THE γ-VECTOR OF A BARYCENTRIC SUBDIVISION

ERAN NEVO, T. KYLE PETERSEN, AND BRIDGET EILEEN TENNER

Abstract. We prove that the γ-vector of the barycentric subdivision of a simplicial sphere is the f-vector of a balanced simplicial complex. The combinatorial basis for this work is the study of certain refinements of Eulerian numbers used by Brenti and Welker to describe the h-vector of the barycentric subdivision of a boolean complex.

1. Introduction

The present work can be motivated by Gal's conjecture, a certain strengthening of the Charney-Davis conjecture phrased in terms of the so-called γ-*vector*.

Conjecture 1 (Gal [6]). *If Δ is a flag homology sphere, then $\gamma(\Delta)$ is nonnegative.*

This conjecture is known to hold for the order complex of a Gorenstein* poset [7], for all Coxeter complexes (see [15], and references therein), and for the (dual simplicial complexes of the) "chordal nestohedra" of [11]—a class containing the associahedron, permutahedron, and other well-studied polytopes.

If Δ has a nonnegative γ-vector, one might ask what is enumerated by the entries in the vector. In certain cases (the type A Coxeter complex, for example), the γ-vector has an explicit combinatorial description.

Recently Nevo and Petersen showed in many cases that not only are the entries in the γ-vector of a flag homology sphere nonnegative, but they satisfy the Frankl-Füredi-Kalai inequalities [10]. In other words, such a γ-vector is the f-vector of a balanced simplicial complex. This suggests a strengthening of Gal's conjecture, given below.

Conjecture 2 (Nevo and Petersen [10]). *If Δ is a flag homology sphere then $\gamma(\Delta)$ is the f-vector of a balanced simplicial complex.*

Our main result, stated below, confirms Conjecture 2 in the case of the barycentric subdivision of a homology sphere.

Theorem 1.1. *If Δ is a boolean complex with a nonnegative and symmetric h-vector, then the γ-vector of the barycentric subdivision of Δ is the f-vector of a balanced simplicial complex.*

In particular, if Δ is a homology sphere, then the Dehn-Sommerville relations guarantee that the h-vector is symmetric. Additionally, the fact that Δ is Cohen-Macaulay means that its h-vector is nonnegative [14]. Hence, Theorem 1.1 implies the conclusion of Conjecture 2 in this case. We remark that the result of Karu [7] implies the nonnegativity of the γ-vector for barycentric subdivisions of homology spheres. However, our approach is quite different.

The paper is organized as follows. Section 2 provides key definitions. Section 3 discusses balanced complexes and presents several key lemmas. In Section 4 we elaborate on work of Brenti and Welker [2] to lay the combinatorial foundation for our main result. The proof of our main theorem, Theorem 1.1, is carried out in Section 5. Finally, in Section 6, we show

how our result has implications for similar results for the h- and g-vectors. In particular we derive a recent result of Murai [9] (who takes an approach similar to ours in building on [2]) that the g-vector of a barycentric subdivision of a homology sphere is the f-vector of a simplicial complex.

2. Preliminaries

The faces of a regular cell complex Δ have a well-defined partial order given by inclusion. More precisely, if F and F' are open cells in Δ, then $F \leq_\Delta F'$ if F is contained in the closure of F'. By convention we assume that the empty face $\emptyset$ is contained in every face of higher dimension. A *boolean cell complex* is a regular cell complex in which every lower interval $[\emptyset, F] = \{G \in \Delta : G \leq F\}$ is isomorphic to a boolean lattice. For example, a simplicial complex is a boolean complex, because the lower interval of a simplex is simply the collection of subsets of the vertices of the simplex.

The *barycentric subdivision* of a boolean complex Δ, denoted $\mathrm{sd}(\Delta)$, is the abstract simplicial complex whose vertex set is identified with the nonempty faces of Δ, and whose i-dimensional faces are strictly increasing flags of the form

$$F_0 <_\Delta F_1 <_\Delta \cdots <_\Delta F_i,$$

where the F_j are nonempty faces in Δ.

The *f-polynomial* of a $(d-1)$-dimensional boolean complex Δ is the generating function for the dimensions of the faces of the complex:

$$f(\Delta; t) = \sum_{F \in \Delta} t^{1+\dim F} = \sum_{i=0}^{d} f_i(\Delta) t^i,$$

where the empty face $\emptyset$ has dimension -1, and so $f_0 = 1$. The *f-vector*

$$f(\Delta) = (f_0(\Delta), f_1(\Delta), \ldots, f_d(\Delta))$$

is the sequence of coefficients of the f-polynomial.

The *h-polynomial* of Δ is a transformation of the f-polynomial:

$$h(\Delta; t) = (1-t)^d f(\Delta; \frac{t}{1-t}) = \sum_{i=0}^{d} h_i(\Delta) t^i,$$

and the *h-vector* is the corresponding sequence of coefficients

$$h(\Delta) = (h_0(\Delta), h_1(\Delta), \ldots, h_d(\Delta)).$$

Although they contain the same information, the h-polynomial is often easier to work with than the f-polynomial. For instance, if Δ is a homology sphere, then the Dehn-Sommerville relations guarantee that the h-vector is symmetric; that is, $h_i = h_{d-i}$ for all $0 \leq i \leq d$.

Whenever a polynomial $h(t) = \sum_{i=0}^{d} h_i t^i$ has symmetric integer coefficients, the polynomial has an integer expansion in the basis $\{t^i(1+t)^{d-2i} : 0 \leq i \leq d/2\}$, and we say that it has "symmetry axis" at degree $\lfloor d/2 \rfloor$. In this case, if we write

$$h(t) = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i t^i (1+t)^{d-2i},$$

then we say $\gamma(h(t)) = (\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor d/2 \rfloor})$ is the *γ-vector* of $h(t)$. In particular if a $(d-1)$-dimensional complex Δ happens to have a symmetric h-vector, then we refer to the γ-vector of $h(\Delta; t)$ as the γ-vector of Δ; that is, $\gamma(\Delta) = \gamma(h(\Delta; t))$.

Another invariant that we will study (though not until Section 6) is the *g-vector* of Δ, $g(\Delta) = (g_0, g_1, \ldots, g_{\lfloor d/2 \rfloor})$, defined by $g_0 = h_0$ and, for $0 \leq i \leq \lfloor d/2 \rfloor$,

$$g_i(\Delta) = h_i(\Delta) - h_{i-1}(\Delta).$$

The polynomial $g(\Delta; t) = \sum_{i=0}^{\lfloor d/2 \rfloor} g_i t^i$ is the generating function for the g-vector.

3. Balanced complexes

A boolean complex Δ on vertex set V is *d-colorable* if there is a coloring of its vertices $c : V \to [d]$ such that for every face $F \in \Delta$, the restriction map $c : F \to [d]$ is injective; that is, every face has distinctly colored vertices. Such coloring is called a *proper coloring*. If a $(d-1)$-dimensional complex Δ is d-colorable, then we say that Δ is *balanced*.

Frankl, Füredi, and Kalai characterized the f-vectors of d-colorable simplicial complexes, for any d, in terms of certain upper bounds on f_{i+1} in terms of f_i [4]. We call these the *Frankl-Füredi-Kalai inequalities*, and a vector satisfying the inequalities with respect to d is a *d-Frankl-Füredi-Kalai vector*, or *d-FFK-vector* for short. We have no need for the explicit inequalities in the present work, but they can be found in [4].

A key ingredient in [4] is the construction of a balanced simplicial complex with a specified f-vector. This relies on the use of a colored version of the idea of a *compressed* simplicial complex. (See, for example, [16, Section 8.5] for a description of compressed complexes as used in the characterization of f-vectors of arbitrary simplicial complexes.)

First, we define a *d-colored k-subset* of $\mathbb{N}$ to be a set of k positive integers such that no two of them are congruent modulo d. Denote by $\binom{\mathbb{N}}{k}_d$ the set of all d-colored k-subsets. The *color* of $S \in \binom{\mathbb{N}}{k}_d$ is the set of remainders modulo d: $\mathrm{col}_d(S) = \{s \bmod d : s \in S\}$. We list d-colored k-subsets in reverse lexicographic ("revlex") order. For example, if $k = 3$ and $d = 4$, then we have

$$123 < 124 < 134 < 234 < 235 < 245 < 345 < \cdots .$$

Notice that the sets $125, 135$, and 145 are *not* 4-colored since $1 \equiv 5 \bmod 4$.

If $f = (1, f_1, \ldots, f_d)$, we let

$$\mathcal{F}_d(f) = \{F_{d,k}(j) : 0 \leq k \leq d, 1 \leq j \leq f_k\},$$

where $F_{d,k}(j)$ denotes the jth element in the revlex order on $\binom{\mathbb{N}}{k}_d$. For example, from above we see $F_{4,3}(1) = 123$, $F_{4,3}(2) = 124$, and so on. In other words, the $(k-1)$-dimensional faces of $\mathcal{F}_d(f)$ are the first f_k of the d-colored k-subsets of $\mathbb{N}$ in revlex order.

Half of the characterization given in [4] states that if f is a d-FFK-vector then $\mathcal{F}_d(f)$ is d-colorable simplicial complex (thus if $f_d \neq 0$ then $\mathcal{F}_d(f)$ is a balanced simplicial complex).

If Δ is simplicial complex with f-vector $f = (1, f_1, \ldots, f_d)$ then we define the *d-compression of* Δ to be $\mathcal{F}_d(f)$. If a $(d-1)$-dimensional simplicial complex Δ is balanced, its d-compression is a balanced simplicial complex, called a *d-compressed complex*.

If Δ and Δ' are simplicial complexes on disjoint vertex sets, define their *join* to be the complex $\Delta * \Delta' = \{F \cup G : F \in \Delta, G \in \Delta'\}$. As a special case, the *cone* over Δ is $\Delta^c = \Delta * \{v\}$, for some vertex $v \notin \Delta$. It is clear by construction that for $k \geq 1$,

$$f_k(\Delta^c) = f_k(\Delta) + f_{k-1}(\Delta).$$

If $f = (1, f_1, \dots, f_d)$ and $f' = (1, f'_1, \dots, f'_d)$ are d-FFK-vectors such that $f_k \geq f'_k$ for all $k = 1, \dots, d$, then we say f *dominates* f'. Clearly if f dominates f' then $\mathcal{F}(f')$ is a subcomplex of $\mathcal{F}(f)$, and, by introducing a new vertex v with color $d+1$, the complex

$$\mathcal{F}(f) \cup \mathcal{F}(f')^c = \mathcal{F}(f) \cup \{F \cup \{v\} : F \in \mathcal{F}(f')\}$$

is a $(d+1)$-colorable d-dimensional complex with f-vector $f + (0, f')$. Applying this argument repeatedly proves the following lemma.

Lemma 3.1. *Consider vectors* $f = (1, f_1, f_2, \dots, f_d)$, $f^{(1)} = (1, f_1^{(1)}, f_2^{(1)}, \dots, f_d^{(1)})$, …, $f^{(k)} = (1, f_1^{(k)}, f_2^{(k)}, \dots, f_d^{(k)})$, *and require them all to be d-FFK-vectors with f dominating $f^{(j)}$ for all j. Then*

$$f + (0, f^{(1)} + \cdots + f^{(k)}) = \left(1, f_1 + k, f_2 + \sum_{j=1}^{k} f_1^{(j)}, \dots, f_d + \sum_{j=1}^{k} f_{d-1}^{(j)}, \sum_{j=1}^{k} f_d^{(j)}\right)$$

is a $(d+1)$-FFK-vector.

Similarly, suppose Δ is a balanced $(d-1)$-dimensional complex on vertex set V with a proper coloring $c : V \to [d]$, and let Δ' be a subcomplex of dimension $(d-2)$, on vertex set V', such that $c(V') \neq [d]$. Let $f = f(\Delta) = (1, f_1, \dots, f_d)$ and $f' = f(\Delta') = (1, f'_1, \dots, f'_{d-1})$. Then by coning over Δ' with a new vertex v colored by the element in $[d] \setminus c(V')$, we see that $\Delta \cup (\Delta')^c$ is a balanced $(d-1)$-dimensional complex with f-vector $f + (0, f')$. Of course, we can do this for any number of such subcomplexes Δ', and so we have the following lemma.

Lemma 3.2. *Let Δ be a balanced $(d-1)$-dimensional complex on vertex set V with a proper coloring $c : V \to [d]$. Furthermore, let Δ have (not necessarily distinct) subcomplexes $\Delta^{(1)}, \dots, \Delta^{(k)}$, each of dimension at most $d-2$, on vertex sets $V^{(1)}, \dots, V^{(k)}$ respectively, such that for every i, we have $c(V^{(i)}) \neq [d]$. Let $f = f(\Delta)$, and $f^{(1)} = f(\Delta^{(1)})$, …, $f^{(k)} = f(\Delta^{(k)})$. Then $\Delta \bigcup_{j=1}^{k} (\Delta^{(j)})^c$ is a balanced $(d-1)$-dimensional complex with f-vector*

$$f + (0, f^{(1)} + \cdots + f^{(k)}).$$

Using the idea of d-compression, we will now exhibit a sufficient condition for the existence of a collection of subcomplexes as in Lemma 3.2.

There is a natural embedding ϕ_d of the revlex order on $\binom{\mathbb{N}}{k}_{d-1}$ in the revlex order on $\binom{\mathbb{N}}{k}_d$, defined as follows. For $s = (d-1)i + j \in \mathbb{N}$ with $0 \leq i$ and $1 \leq j \leq d-1$, let $\phi_d(s) = s + i = di + j$. Then if S is a $(d-1)$-colored k-subset, let $\phi_d(S) = \{\phi_d(s) : s \in S\}$. Observe that the color of S is preserved under ϕ_d; that is, $\mathrm{col}_{d-1}(S) = \mathrm{col}_d(\phi_d(S))$. Moreover, if $S \leq T$ in $\binom{\mathbb{N}}{k}_{d-1}$, then $\phi_d(S) \leq \phi_d(T)$ in $\binom{\mathbb{N}}{k}_d$.

Define the function $r_{d,k}(a) = b$ as the position that the ath element in $\binom{\mathbb{N}}{k}_{d-1}$ gets mapped to in $\binom{\mathbb{N}}{k}_d$. That is, $\phi_d(F_{d-1,k}(a)) = F_{d,k}(b)$.

Proposition 3.3. *For all $a, k, d \in \mathbb{N}$ such that $k < d$, we have $r_{d,k}(a) \leq (k+1)a$.*

Proof. Let $b = r_{d,k}(a)$, and let $A = \{S : S \leq F_{d,k}(b), d \notin \mathrm{col}_d(S)\}$ and $B = \{T : T < F_{d,k}(b), d \in \mathrm{col}_d(T)\}$. Notice $A = \{\phi_d(S) : S \leq F_{d-1,k}(a)\}$, and so $|A| = a$.

Define $\pi : B \to A$ by

$$\pi(T) = (T \setminus \{r\}) \cup \{r'\},$$

where r is the element in T of the form $r = di$ and $r' = d(i-1)+j$ for the greatest j such that $1 \le j \le d-1$ and $j \notin \mathrm{col}_d(T)$. Since $k \le d-1$, there is always some color not in $\mathrm{col}_d(T)$. By construction, $\pi(T) < T$. Thus $\pi(T) \in A$ and π is well-defined. For example, with $d = 5$, $\pi(\{1,4,5,8\}) = \{1,2,4,8\}$.

Now for any element $S \in A$ consider $T \in \pi^{-1}(S)$. We obtain such a T by replacing s in S with a number r such that $s < r < s+d$ and r is a multiple of d. Since there is at most one such r for each element of S, we have $|\pi^{-1}(S)| \le |S| = k$.

Now,

$$r_{d,k}(a) = |A| + |B| \le |A| + k|A| = (k+1)a,$$

as desired. □

We now have the following corollary.

Corollary 3.4. *Let $f = (1, f_1, \ldots, f_{d-1})$ be a $(d-1)$-FFK-vector and $f' = (1, f'_1, \ldots, f'_d)$ a d-FFK-vector satisfying $(k+1)f_k \le f'_k$ for $1 \le k \le d-1$. Then there is a color-preserving isomorphism between $\mathcal{F}_{d-1}(f)$ and a subcomplex of $\mathcal{F}_d(f')$.*

Proof. By Proposition 3.3, we have that $r_{d,k}(f_k) \le (k+1)f_k \le f'_k$. Thus, from the definition of d-compressed complexes (and the function $r_{d,k}(i)$), if $S \in \mathcal{F}_{d-1}(f)$, then $\phi_d(S) \in \mathcal{F}_d(f')$. That is, $\phi_d(\mathcal{F}_{d-1}(f))$ is a subcomplex of $\mathcal{F}_d(f')$. As $\mathrm{col}_{d-1}(S) = \mathrm{col}_d(\phi_d(S))$, we have that $\mathcal{F}_{d-1}(f) \to \phi_d(\mathcal{F}_{d-1}(f))$ is a color-preserving isomorphism of balanced simplicial complexes, yielding the desired result: $\mathcal{F}_{d-1}(f) \cong \phi_d(\mathcal{F}_{d-1}(f)) \subset \mathcal{F}_d(f')$. □

Repeatedly using this fact, along with Lemma 3.2, yields the following companion to Lemma 3.1.

Lemma 3.5. *Let $f = (1, f_1, f_2, \ldots, f_d)$ be a d-FFK-vector, and $f^{(1)} = (1, f_1^{(1)}, f_2^{(1)}, \ldots, f_{d-1}^{(1)})$, $\ldots$, $f^{(k)} = (1, f_1^{(k)}, f_2^{(k)}, \ldots, f_{d-1}^{(k)})$ be $(d-1)$-FFK-vectors such that $f_i \ge (i+1)f_i^{(j)}$ for all i and all j. Then*

$$f + (0, f^{(1)} + \cdots + f^{(k)}) = \left(1, f_1 + k, f_2 + \sum_{j=1}^{k} f_1^{(j)}, \ldots, f_d + \sum_{j=1}^{k} f_{d-1}^{(j)}\right)$$

is a d-FFK-vector.

Let us now define terms for f-vectors satisfying the conditions of these lemmas.

Definition 3.6. *Let $f = (1, f_1, \ldots, f_d)$ be the f-vector of a $(d-1)$-dimensional balanced complex.*

1. ***$(d+1)$-good**. As in Lemma 3.1, let $g = (g_1, \ldots, g_d, g_{d+1})$ be a sum of d-FFK-vectors, each of which is dominated by f. Some, but not all, of these d-FFK-vectors may be shorter than f; that is, $g_{d+1} \neq 0$. Then we say that $(0, g)$ is* $(d+1)$-good *for f. Note that $f + (0, g)$ is a $(d+1)$-FFK-vector.*
2. ***d-good**. Let $g = (g_1, \ldots, g_d) = f^{(1)} + \cdots + f^{(k)}$, with $g_d \neq 0$, be a sum of $(d-1)$-FFK-vectors such that $f_i \ge (i+1)f_i^{(j)}$ for all i and all j. Then we say that $(0, g)$ is* d-good *for f. Note that $f + (0, g)$ is a d-FFK-vector.*

The following is immediate from the definitions.

Observation 3.7. *Let $f = (1, f_1, \ldots, f_d)$ be the f-vector of a $(d-1)$-dimensional balanced complex. If $(0, g)$ is d-good for f and $(0, g')$ is $(d+1)$-good for f, then $(0, g+g')$ is $(d+1)$-good for f. Recall that $f + (0, g+g')$ is a $(d+1)$-FFK-vector.*

4. Restricted Eulerian polynomials

In [2], Brenti and Welker provide, among other things, explicit combinatorial formulas relating $h(\Delta)$ and $h(\mathrm{sd}(\Delta))$, which we now describe.

Let S_n denote the symmetric group on $\{1, 2, \ldots, n\}$. For a permutation $w \in S_n$, the *descent number* of w is defined as

$$d(w) = |\{i : w(i) > w(i+1)\}|.$$

Let $A(n, i, j)$ denote the number of permutations $w \in S_n$ such that $w(1) = j$ and $d(w) = i$.

Theorem 4.1 (Brenti and Welker [2, Theorem 2.2])**.** *Let Δ be a $(n-1)$-dimensional boolean complex. Then for $0 \le i \le n$,*

$$h_i(\mathrm{sd}(\Delta)) = \sum_{j=0}^{n} A(n+1, i, j+1) h_j(\Delta). \tag{1}$$

We shall return to this characterization of the h-vector of $\mathrm{sd}(\Delta)$ in Section 5. First we will investigate various properties of the coefficients $A(n, i, j)$ and related generating functions.

Let

$$S_{n,j} = \{w \in S_n : w(1) = j\},$$

and define the *restricted Eulerian polynomials* as the descent generating functions for these subsets of S_n:

$$A_{n,j}(t) = \sum_{w \in S_{n,j}} t^{d(w)} = \sum_{i=0}^{n-1} A(n, i, j) t^i.$$

Note that the usual Eulerian polynomial is the descent generating function for all of S_n:

$$A_n(t) = \sum_{w \in S_n} t^{d(w)} = \sum_{j=1}^{n} A_{n,j}(t).$$

For example,

$$\begin{aligned} A_{4,1}(t) &= 1 + 4t + t^2, \\ A_{4,2}(t) &= 4t + 2t^2, \\ A_{4,3}(t) &= 2t + 4t^2, \\ A_{4,4}(t) &= t + 4t^2 + t^3, \end{aligned}$$

and $A_4(t) = 1 + 11t + 11t^2 + t^3$.

The following observation is immediate by considering that if $w(1) = 1$ there is never a descent in the first position, and if $w(1) = n$ there is always a descent in the first position.

Observation 4.2. *We have $A_{n,1}(t) = A_{n-1}(t)$ and $A_{n,n}(t) = tA_{n-1}(t)$.*

Tracking the effect of removing the letter j from a permutation in $S_{n,j}$ yields the following recurrence for restricted Eulerian polynomials.

Observation 4.3 (Brenti and Welker [2, Lemma 2.5(i)]). *We have*

$$A(n,i,j) = \sum_{k=1}^{j-1} A(n-1,i-1,k) + \sum_{k=j}^{n-1} A(n-1,i,k),$$

and thus

$$A_{n,j}(t) = t\sum_{k=1}^{j-1} A_{n-1,k}(t) + \sum_{k=j}^{n-1} A_{n-1,k}(t).$$

A standard involution on permutations in S_n is given by mapping each i to $n+1-i$. This involution has the effect of exchanging ascents and descents. Hence the following symmetries hold.

Observation 4.4 (Brenti and Welker [2, Lemma 2.5(ii)]). *We have*

$$A(n,i,j) = A(n, n-1-i, n+1-j),$$

and thus

$$A_{n,j}(t) = t^{n-1} A_{n,n+1-j}(1/t).$$

We now define the *symmetric restricted Eulerian polynomials* by lumping together classes fixed by the involution just described, namely all permutations beginning with either j or $n+1-j$:

$$\boldsymbol{A}_{n,j}(t) = \sum_{w \in S_{n,j} \cup S_{n,n+1-j}} t^{d(w)}.$$

Observe that

$$\boldsymbol{A}_{n,j}(t) = \begin{cases} A_{n,j}(t) + A_{n,n+1-j}(t) & \text{if } j \neq (n+1)/2, \text{ and} \\ A_{n,j}(t) & \text{if } j = (n+1)/2. \end{cases}$$

By Observation 4.4, the polynomials $\boldsymbol{A}_{n,j}(t)$ have symmetric coefficients, and hence a γ-vector. Clearly $\boldsymbol{A}_{n,j}(t)$ has symmetry axis at degree $\lfloor \frac{n-1}{2} \rfloor$. If

$$\boldsymbol{A}_{n,j}(t) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \gamma_i^{(n,j)} t^i (1+t)^{n-1-2i},$$

let $\gamma^{(n,j)} = (\gamma_0^{(n,j)}, \gamma_1^{(n,j)}, \ldots, \gamma_{\lfloor (n-1)/2 \rfloor}^{(n,j)})$ denote the corresponding γ-vector.

We will have reason to consider the following polynomials as well, where $1 \le j < (n+1)/2$:

$$\boldsymbol{A}'_{n,j}(t) = tA_{n,j}(t) + A_{n,n+1-j}(t).$$

Note that, by Observation 4.4, the integer coefficients of $\boldsymbol{A}'_{n,j}(t)$ are symmetric, and so $\boldsymbol{A}'_{n,j}(t)$ has a γ-vector. Clearly $\boldsymbol{A}'_{n,j}(t)$ has symmetry axis at degree $\lfloor n/2 \rfloor$.

Define

$$\gamma'^{(n,j)} = (\gamma_0'^{(n,j)}, \gamma_1'^{(n,j)}, \ldots, \gamma_{\lfloor n/2 \rfloor}'^{(n,j)})$$

to be the γ-vector for the polynomial $\boldsymbol{A}'_{n,j}(t)$.

Using Observation 4.3, we obtain the following recurrences.

Lemma 4.5. *We have the following recurrences for the* $\gamma^{(n,j)}$ *and* $\gamma'^{(n,j)}$*:*

(1) *If* $j = (n+1)/2$, *then*

$$\gamma^{(n,(n+1)/2)} = \gamma'^{(n-1,1)} + \gamma'^{(n-1,2)} + \cdots + \gamma'^{(n-1,(n-1)/2)}.$$

(2) *For* $j < (n+1)/2$,

$$\gamma^{(n,j)} = 2\sum_{k=1}^{j-1} \gamma'^{(n-1,k)} + \sum_{k=j}^{\lfloor n/2 \rfloor} \gamma^{(n-1,k)}.$$

(3) *For* $j < (n+1)/2$,

$$\gamma'^{(n,j)} = \sum_{k=1}^{j-1} \gamma'^{(n-1,k)} + 2\sum_{k=j}^{\lfloor n/2 \rfloor} (0, \gamma^{(n-1,k)}).$$

Proof. Statement (1) follows because, for $j = (n+1)/2$ we have, by Observation 4.3,

$$\begin{aligned}
\boldsymbol{A}_{n,(n+1)/2} = A_{n,(n+1)/2}(t) &= t\sum_{k=1}^{(n-1)/2} A_{n-1,k}(t) + \sum_{k=(n+1)/2}^{n-1} A_{n-1,k}(t) \\
&= t\sum_{k=1}^{(n-1)/2} A_{n-1,k}(t) + \sum_{k=1}^{(n-1)/2} A_{n-1,n-k}(t) \\
&= \sum_{k=1}^{(n-1)/2} \boldsymbol{A}'_{n-1,k}(t).
\end{aligned}$$

If $j < (n+1)/2$, then Observation 4.3 implies

$$\begin{aligned}
\boldsymbol{A}_{n,j}(t) &= A_{n,j}(t) + A_{n,n+1-j}(t) \\
&= t\sum_{k=1}^{j-1} A_{n-1,k}(t) + \sum_{k=j}^{n-1} A_{n-1,k}(t) + t\sum_{k=1}^{n-j} A_{n-1,k}(t) + \sum_{k=n+1-j}^{n-1} A_{n-1,k}(t) \\
&= 2t\sum_{k=1}^{j-1} A_{n-1,k}(t) + (1+t)\sum_{k=j}^{n-j} A_{n-1,k}(t) + 2\sum_{k=n+1-j}^{n-1} A_{n-1,k}(t) \\
&= 2\sum_{k=1}^{j-1} (tA_{n-1,k}(t) + A_{n-1,n-k}(t)) + (1+t)\sum_{k=j}^{\lfloor \frac{n-1}{2} \rfloor} (A_{n-1,k}(t) + A_{n-1,n-k}(t)) \\
&\qquad + \begin{cases} (1+t)A_{n-1,n/2} & \text{if } 2|n \\ 0 & \text{if } 2 \nmid n. \end{cases} \\
&= 2\sum_{k=1}^{j-1} \boldsymbol{A}'_{n-1,k}(t) + (1+t)\sum_{k=j}^{\lfloor n/2 \rfloor} \boldsymbol{A}_{n-1,k}(t),
\end{aligned}$$

which yields statement (2).

Lastly we consider $\boldsymbol{A}'_{n,j}(t)$ in terms of Observation 4.3. We have

$$\begin{aligned}
\boldsymbol{A}'_{n,j}(t) &= tA_{n,j}(t) + A_{n,n+1-j}(t) \\
&= t^2 \sum_{k=1}^{j-1} A_{n-1,k}(t) + t \sum_{k=j}^{n-1} A_{n-1,k}(t) + t \sum_{k=1}^{n-j} A_{n-1,k}(t) + \sum_{k=n+1-j}^{n-1} A_{n-1,k}(t) \\
&= t^2 \sum_{k=1}^{j-1} A_{n-1,k}(t) + t \sum_{k=1}^{n-j} A_{n-1,n-k}(t) + t \sum_{k=1}^{n-j} A_{n-1,k}(t) + \sum_{k=1}^{j-1} A_{n-1,n-k}(t) \\
&= t \sum_{k=1}^{j-1} \boldsymbol{A}'_{n-1,k}(t) + t \sum_{k=j}^{n-j} A_{n-1,n-k}(t) + \sum_{k=1}^{j-1} \boldsymbol{A}'_{n-1,k}(t) + t \sum_{k=j}^{n-j} A_{n-1,k}(t) \\
&= (t+1) \sum_{k=1}^{j-1} \boldsymbol{A}'_{n-1,k}(t) + t \sum_{k=j}^{n-j} (A_{n-1,k}(t) + A_{n-1,n-k}(t)) \\
&= (t+1) \sum_{k=1}^{j-1} \boldsymbol{A}'_{n-1,k}(t) + 2t \sum_{k=j}^{\lfloor n/2 \rfloor} \boldsymbol{A}_{n-1,k}(t).
\end{aligned}$$

This confirms (3), completing the proof. □

Remark 4.6. *In the case of the ordinary Eulerian polynomials, the coefficients of the γ-vector have the following interpretation, first described by Foata and Schützenberger* [3]*. Let $\widehat{S}_n = \{w \in S_n : w_{n-1} < w_n,\ \text{and if } w_{i-1} > w_i \text{ then } w_i < w_{i+1}\}$. Then $\gamma_i = |\{w \in \widehat{S}_n : d(w) = i\}|$, so that $A_n(t) = \sum \gamma_i t^i (1+t)^{n-1-2i}$. It would be interesting to have a similar combinatorial interpretation for the coefficients of the vectors $\gamma^{(n,j)}$.*

5. Symmetric h-vectors

If Δ has a symmetric h-vector (that is, if $h_i = h_{n-i}$ for all i, where Δ has dimension $n-1$), then from Theorem 4.1 and Observation 4.4 it follows that $h(\mathrm{sd}(\Delta))$ is symmetric as well.

Corollary 5.1 (Brenti and Welker [2, Corollary 2.6])**.** *If $h(\Delta)$ is symmetric, then $h(\mathrm{sd}(\Delta))$ is symmetric.*

Moreover, the following proposition holds.

Proposition 5.2. *If Δ is a boolean complex of dimension $n-1$, with $h(\Delta) = (h_0, h_1, \ldots, h_n)$ symmetric, then*

$$h_i(\mathrm{sd}(\Delta)) = \sum_{j=0}^{\lfloor n/2 \rfloor} (A(n+1, i, j+1) + A(n+1, i, n+1-j))h_j,$$

and thus

$$h(\mathrm{sd}(\Delta); t) = \sum_{i=0}^{\lfloor n/2 \rfloor} h_i\, \boldsymbol{A}_{n+1,i+1}(t).$$

In terms of γ-vectors,

$$\gamma(\mathrm{sd}(\Delta)) = \sum_{i=0}^{\lfloor n/2 \rfloor} h_i \gamma^{(n+1,i+1)}.$$

For example, if $n = 5$ and $h(\Delta) = (h_0, h_1, h_2, h_3 = h_2, h_4 = h_1, h_5 = h_0)$, then

$$h(\mathrm{sd}(\Delta))^t = \begin{pmatrix} h_0 \\ 27h_0 + 18h_1 + 12h_2 \\ 92h_0 + 102h_1 + 108h_2 \\ 92h_0 + 102h_1 + 108h_2 \\ 27h_0 + 18h_1 + 12h_2 \\ h_0 \end{pmatrix}$$
$$= h_0 \begin{pmatrix} 1 \\ 5 \\ 10 \\ 10 \\ 5 \\ 1 \end{pmatrix} + (22h_0 + 18h_1 + 12h_2) \begin{pmatrix} 0 \\ 1 \\ 3 \\ 3 \\ 1 \\ 0 \end{pmatrix} + (16h_0 + 48h_1 + 72h_2) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Equivalently,

$$h(\mathrm{sd}(\Delta); t) = h_0\, \boldsymbol{A}_{6,1}(t) + h_1\, \boldsymbol{A}_{6,2}(t) + h_2\, \boldsymbol{A}_{6,3}(t),$$

or

$$\begin{aligned} \gamma(\mathrm{sd}(\Delta)) &= h_0\gamma^{(6,1)} + h_1\gamma^{(6,2)} + h_2\gamma^{(6,3)} \\ &= h_0(1, 22, 16) + h_1(0, 18, 48) + h_2(0, 12, 72). \end{aligned}$$

If $f = (f_0, f_1, \ldots, f_d)$ is a d-FFK vector and $f_d \neq 0$, we simply say that f is an FFK-vector, that is, f is the f-vector of a balanced complex. Our goal now is to show that $\gamma(\mathrm{sd}(\Delta))$ is an FFK-vector.

5.1. **Proof of Theorem 1.1.** Since $\boldsymbol{A}_{n,1}(t) = (1+t)A_{n-1}(t)$ (Observation 4.2), where $A_{n-1}(t)$ is the usual Eulerian polynomial, we see that $\boldsymbol{A}_{n,1}(t)$ and $A_{n-1}(t)$ have the same γ-vector. By [10, Theorem 6.1 (1)], the γ-vector of $A_{n-1}(t)$ is an FFK-vector, and thus $\gamma^{(n,1)}$ is an FFK-vector for any n. As $A_{n-1}(t)$ has symmetry axis at degree $\lfloor \frac{n}{2} \rfloor - 1$, $\gamma^{(n,1)} = (1, f_1, \ldots, f_d)$ where $d = \lfloor \frac{n}{2} \rfloor - 1$. Since $h_0 = 1$ for any boolean complex Δ, we have (if $\dim \Delta = n - 1$):

$$\gamma(\mathrm{sd}(\Delta)) = \gamma^{(n+1,1)} + h_1\gamma^{(n+1,2)} + \cdots$$

Observe that $\gamma_0^{(n+1,j)} = 0$ for all $j > 1$. What we will show is that the vectors $h_i\gamma^{(n+1,i+1)}$ are d- or $(d+1)$-good for $\gamma^{(n+1,1)}$, in the sense of Definition 3.6, for all $i \geq 1$. More precisely, we will prove the following.

Proposition 5.3. *Let* $\gamma^{(n,1)} = (1, f_1, \ldots, f_d)$*, where* $d = \lfloor n/2 \rfloor - 1$*.*

(1) *If* n *is even, i.e.,* $n = 2d + 2$*, then*
  (a) $\gamma^{(n,j)}$, $1 < j \leq n/2$, *is* d*-good for* $\gamma^{(n,1)}$*, and*
  (b) $\gamma'^{(n,j)}$, $1 \leq j \leq n/2$, *is* $(d+1)$*-good for* $\gamma^{(n,1)}$*.*

(2) *If* n *is odd, i.e.,* $n = 2d + 3$*, then*
  (a) $\gamma^{(n,j)}$, $1 < j \leq (n+1)/2$, *is* $(d+1)$*-good for* $\gamma^{(n,1)}$*, and*
  (b) $\gamma'^{(n,j)}$, $1 \leq j < (n+1)/2$, *is* $(d+1)$*-good for* $\gamma^{(n,1)}$*.*

In proving Proposition 5.3, it will be helpful to recall a definition of [10] and collect some preliminary results.

Let

$$\widehat{S}_n = \{w \in S_n : w(n-1) < w(n), \text{ and if } w(i-1) > w(i) \text{ then } w(i) < w(i+1)\}.$$

In other words, $\widehat{S}_n$ is the set of permutations in S_n with no double descents and no final descent. We write elements from this set as permutations in one-line notation with bars at the descent positions, or as certain ordered lists of blocks written in increasing order. For example,

$$235|1479|68, \quad 8|34|19|27|56, \quad 1235|46789$$

are elements of $\widehat{S}_9$. Notice that the leftmost block can have one element, but all other blocks have size at least two. Further, if $w = B_1|\cdots|B_k$, then $\max B_i > \min B_{i+1}$ for $i = 1, \ldots, k-1$. That is, the number of bars in w is $d(w)$.

Define $\Gamma(n)$ to be the simplicial complex whose faces are the elements of $\widehat{S}_n$, with $\dim w = d(w) - 1$. We have $w \subseteq v$ if v can be obtained from w by refinement of blocks. Vertices are elements with only one bar. This is a balanced simplicial complex of dimension $\lfloor \frac{n-1}{2} \rfloor - 1$. The color set of a face w is $\mathrm{col}(w) = \{\lceil i/2 \rceil : w(i) > w(i+1)\}$. A result of [10] has

$$f(\Gamma(n)) = \gamma^{(n+1,1)}.$$

With this interpretation for $\gamma^{(n+1,1)}$, we can prove the following.

Lemma 5.4. *For all $n \geq 1$, $1 \leq i \leq \lfloor n/2 \rfloor - 1$, we have*

$$(i+1)\gamma_i^{(n,1)} \leq \gamma_i^{(n+1,1)}.$$

Proof. Consider any face of $\Gamma(n-1)$ with i bars, say $w = B_1|\cdots|B_{i+1}$. Then we can associate to w a face of $\Gamma(n)$ by adding the number n to any of the $i+1$ blocks of w. If n is inserted at the end of a block, no new descents will be created and hence this insertion takes $(i-1)$-dimensional faces to $(i-1)$-dimensional faces. If $w \neq v$ are in $\Gamma(n-1)$ then it is clear that the set of faces formed from w cannot intersect the set of faces formed from v. The inequality follows. □

The next lemma is crucial to later analyses.

Lemma 5.5. *Let $n = 2d + 2$. Then:*

1. *If $(0, f)$ is d-good for $\gamma^{(n-1,1)}$, then $(0, f)$ is d-good for $\gamma^{(n,1)}$.*
2. *If $(0, f)$ is $(d+1)$-good for $\gamma^{(n,1)}$, then $(0, f)$ is $(d+1)$-good for $\gamma^{(n+1,1)}$.*

Proof. To prove (1), we consider $f = (f_1, \ldots, f_d) = f^{(1)} + \cdots + f^{(k)}$, where each $f^{(j)} = (1, f_1^{(j)}, \cdots, f_{d-1}^{(j)})$ is a $(d-1)$-FFK vector and $f_i^{(j)} \leq \gamma_i^{(n-1,1)}$ for all i and all j. Then by Lemma 5.4, we have that for all $1 \leq i \leq d-1$

$$(i+1)f_i^{(j)} \leq (i+1)\gamma_i^{(n-1,1)} \leq \gamma_i^{(n,1)},$$

and thus from Lemma 3.5 we conclude that $(0, f)$ is d-good for $\gamma^{(n,1)}$.

We now consider (2). If $(0, f)$ is $(d+1)$-good for $\gamma^{(n,1)}$, then we can write $f = f^{(1)} + \cdots + f^{(k)}$, for some d-FFK-vectors $f^{(j)}$ such that $\gamma_i^{(n,1)} \geq f_i^{(j)}$ for all i and j. By Lemma 5.4 we find $\gamma^{(n+1,1)}$ dominates the $f^{(j)}$ as well, and so by Lemma 3.1 we conclude that $(0, f)$ is $(d+1)$-good for $\gamma^{(n+1,1)}$ (note that $\gamma^{(n+1,1)}$ has degree d). □

In particular, since $(0, \gamma^{(n-1,1)})$ is clearly d-good for $\gamma^{(n-1,1)}$, Lemma 5.5 implies $(0, \gamma^{(n-1,1)})$ is d-good for $\gamma^{(n,1)}$. Similarly, $(0, \gamma^{(n,1)})$ is $(d+1)$-good for $\gamma^{(n+1,1)}$. We are now ready to prove Proposition 5.3.

Proof of Proposition 5.3. We will proceed by induction on $d = \lfloor n/2 \rfloor - 1$. If $d = 0$, we have:

$$\begin{array}{llll} \gamma^{(2,1)} = (1) & \gamma'^{(2,1)} = (0,2) & \gamma^{(3,1)} = (1) & \gamma'^{(3,1)} = (0,2) \\ & & \gamma^{(3,2)} = (0,2) & \end{array}$$

and the claims are trivially verified. That is, $(0, 2)$ is 1-good for (1). With $d = 1$, we have:

$$\begin{array}{llll} \gamma^{(4,1)} = (1,2) & \gamma'^{(4,1)} = (0,2,4) & \gamma^{(5,1)} = (1,8) & \gamma'^{(5,1)} = (0,2,16) \\ \gamma^{(4,2)} = (0,6) & \gamma'^{(4,2)} = (0,2,4) & \gamma^{(5,2)} = (0,10,8) & \gamma'^{(5,2)} = (0,2,16) \\ & & \gamma^{(5,3)} = (0,4,8) & \end{array}$$

From here we see the first instance of part (1a) of the proposition.

Now suppose that the claims of the proposition hold for $d-1$ and we will prove it for d.

Case 1 (n even). Let $n = 2d+2$ and consider $\gamma^{(n,j)}$ for some $1 < j \leq n/2$. We wish to show that $\gamma^{(n,j)}$ is d-good for $\gamma^{(n,1)}$. By Lemma 4.5, we have

$$\gamma^{(n,j)} = 2\sum_{k=1}^{j-1} \gamma'^{(n-1,k)} + \sum_{k=j}^{n/2} \gamma^{(n-1,k)}.$$

Since $n-1 = 2(d-1)+3$ is odd, our induction hypothesis has each summand d-good for $\gamma^{(n-1,1)}$. The sum of d-good vectors is again d-good, so $\gamma^{(n,j)}$ is d-good for $\gamma^{(n-1,1)}$. By Lemma 5.5 (1), we conclude that $\gamma^{(n,j)}$ is also d-good for $\gamma^{(n,1)}$, proving part (1a).

Now we wish to show $\gamma'^{(n,j)}$, with $1 \leq j \leq n/2$, is $(d+1)$-good for $\gamma^{(n,1)}$. From Lemma 4.5 we have

$$\gamma'^{(n,j)} = \sum_{k=1}^{j-1} \gamma'^{(n-1,k)} + 2\sum_{k=j}^{n/2} (0, \gamma^{(n-1,k)}).$$

In the degenerate case $j = 1$, this gives $\gamma'^{(n,1)} = (0, 2\gamma^{(n,1)})$, which is clearly $(d+1)$-good for $\gamma^{(n,1)}$. If $j > 1$ we can rewrite this as:

$$\gamma'^{(n,j)} = \sum_{k=2}^{j-1} \gamma'^{(n-1,k)} + (0, 2f),$$

where

$$f = \gamma^{(n-1,1)} + \sum_{k=j}^{\lfloor n/2 \rfloor} \gamma^{(n-1,k)}.$$

All the terms $\gamma'^{(n-1,k)}$ in the left summation are d-good for $\gamma^{(n-1,1)}$ by our induction hypothesis. By Lemma 5.5 (1), we have that these terms are d-good for $\gamma^{(n,1)}$. Also, by our induction hypothesis, $f = (1, f_1, \ldots, f_{d-1})$ is an FFK-vector, and as

$$\gamma^{(n,1)} = \sum_{k=1}^{n/2} \gamma^{(n-1,k)},$$

we see that $\gamma^{(n,1)}$ dominates f. Thus, by Lemma 3.1, we see that $(0,2f)$ is $(d+1)$-good for $\gamma^{(n,1)}$. The sum of d- and $(d+1)$-good vectors is $(d+1)$-good by Observation 3.7, so we conclude that $\gamma'^{(n,j)}$ is $(d+1)$-good for $\gamma^{(n,j)}$, proving 1(b) as desired.

Case 2 (n odd). Let $n = 2d+3$, and consider $\gamma^{(n,j)}$ with $1 < j \le (n+1)/2$. We wish to show $\gamma^{(n,j)}$ is $(d+1)$-good for $\gamma^{(n,1)}$. As a special case, if $j = (n+1)/2$, then Lemma 4.5 (1) has

$$\gamma^{(n,(n+1)/2)} = \sum_{k=1}^{(n-1)/2} \gamma'^{(n-1,k)}.$$

As shown in Case 1, each term in the sum is $(d+1)$-good for $\gamma^{(n-1,1)}$, and so by Lemma 5.5, we conclude that $\gamma^{(n,(n+1)/2)}$ is $(d+1)$-good for $\gamma^{(n,1)}$.

Now suppose $1 < j < (n+1)/2$. By Lemma 4.5 part (2),

$$\gamma^{(n,j)} = 2\sum_{k=1}^{j-1} \gamma'^{(n-1,k)} + \sum_{k=j}^{(n-1)/2} \gamma^{(n-1,k)}.$$

It was shown in Case 1 that all the terms in the left summation are $(d+1)$-good for $\gamma^{(n-1,1)}$ and the terms in the right summation are d-good for $\gamma^{(n-1,1)}$. Thus, $\gamma^{(n,j)}$ is $(d+1)$-good for $\gamma^{(n-1,1)}$, and by Lemma 5.5 $\gamma^{(n,j)}$ is $(d+1)$-good for $\gamma^{(n,1)}$. This proves part (2a).

For part (2b), our analysis is identical to Case (1b). That is, in the $j = 1$ case we get $\gamma'^{(n,1)} = (0, 2\gamma^{(n-1,1)})$, which is clearly $(d+1)$-good for $\gamma^{(n-1,1)}$, and hence for $\gamma^{(n,1)}$. If $1 < j < (n+1)/2$ we have

$$\gamma'^{(n,j)} = \sum_{k=2}^{j-1} \gamma'^{(n-1,k)} + (0, 2f),$$

and it follows that all terms involved are $(d+1)$-good for $\gamma^{(n-1,1)}$; hence, for $\gamma^{(n,1)}$. This proves part (2b), and completes the proof of the proposition. □

As discussed above, we have now proved Theorem 1.1.

In particular, as barycentric subdivisions are flag, and the operation of taking barycentric subdivisions leaves the topology of the underlying space unchanged, we confirm the following case of Conjecture 1.4 of [10]. (Indeed, the h-vector of a homology sphere is symmetric and nonnegative [14].)

Corollary 5.6. *If Δ is a homology sphere, then $\gamma(\mathrm{sd}(\Delta))$ is an FFK-vector. In other words, $\gamma(\mathrm{sd}(\Delta))$ is the f-vector of a balanced simplicial complex.*

Frohmader [5] proved that the f-vectors of flag simplicial complexes form a (proper) subset of the f-vectors of balanced complexes. In [10], all the γ-vectors in question were shown to be the f-vectors of flag complexes. This suggests the following problem, for which a positive answer would imply stronger conditions on $\gamma(\mathrm{sd}(\Delta))$ than Corollary 5.6.

Problem 5.7. *Let Δ be a homology sphere. Is $\gamma(\mathrm{sd}(\Delta))$ the f-vector of a flag simplicial complex?*

Remark 5.8. *It would be interesting if one could explicitly construct a simplicial complex $\Gamma(\mathrm{sd}(\Delta))$ such that $f(\Gamma) = \gamma(\mathrm{sd}(\Delta))$, as in Section 4 of* [10]. *Such a construction may follow from a "nice" combinatorial interpretation for the entries of the vectors $\gamma^{(n,j)}$.*

6. Implications for h- and g-vectors

It is easy to see that γ-nonnegativity implies h-nonnegativity, as well as g-nonnegativity. Moreover, if the polynomial $\gamma(t)$ has only real roots then $h(t)$ and $g(t)$ have only real roots as well. (This follows from the fact that $h(t)$ and $g(t)$ are obtained by totally nonnegative linear transformations of $\gamma(t)$. See work of Brändén [1] for an introduction to these type of results.) In the context of this work, it is natural to ask whether the fact that γ is the f-vector of a simplicial complex implies that the same is true of the h- or g-vector. We show here that the answer is affirmative.

Remark 6.1. *Brenti and Welker show* [2, Theorem 3.1] *that $h(\mathrm{sd}(\Delta); t)$ has only real roots. This is another way to prove the Charney-Davis conjecture for barycentric subdivisions. See, for example, the discussion after Example 3.7 in* [2]. *We have not attempted to investigate whether it is true that $\gamma(\mathrm{sd}(\Delta); t)$ has only real roots, but if true, this would imply Brenti and Welker's Theorem 3.1 as well. However, as already mentioned in the introduction, our Theorem 1.1 implies Gal's conjecture for barycentric subdivisions, which in turn implies the Charney-Davis conjecture in this case.*

Suppose throughout this section that, respectively, $h = (h_0, h_1, \ldots, h_d)$ (with $h_i = h_{d-i}$), $g = (g_0, g_1, \ldots, g_{\lfloor d/2 \rfloor})$, and $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor d/2 \rfloor})$ are the h-, g-, and γ-vectors of some $(d-1)$-dimensional boolean complex with a nonnegative and symmetric h-vector.

First, we have the following observation for relating the h- and g-vectors to the γ-vector.

Observation 6.2. *We have, for $0 \leq i \leq \lfloor d/2 \rfloor$:*

$$h_i = h_{d-i} = \sum_{0 \leq j \leq i} \gamma_j \binom{d-2j}{i-j},$$

and

$$g_i = \sum_{0 \leq j \leq i} \gamma_j \left(\binom{d-2j}{i-j} - \binom{d-2j}{i-j-1} \right).$$

We can transform a γ-vector into its corresponding h- or g-vector with the following transformations:

$$A := \left[\binom{d-2j}{i-j} \right]_{0 \leq i,j \leq d/2}, \quad B := \left[\binom{d-2j}{i-j} - \binom{d-2j}{i-j-1} \right]_{0 \leq i,j \leq d/2}.$$

We have $A\gamma = (h_0, \ldots, h_{\lfloor d/2 \rfloor})$ (construct the rest of h by symmetry) and $B\gamma = g$. We remark that matrices A and B are *totally nonnegative*, i.e., the determinant of every minor of these matrices are nonnegative. (It is straightforward to prove this with a "Lindstrom-Gessel-Viennot"-type argument involving planar networks.)

Notice that at the extremes we have $g_0 = \gamma_0$ and

$$g_{\lfloor d/2 \rfloor} = \sum_{0 \leq j \leq d/2} \gamma_j C_{\lceil d/2 \rceil - j},$$

where $C_r = \binom{2r}{r} - \binom{2r}{r-1} = \binom{2r}{r}/(r+1)$ is a Catalan number.

Proposition 6.3. *If γ is an f-vector of a simplicial complex then h is an f-vector of a simplicial complex.*

Proof. Let $\mathcal{F}(\gamma)$ denote the standard compressed complex for γ (take the definition for $\mathcal{F}_d(\gamma)$ and let $d \to \infty$). Then $\mathcal{F}(\gamma)$ is a simiplicial complex. Let

$$\Delta = \{F \cup G : F \in \mathcal{F}(\gamma), G \in 2^{[d-2|F|]}\},$$

where $2^{[k]}$ denotes a $(k-1)$-simplex on a vertex set $\{1, 2, \dots, k\}$ disjoint from the vertex set of $\mathcal{F}(\gamma)$. It is straightforward to see that Δ is in fact a simplicial complex. Indeed, if $\bar{F} = F \cup G$ is a face of Δ, then all subsets of $\bar{F}$ are of the form $\bar{H} = F' \cup G'$, where $F' \subseteq F$ and $G' \subseteq G$. As $F' \in \mathcal{F}(\gamma)$ and $G' \in 2^{[k]}$ for all $k \geq |G'|$, we see that $\bar{H} \in \Delta$.

We compute

$$\begin{aligned} f_i(\Delta) = \sum_{|F \cup G| = i} 1 = \sum_{0 \leq j \leq i} \sum_{|F|=j} \sum_{|G|=i-j} 1 &= \sum_{0 \leq j \leq i} f_j(\mathcal{F}(\gamma)) \cdot f_{i-j}(2^{[d-2j]}), \\ &= \sum_{0 \leq j \leq i} \gamma_j \binom{d-2j}{i-j}. \end{aligned}$$

By Observation 6.2, we conclude that $h_i = f_i(\Delta)$, completing the proof. □

The corresponding result for g-vectors follows the same approach, but first requires the construction of an auxillary complex $\mathcal{B}(k)$ defined below.

A *ballot path* is a lattice path that starts at $(0, 0)$, takes steps of the form $(0, 1)$ (north) and $(1, 0)$ (east), and does not go above the line $y = x$ (but it can touch the line). We write ballot paths as words p on $\{N, E\}$ such that any initial subword of p has at least as many letters E as letters N. Let $B(k) = \{\text{ballot paths of length } k\}$, and for $p = p_1 \cdots p_k \in B(k)$, let $S(p) = \{k + 1 - i : p_i = N\}$ denote the set of positions of letters N in p (read from right to left). For example, the ballot path $p = ENEENEE$ has $S(p) = \{3, 6\}$.

Let $\mathcal{B}(k) = \{S(p) : p \in B(k)\}$ denote the set of sets of north steps for ballot paths of length k.

Proposition 6.4. *For any $k \geq 0$, $\mathcal{B}(k)$ is a simplicial complex on vertex set $[k-1]$. Moreover,* $\dim \mathcal{B}(k) = \lfloor k/2 \rfloor - 1$, *and* $f(\mathcal{B}(k)) = (f_0, f_1, \dots, f_{\lfloor k/2 \rfloor})$ *is given by*

$$f_i(\mathcal{B}(k)) = \binom{k}{i} - \binom{k}{i-1}.$$

Proof. To see that $\mathcal{B}(k)$ is a simplicial complex, simply observe that if p is ballot path, changing any north step to an east step results in a path that remains below $y = x$. That the vertex set is $\{1, 2, \dots, k-1\}$, follows from observing that the first step of a ballot path must be east.

The dimension of a face is simply one less than the number of north steps taken in the corresponding path. Thus,

$$\begin{aligned} f_i(\mathcal{B}(k)) &= |\{p \in B(k) : |S(p)| = i\}| \\ &= |\{\text{lattice paths from } (0,0) \text{ to } (k-i, i) \text{ not surpassing the line } y = x\}| \\ &= \binom{k}{i} - \binom{k}{i-1}. \end{aligned}$$

The final equality is a straightforward counting argument described, e.g., in [13, Exercise 6.20]. (Show there are $\binom{k}{i-1}$ paths from $(0, 0)$ to $(k - i, i)$ that go above $y = x$.)

To complete the proof, observe that a maximal face corresponds to a path from $(0,0)$ to $(k/2,k/2)$ if k is even, or from $(0,0)$ to $(\frac{k+1}{2},\frac{k-1}{2})$ if k is odd. Thus, $\dim \mathcal{B}(k) = \lfloor k/2 \rfloor - 1$. □

We remark that as $f_{\lfloor k/2 \rfloor} = \binom{k}{\lfloor k/2 \rfloor} - \binom{k}{\lfloor k/2 \rfloor - 1} = C_{\lceil k/2 \rceil}$, the number of facets of $\mathcal{B}(k)$ is a Catalan number.

Proposition 6.5. *If γ is an f-vector of a simplicial complex then g is an f-vector of a simplicial complex.*

Proof. Similarly to the proof of Proposition 6.3, let $\mathcal{F}(\gamma)$ denote the standard compressed complex for γ. Now, let

$$\Delta = \{F \cup G : F \in \mathcal{F}(\gamma), G \in \mathcal{B}(d - 2|F|)\},$$

where we take the vertex set $\{1, 2, \ldots, k-1\}$ for $\mathcal{B}(k)$ to be disjoint from the vertex set of $\mathcal{F}(\gamma)$. To see that Δ is in fact a simplicial complex suppose $\bar{F} = F \cup G$ is a face of Δ. All subsets of $\bar{F}$ are of the form $\bar{H} = F' \cup G'$, where $F' \subseteq F$ and $G' \subseteq G$. That $F' \in \mathcal{F}(\gamma)$ is obvious. To prove $\bar{H} \in \Delta$, it remains to show that $G' \in \mathcal{B}(k)$ for all $k \geq |G'|$. To see this, consider path $p \in B(|G'|)$ such that $S(p) = G'$. For any $i \geq 0$, we can extend this path to a path p' in $B(|G'| + i)$ by prepending i east steps to p. In other words,

$$p' = \underbrace{EE \cdots E}_{i} p \in B(|G'| + i)$$

and $S(p') = S(p) = G'$.

We conclude

$$f_i(\Delta) = \sum_{|F \cup G| = i} 1 = \sum_{0 \leq j \leq i} \sum_{|F| = j} \sum_{|G| = i-j} 1 = \sum_{0 \leq j \leq i} f_j(\mathcal{F}(\gamma)) \cdot f_{i-j}(\mathcal{B}(d - 2j)),$$

$$= \sum_{0 \leq j \leq i} \gamma_j \left(\binom{d-2j}{i-j} - \binom{d-2j}{i-j-1} \right).$$

By Observation 6.2, we conclude that $g_i = f_i(\Delta)$, completing the proof. □

Considering the results of this paper and those of [10, Theorem 6.1], we have the following consequences of Propositions 6.3 and 6.5.

Corollary 6.6. *We have that $h(\Delta)$ and $g(\Delta)$ are f-vectors of simplicial complexes if:*

(a) *Δ is the barycentric subdivision of a boolean complex with symmetric and nonnegative h-vector (e.g., Δ is the barycentric subdivision of a simplicial sphere).*
(b) *Δ is a Coxeter complex.*
(c) *Δ is the simplicial complex dual to an associahedron.*
(d) *Δ is the simplicial complex dual to a cyclohedron.*
(e) *Δ is a flag homology sphere with $\gamma_1(\Delta) \leq 3$.*

We remark that it is known that $h(\Delta)$ is the f-vector of a balanced simplicial complex whenever Δ is a balanced Cohen-Macaulay complex [12, Theorem 4.6]. As mentioned in the introduction, recent work of Murai [9] proves Corollary 6.6(a) (for g-vectors) directly, building on the approach of Brenti and Welker. That the g-vector of the barycentric subdivision of a Cohen-Macaulay complex is an M-sequence was proved earlier by algebraic means [8].

One may ask whether the results of Propositions 6.3 and 6.5 can be strengthened to f-vectors of *balanced* simplicial complexes. As we will show below, the answer is Yes for the h-vector and No for the g-vector.

First let us remark that for flag spheres of dimension at most 3, it is known that the γ-vector is an FFK-vector. Then a simple computation shows that the g-vector is an FFK-vector as well. However, this does not hold in higher dimensions.

Example 6.7. *Let Δ be the triangulation of the 4-sphere obtained by taking the suspension of the join of two $(k+2)$-gons, where $k \geq 2$. Then Δ is a flag sphere with*

$$h(\Delta) = (1, 2k+1, (k+1)^2+1, (k+1)^2+1, 2k+1, 1).$$

Its γ-vector is $(1, 2(k-2), (k-2)^2)$, which is 2-FFK, while $g(\Delta) = (1, 2k, k^2+1)$, which is not 2-FFK.

(We see that γ is the f-vector the complete bipartite graph on $2(k-2)$ vertices, while there can be no bipartite graph with $2k$ vertices and more than k^2 edges.)

Proposition 6.8. *If γ is the f-vector of a balanced simplicial complex then h is the f-vector of a balanced simplicial complex.*

Proof. Let γ be the f-vector of a balanced complex. In particular, γ is d-FKK (as it is $\lfloor \frac{d}{2} \rfloor$-FFK). Let Δ be a d-colorable simplicial complex with $f(\Delta) = \gamma$ on a vertex set V, and with a proper coloring $c : V \to [d]$ (assume $V \cap [d] = \emptyset$). For $F \in \Delta$, let $d(F)$ be the set of the first $|F|$ integers in $[d] - c(F)$. Consider the following simplicial complex

$$\Gamma = \{F \cup G : \ F \in \Delta, G \subset 2^{[d]-(c(F)\cup d(F))}\}.$$

Then Γ is a d-colorable simplicial complex with $f(\Gamma) = h$. Indeed, each face $F \cup G$ of Γ has distinctly colored vertices by construction. To see that Γ is indeed a simplicial complex, note that $F' \subseteq F \subseteq V$ implies $c(F') \subseteq c(F)$, so if $F \cup G \in \Gamma$ then $G \subseteq [d] - (c(F) \cup d(F)) \subseteq [d] - (c(F') \cup d(F'))$, hence $F' \cup G \in \Gamma$. □

Thus, a consequence of Conjecture 2 would be the following.

Conjecture 3. *If Δ is a flag homology sphere then $g(\Delta)$ is the f-vector of a simplicial complex and $h(\Delta)$ is the f-vector of a balanced simplicial complex.*

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Department of Mathematics, Cornell University, Ithaca, NY 14853, USA
E-mail address: `eranevo@math.cornell.edu`

Department of Mathematical Sciences, DePaul University, Chicago, IL 60614, USA
E-mail address: `tkpeters@math.depaul.edu, bridget@math.depaul.edu`