it is natural to ask which FFK vector is realizable as the $d$-polynomial of a Gorenstein* poset. The next result shows that not all $\left\lfloor \frac{n}{2} \right\rfloor$-FFK vectors are realizable as the $d$-polynomial of a Gorenstein* poset of rank $n + 1$.

First recall that the ordinal sum $Q_1 + Q_2$ of two disjoint posets $Q_1$ and $Q_2$ is the poset whose elements are the union of elements in $Q_1$ and $Q_2$ and whose relations
are those in $Q_1$ union those in $Q_2$ union all $q_1 < q_2$ where $q_1 \in Q_1$ and $q_2 \in Q_2$.

For Gorenstein* posets $Q_1$ and $Q_2$, the poset $Q_1 \ast Q_2 = (Q_1 - \{1\}) + (Q_2 - \{0\})$ is called the join of $Q_1$ and $Q_2$, and $\Sigma Q_1 = Q_1 \ast B_2$, where $B_2$ is a Boolean algebra of rank 2, is called the suspension of $Q_1$. By [11], $\Phi_{Q_1 \ast Q_2}(c, d) = \Phi_{Q_1}(c, d) \cdot \Phi_{Q_2}(c, d)$.

**Proposition 4.6.** Let $P$ be a Gorenstein* poset of rank 5, and let

$$\Phi_P(c, d) = c^4 + \alpha_{\{1\}}c^2d + \alpha_{\{2\}}cdc + \alpha_{\{3\}}dc^2 + \alpha_{\{1,3\}}d^2$$

be its cd-index. Suppose $\alpha_{\{2\}} = 0$. Then there are Gorenstein* posets $P_1$ and $P_2$ of rank 3 such that $P = P_1 \ast P_2$. In particular, $\alpha_{\{1,3\}} = \alpha_{\{1\}}\alpha_{\{3\}}$.

**Proof.** Let $r$ denote the rank function $r : P \to \{0, 1, \ldots, 5\}$ ($r(\emptyset) = 0$, $r(\hat{1}) = 5$). Let $P_1 := \{ F \in P : r(F) \leq 2 \}$ and $P_2 := \{ F \in P : r(F) \geq 3 \}$.

As $P$ is Gorenstein*, to show that $P = P_1 \cup P_2$ it is enough to show that $P_2 \cup \{\emptyset\}$ is Gorenstein* (as a Gorenstein* poset contains no proper subposet which is Gorenstein* of the same rank, and each interval $[F, \hat{1}]$ with $r(F) = 2$ in $P$ is Gorenstein*). For this, it is enough to show that any rank 4 element in $P$ covers exactly two rank 3 elements in $P$. Indeed, this guarantees that the dual poset to $P_2$, denoted $P_2^*$, is the face poset of a union of CW 1-spheres, and as $P$ is Gorenstein* so is its dual $P^*$, hence $P_2^*$ is Cohen-Macaulay since $P_2^*$ is a rank selected poset [St2] III, Theorem 4.5, which implies that $P_2^*$ is the face poset of one CW 1-sphere, i.e. $P_2 \cup \{\emptyset\}$ is Gorenstein*.

Let $F$ be a rank 4 element of $P$. Then $P$ is a subdivision of $\Sigma(\emptyset, F)$ (Recalling [EK] Definition 2.6), this is shown by the map $\phi : P \to \Sigma(\emptyset, F)$, $\phi(\sigma) = \sigma$ if $\sigma < F$, $\phi(\sigma) = \sigma_1$ if $\sigma$ and $F$ are incomparable, and $\phi(\sigma) = \sigma_2$, where $\sigma_1, \sigma_2$ are the rank 4 elements in $\Sigma(\emptyset, F)$. Thus, by Lemma 2.4, the coefficient of $cdc$ in the cd-index of $\Sigma(\emptyset, F)$ is zero, hence the coefficient of the monomial $cd$ in the cd-index of $\emptyset, F)$ is zero.

This fact implies, when expanding the cd-index of $\emptyset, F)$ in terms of $a, b$, that $h_{[3]}(\emptyset, F)$ equals the coefficient of $c^3$, namely $h_{[3]}(\emptyset, F) = 1$. Switching to the flag $f$-vector of $\emptyset, F)$ we get $f_{[3]}(\emptyset, F') = h_{[3]}(\emptyset, F') + h_{[3]}(\emptyset, F) = 1 + 1 = 2$. Thus, $F$ covers exactly two rank 3 elements in $P$. \hfill \Box

**Example 4.7.** Consider the 2-FFK vector $(1, 6, 7)$. We claim that $\Phi_P(1, d) \neq 1 + 6d + 7d^2$ for all Gorenstein* poset $P$ of rank 5. Indeed, if $\Phi_P(1, d) = 1 + 6d + 7d^2$, then $\alpha_{\{1,3\}} = 7$. Then $\alpha_{\{1\}} + \alpha_{\{3\}} = 6$ and $\alpha_{\{2\}} = 0$ by Proposition 4.4 which contradicts Proposition 4.6.

A similar argument shows that $(1, 2a, a^2 - 2)$, where $a \geq 3$, is 2-FFK, but not realizable as the $d$-polynomial of a Gorenstein* poset of rank 5.

**References**


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