Contemporary Mathematics

Restriction of Saito-Kurokawa representations

Wee Teck Gan and Nadya Gurevich
with an appendix by Gordan Savin

to Professor Steve Gelbart
on the occasion of his sixtieth birthday

Abstract. We study the restriction of the Saito-Kurokawa representations of SO_5 to various subgroups SO_4, giving a precise determination of which representations of SO_4 occurs this restriciton. Locally, the answer is determined by an epsilon factor condition, whereas globally it is controlled by the non-vanishing of an L-function. This is the simplest example of an extension of the Gross-Prasad conjecture from the setting of tempered L-packets to A-packets.

1. Introduction

In [GP], Gross and Prasad formulated a very precise conjecture describing the branching of an irreducible representation of SO_n when restricted to SO_{n-1} over a local field. Their conjecture, however, assumes the local Langlands correspondence for special orthogonal groups and so can only be checked in cases where one has (at least partially) such a correspondence. This is the case, for example, in many low rank groups, or for certain tamely ramified Langlands parameters. Investigations of the local Gross-Prasad conjecture can be found in a number of papers, such as [P1], [P2] and [GR].

In addition to the local conjecture, there is also a global Gross-Prasad conjecture regarding SO_{n-1}-periods of cusp forms on $SO_n \times SO_{n-1}$. When there are no local obstructions, the non-vanishing of the global period should be controlled by the non-vanishing of a relevant Rankin-Selberg L-function. There have been much significant progress and refinements on this global conjecture recently; see for example [GJR], [BFS] and [II].

The local conjecture of [GP] focuses on addressing the branching problem from SO_n to SO_{n-1} as the representations involved vary over a tempered L-packet; the answer is governed by a condition on epsilon factors. In view of global applications, it is natural to ask how the branching problem would behave if the representations were to vary over a (non-tempered) Arthur packet. The goal of this short paper is to investigate this for one of the best-understood non-tempered Arthur packets, namely the Saito-Kurokawa packets for SO_5.

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We shall recall the definition and construction of the Saito-Kurokawa packets in Section 2. At this point, we simply note that each irreducible infinite dimensional representation π of PGL_2 determines a packet of (at most) two representations of the split group $SO(3,2)$, which will be denoted by $\eta^+(\pi)$ and $\eta^-(\pi)$. We are interested in the restriction of $\eta^\epsilon(\pi)$ to the subgroup $SO(2,2) \subset SO(3,2)$. Since $GSO(2,2) \cong (GL_2 \times GL_2)/\Delta\mathbb{G}_m$, one sees that an L-packet on $SO(2,2)$ is indexed by a representation $\tau_1 \boxtimes \tau_2$ of $GSO(2,2)$. The elements of the L-packet are simply the irreducible constituents of the restriction of $\tau_1 \boxtimes \tau_2$ to $SO(2,2)$. With these notations in place, our main local theorem is:

Main Local Theorem

Over a non-archimedean local field of charateristic zero, we have:

(i) $\mathrm{Hom}_{SO(2,2)}(\eta^\epsilon(\pi), \tau_1 \boxtimes \tau_2) = 0$ if $\tau_1 \neq \tau_2^\vee$.

(ii) $Hom_{SO(2,2)}(\eta^\epsilon(\pi), \tau \boxtimes \tau^\vee) \neq 0$ if and only if $\epsilon = \epsilon(1/2, \pi \otimes \tau \otimes \tau^\vee)$, in which case the dimension of the Hom space is 1.

After recalling some basic properties of the theta correspondence for similitude groups in Section 3, we give the proof of the main theorem in Section 4 and describe variants of the theorem for arbitrary forms of SO_5 and SO_4 in Section 5. The restriction to $SO(3,1)$ is especially interesting, but the result is too intricate to state precisely here. We should stress that all our results about epsilon dichotomy have their roots in Prasad's thesis [P1]; we have simply percolated his results to higher rank cases. In Section 6, we discuss the archimedean analog of the main theorem which has been studied by Savin [Sa], who has kindly provided us with an appendix. Using these local results, we shall prove in Section 7 a precise global analog relating the non-vanishing of $SO(2,2)$-periods with the non-vanishing of a suitable L-function:

Main Global Theorem

Let π be a cuspidal representation of PGL_2 and τ a cuspidal representation of GL_2. Let $\epsilon_v = \epsilon(1/2, \pi_v \otimes \tau_v \otimes \tau_v^\vee)$ and let $\eta^{\underline{\epsilon}}(\pi) = \bigotimes_v \eta^{\epsilon_v}(\pi_v)$ be the corresponding representation in the global Saito-Kurokawa packet associated to π. Then the following are equivalent:

(a) the representation $\eta^{\underline{\epsilon}}(\pi) \otimes (\tau \otimes \tau^\vee)$ of $SO(3,2) \times SO(2,2)$ occurs in the discrete spectrum and has non-vanishing period integral over the diagonal subgroup $SO(2,2)$;

(b) the following non-vanishing result holds:

$$L(1/2, \pi \times Ad(\tau)) \neq 0.$$

For any other representation in the global Saito-Kurokawa packet, the period integral is zero.

We should mention that in [I], Ichino has given an explicit formula relating the special value of the L-function to the square of the absolute value of the period integral above, when the cuspidal representations involved are associated to holomorphic modular forms of level 1. It will be very interesting to prove such a formula in general, in the style of the refinement of the global Gross-Prasad conjecture given by Ichino-Ikeda in [II].

To see that the special L-value in the global theorem is indeed the one predicted by the global Gross-Prasad conjecture, or rather its refinement given in [II], we recall that to an A-parameter ψ, one can naturally associate an L-parameter ϕ_ψ. If ψ is the A-parameter of the Saito-Kurokawa packet attached to a cuspidal representation π of PGL_2, then the associated L-parameter is given by:

$$\phi_\psi = \phi_\pi \oplus |-|^{1/2} \oplus |-|^{-1/2},$$

where ϕ_π is the L-parameter of π. According to [II], the L-value which should control the non-vanishing of the period integral in the above global theorem is the value at $s = 1/2$ of:

$$\mathcal{P}(s) = \frac{L(s, \phi_\psi \otimes \phi_\tau \otimes \phi_\tau^\vee)}{L(s+1/2, Ad \circ \phi_\psi) \cdot L(s+1/2, Ad \circ \phi_\tau) \cdot L(s+1/2, Ad \circ \phi_\tau)}.$$

Expressing the L-functions occurring in $\mathcal{P}(s)$ in terms of automorphic L-functions and evaluating at $s = 1/2$, we see after a short computation that

$$\mathcal{P}(1/2) = \frac{L(1/2, \pi \times Ad(\tau))}{\zeta_F(2) \cdot L(3/2, \pi) \cdot L(1, Ad(\pi))}.$$

Since the denominator is harmless, we see that the non-vanishing of $\mathcal{P}(1/2)$ is equivalent to that of $L(1/2, \pi \times Ad(\tau))$.

Finally, we end the paper by resolving a couple of miscellaneous problems for the Saito-Kurokawa representations, such as if their pullbacks to $Spin_5 = Sp_4$ remain irreducible and what are the local Bessel models that they support.

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2. Saito-Kurokawa Representations

Let F be a non-archimedean local field and fix a non-trivial additive character ψ of F. We begin by recalling the definition and construction of the Saito-Kurokawa A-packets on $PGSp_4$.

The Saito-Kurokawa packets are indexed by irreducible infinite dimensional (unitary) representations of $PGL_2(F)$. Given such a representation π of PGL_2, Waldspurger has associated a packet $\tilde{A}_\pi$ of irreducible genuine (unitary) representations of the metaplectic group $\widetilde{SL}_2(F)$. The local packet $\tilde{A}_\pi$ has two or one element, depending on whether π is a discrete series representation or not. Thus $\tilde{A}_\pi$ has the form

$$\tilde{A}_\pi = \begin{cases} \{\sigma^+, \sigma^-\}, & \text{if } \pi \text{ is a discrete series representation,} \\ \{\sigma^+\} & \text{otherwise.} \end{cases}$$

While the packets themselves are canonical, their parametrization by the representations of PGL_2 depends on the choice of the additive character ψ. With ψ fixed, we shall write

$$\pi = Wd_\psi(\sigma) \quad \text{if } \sigma \in \tilde{A}_\pi.$$

Moreover, if Z is the center of SL_2, its inverse image $\tilde{Z}$ is the center of $\tilde{SL}_2$ and the central character ω_σ of σ has the form

$$\omega_\sigma = \chi_\psi|_{\tilde{Z}} \cdot \epsilon_\psi(\sigma)$$

where χ_ψ is a canonical genuine character (defined in [W1]) of the diagonal torus in $\tilde{SL}_2$ and $\epsilon_\psi(\sigma)$ is a character of Z. We shall regard $\epsilon_\psi(\sigma)$ as ± 1, depending on whether this character is trivial or not. If $\sigma^\epsilon \in \tilde{A}_\pi$, then

$$\epsilon_\psi(\sigma^\epsilon) = \epsilon \cdot \epsilon(1/2, \pi).$$

Thus the representations in $\tilde{A}_\pi$ can be distinguished by their central characters. Suppose that K is an étale quadratic algebra, corresponding to $a_K \in F^\times/F^{\times 2}$, let ψ_K denote the additive character $\psi_K(x) = \psi(a_K x)$. Then

$$\begin{cases} Wd_{\psi_K}(\sigma) = Wd_\psi(\sigma) \otimes \chi_K \\ \epsilon_{\psi_K}(\sigma) = \epsilon_\psi(\sigma) \cdot \chi_K(-1). \end{cases}$$

The packet $\tilde{A}_\pi$ is constructed by using the local theta lift (associated to ψ) furnished by the dual pairs:

$$PGL_2 \times \tilde{SL}_2 \quad \text{and} \quad PD^\times \times \tilde{SL}_2,$$

where D denotes the unique quaternion division algebra over F. Indeed, we have:

$$\sigma^+ = \theta_\psi(\pi) \quad \text{and} \quad \sigma^- = \theta_\psi(JL(\pi))$$

where $JL(\pi)$ is the Jacquet-Langlands lift of π to $PD^\times$.

Now to construct the Saito-Kurokawa A-packet $SK(\pi)$ of $PGSp_4 \cong SO_5$ associated to π, one considers the theta correspondence furnished by the dual pair

$$\tilde{SL}_2 \times SO_5 \subset \tilde{Sp}_{10}$$

and set

$$\eta^+(\pi) = \theta_\psi(\sigma^+) \quad \text{and} \quad \eta^-(\pi) = \theta_\psi(\sigma^-).$$

Then the Saito-Kurokawa packet (which is independent of ψ) is:

$$SK(\pi) = \{\eta^+(\pi), \eta^-(\pi)\}.$$

The following proposition describes these representations more precisely (cf. [G]):

PROPOSITION 2.1. *(i) Let $P = MN$ be the Siegel parabolic of SO_5, with Levi factor $M = PGL_2 \times GL_1$. Let $J_P(\pi, 1/2)$ be the unique irreducible quotient of the normalized induced representation*

$$I_P(\pi, 1/2) = Ind_P^{SO_5} \pi \boxtimes |-|^{1/2}.$$

Then we have

$$\eta^+(\pi) = J_P(\pi, 1/2).$$

(ii) Suppose that $\pi = St$ is the Steinberg representation. Let Q be the other maximal parabolic of SO_5, with Levi factor $L = GL_2$. Then $\eta^-(St)$ is the unique non-generic summand in the normalized induced representation $I_Q(St)$ (which is semisimple with two summands).

(iii) When π is supercuspidal or a twisted Steinberg representation St_χ (with χ a nontrivial quadratic character), $\eta^-(\pi)$ is supercuspidal.

The above proposition describes the representations in $SK(\pi)$ except when π is supercuspidal or twisted Steinberg, in which case it does not offer any information on $\eta^-(\pi)$ (other than supercuspidality). However, there is another way of constructing the packet $SK(\pi)$. We shall describe this alternative construction at the end of the next section.

3. Theta Correspondences for Similitudes

In this section, we shall describe some basic properties of theta correspondences for similitudes; in particular, we shall relate it to the usual theta correspondences for isometric groups. The definitive reference for this material is the paper [Ro1] of B. Roberts.

Suppose that $O(V) \times Sp(W)$ is a dual pair; for simplicity, we have assumed that $\dim V$ is even. For each non-trivial additive character ψ, let ω_ψ be the Weil representation for $O(V) \times Sp(W)$. If π is an irreducible representation of $O(V)$ (resp. $Sp(W)$), the maximal π-isotypic quotient has the form

$$\pi \boxtimes \theta_{\psi,0}(\pi)$$

for some smooth representations of $Sp(W)$ (resp. $O(V)$). It is known that $\theta_{\psi,0}(\pi)$ is of finite length and hence is admissible. Let $\theta_\psi(\pi)$ be the maximal semisimple quotient of $\theta_{\psi,0}(\pi)$. Then it was a conjecture of Howe that

- $\theta_\psi(\pi)$ is irreducible whenever $\theta_{\psi,0}(\pi)$ is non-zero.
- the map $\pi \mapsto \theta_\psi(\pi)$ is injective on its domain.

This has been proved by Waldspurger when the residual characteristic of F is not 2, as well as for all supercuspidal representations π. It can also be checked in many low-rank cases, regardless of the residual characteristic of F. In particular, it holds in all cases considered in this paper. Henceforth, we assume that the Howe conjecture for isometry groups holds.

Let λ_V and λ_W be the similitude factors of $GO(V)$ and $GSp(W)$ respectively. We shall consider the group

$$R = GO(V) \times GSp(W)^+$$

where $GSp(W)^+$ is the subgroup of $GSp(W)$ consisting of elements g such that $\lambda_W(g)$ is in the image of λ_V. The group R contains the subgroup

$$R_0 = \{(h, g) \in R : \lambda_V(h) \cdot \lambda_W(g) = 1\}.$$

The Weil representation ω_ψ extends naturally to the group R_0. Now consider the (compactly) induced representation

$$\Omega = ind_{R_0}^R \omega_\psi.$$

As a representation of R, Ω depends only on the orbit of ψ under the evident action of $Im\lambda_V \subset F^\times$. For example, if λ_V is surjective, then Ω is independent of ψ. For any irreducible representation π of $GO(V)$ (resp. $GSp(W)^+$), the maximal π-isotypic quotient of Ω has the form

$$\pi \otimes \theta_0(\pi)$$

where $\theta_0(\pi)$ is some smooth representation of $GSp(W)^+$ (resp. $GO(V)$). Further, we let $\theta(\pi)$ be the maximal semisimple quotient of $\theta_0(\pi)$. The extended Howe conjecture for similitudes says that $\theta(\pi)$ is irreducible whenever $\theta_0(\pi)$ is non-zero, and the map $\pi \mapsto \theta(\pi)$ is injective on its domain. It was shown by Roberts [Ro1] that this essentially follows from the Howe conjecture for isometry groups. In particular, we have the following lemma which relates the theta correspondence for isometries and similitudes:

Lemma 3.1. *Assume that the Howe conjecture for isometry groups holds.*

(i) Suppose that

$$\mathrm{Hom}_R(\Omega, \pi_1 \boxtimes \pi_2) \neq 0.$$

Then there is a bijection

$$f : \{\textit{irreducible summands of } \pi_1|_{O(V)}\} \longrightarrow \{\textit{irreducible summands of } \pi_2|_{Sp(W)}\}.$$

such that for any irreducible summand τ_i in the restriction of π_i to the relevant isometry group,

$$\mathrm{Hom}_{O(V)\times Sp(W)}(\omega_\psi, \tau_1 \boxtimes \tau_2) \neq 0$$

if and only if

$$\tau_2 = f(\tau_1).$$

(ii) If τ is a representation of $GO(V)$ (resp. $GSp(W)^+$) and the restriction of τ to the relevant isometry group is $\oplus_i \tau_i$, then as representations of $Sp(W)$ (resp. $O(V)$),

$$\theta_0(\tau) \cong \bigoplus_i \theta_{\psi,0}(\tau_i).$$

In particular, if $\theta_{\psi,0}(\tau_i) = \theta_\psi(\tau_i)$ for each i, then

$$\theta_0(\tau) = \theta(\tau)$$

is irreducible.

Proof. (i) This is essentially [Ro1, Lemma 4.2]. We include the proof for the convenience of the reader. In [AP], it was shown that restrictions of irreducible representations from similitude groups to isometry groups are multiplicity-free. Thus we can write

$$\pi_1|_{O(V)} = \bigoplus_i \tau_i \quad \text{and} \quad \pi_2|_{Sp(W)} = \bigoplus_j \sigma_j.$$

Since $\mathrm{Hom}_R(\Omega, \pi_2 \otimes \pi_2) \neq 0$, one sees by Frobenius reciprocity that

$$\mathrm{Hom}_{O(V)\times Sp(W)}(\omega_\psi, \pi_1 \boxtimes \pi_2) \neq 0.$$

Hence, there are two irreducible constituents, say τ_1 and σ_1, such that

$$\mathrm{Hom}_{O(V)\times Sp(W)}(\omega_\psi, \tau_1 \boxtimes \sigma_1) \neq 0.$$

Now recall that the group R_0 normalizes $O(V,F) \times_{\mu_2} Sp(W,F)$ and the Weil representation ω_ψ extends to R_0. If $r \in R_0$ and L is a non-zero element of $\mathrm{Hom}_{O(V)\times Sp(W)}(\omega_\psi, \tau_1 \boxtimes \sigma_1)$, then the map $v \mapsto L(r \cdot v)$ defines a non-zero element of $\mathrm{Hom}_{O(V)\times Sp(W)}(\omega_\psi, r(\tau_1 \boxtimes \sigma_1))$.

Now the group R_0 acts transitively on the irreducible constituents of $\pi_1|_{O(V)}$, as well as on those of $\pi_2|_{Sp(W)}$, since the projections of R_0 to $GO(V)$ and $GSp(W)^+$ are surjective. Thus, for each τ_i, there is a σ_i such that

$$\mathrm{Hom}_{O(V)\times Sp(W)}(\omega_\psi, \tau_i \boxtimes \sigma_i) \neq 0,$$

and vice versa. Moreover, the equivalence classes of τ_i and σ_i determine each other by the Howe conjecture for isometry groups. Thus we have the desired bijection.

(ii) By symmetry, let us suppose that τ is a representation of $GSp(W)^+$. Then we have the following sequence of $O(V)$-equivariant isomorphisms:

$$\begin{aligned}\theta_0(\tau)^* &\cong \mathrm{Hom}_{GSp(W)^+}(\Omega, \tau) \\ &\cong \mathrm{Hom}_{Sp(W)}(\omega_\psi, \tau|_{Sp(W)}) \quad \text{(by Frobenius reciprocity)} \\ &\cong \bigoplus_i \mathrm{Hom}_{Sp(W)}(\omega_\psi, \tau_i) \\ &\cong \bigoplus_i \theta_{\psi,0}(\tau_i)^*.\end{aligned}$$

Thus, we have an $O(V)$-equivariant isomorphism of smooth vectors

$$\theta_0(\tau)^\vee \cong \bigoplus_i \theta_{\psi,0}(\tau_i)^\vee$$

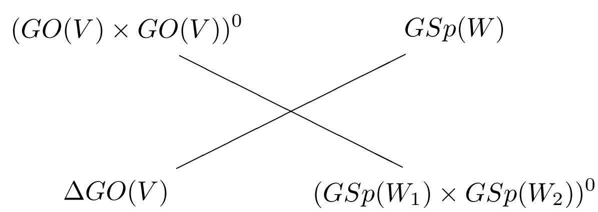
and the desired result follows by taking contragredient (and using the fact that the $\theta_{\psi,0}(\tau_i)$'s are admissible).

Now if $\theta_{\psi,0}(\tau_i) = \theta_\psi(\tau_i)$ is irreducible, then by (i), we see that any irreducible constituent π of $\theta(\tau)$ satisfies:

$$\pi|_{O(V)} = \bigoplus_i \theta_\psi(\tau_i).$$

In view of the above, we see that $\theta_0(\tau) = \theta(\tau)$ is irreducible. □

Now we consider the extension of the see-saw identity to similtiude groups. Asume for simplicity that λ_V is surjective so that $GSp(W)^+ = GSp(W)$. Suppose that $W = W_1 \oplus W_2$. Then one has the see-saw diagram:



Here,

$$(GSp(W_1) \times GSp(W_2))^0 = \{(g_1, g_2) : \lambda_{W_1}(g_1) = \lambda_{W_2}(g_2)\}$$

and similarly for $(GO(V) \times GO(V))^0$. The see-saw identity states that for irreducible representations σ and τ of $GO(V)$ and $(GSp(W_1) \times GSp(W_2))^0$ respectively,

$$\dim \operatorname{Hom}_{GO(V)}(\theta_0(\tau), \sigma) = \dim \operatorname{Hom}_{(GSp(W_1)\times GSp(W_2))^0}(\theta_0(\sigma), \tau).$$

Now suppose we take an irreducible representation $\tau_1 \boxtimes \tau_2$ of $GSp(W_1) \times GSp(W_2)$ and consider its restriction to $(GSp(W_1) \times GSp(W_2))^0$, say:

$$\tau_1 \boxtimes \tau_2 = \bigoplus_i \pi_i.$$

For each π_i, we have the representation $\theta_0(\pi_i)$ of $(GO(V) \times GO(V))^0$.

Lemma 3.2. *We have:*

$$\bigoplus_i \theta_0(\pi_i) \cong \theta_0(\tau_1) \boxtimes \theta_0(\tau_2)$$

as representations of $(GO(V) \times GO(V))^0$.

Proof. This is similar to the proof of Lemma 3.1(ii). □

Corollary 3.3. *In the setting of the lemma,*

$$\dim \operatorname{Hom}_{(GSp(W_1)\times GSp(W_2))^0}(\theta_0(\sigma), \tau_1 \boxtimes \tau_2) = \dim \operatorname{Hom}_{GO(V)}(\theta_0(\tau_1) \boxtimes \theta_0(\tau_2), \sigma).$$

PROOF. This follows from the see-saw identity and the lemma above. □

Let us conclude this section with the alternative construction of the Saito-Kurokawa packets. For this, one considers the theta correspondence for similitudes furnished by the dual pairs:

$$GSO(2,2) \times GSp_4 \cong (GL_2 \times GL_2)/\Delta\mathbb{G}_m \times GSp_4$$

and

$$GSO(4) \times GSp_4 \cong (D^\times \times D^\times)/\Delta\mathbb{G}_m \times GSp_4.$$

These correspondences have been studied in detail by B. Roberts in [Ro2].

Given any representation $\pi_1 \boxtimes \pi_2$ of $GSO(V) = GSO(2,2)$ or $GSO(4)$, let $(\pi_1 \boxtimes \pi_2)^+$ denote $ind_{GSO(V)}^{GO(V)}(\pi_1 \boxtimes \pi_2)$ if $\pi_1 \neq \pi_2^\vee$. If $\pi_1 = \pi_2^\vee$, there will be two extensions of $\pi_1 \boxtimes \pi_2$ to $GO(V)$, but exactly one of them will participate in the theta correspondence with GSp_4 (cf. [Ro2]). We let $(\pi_1 \boxtimes \pi_2)^+$ denote this unique extension of $\pi_1 \boxtimes \pi_2$ to $GO(V)$ which participates in the theta correspondence with GSp_4.

Now one has the following result of R. Schmidt [Sch]:

PROPOSITION 3.4. *Let π be an irreducible infinite-dimensional representation of PGL_2. We have:*

$$\eta^+(\pi) = \theta((\pi \boxtimes \mathbf{1})^+) \quad \textit{and} \quad \eta^-(\pi) = \theta_D((JL(\pi) \boxtimes \mathbf{1_D})^+).$$

4. Proof of the Main Local Theorem

We are now ready to consider the restriction of the representations $\eta^\pm(\pi)$ to the subgroup

$$H = SO(2,2) = (GL_2 \times GL_2)^0/\Delta\mathbb{G}_m \subset SO_5$$

and to give the proof of the main local theorem stated in the introduction. More precisely, given a pair of irreducible infinite-dimensional representations τ_1 and τ_2 of $GL_2(F)$ whose central characters are inverses of each other, we would like to compute

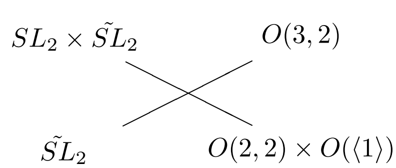
$$\dim \mathrm{Hom}_H(\eta^\pm(\pi), \tau_1 \boxtimes \tau_2).$$

Note that the restriction of $\tau_1 \boxtimes \tau_2$ from $(GL_2 \times GL_2)/\Delta\mathbb{G}_m$ to H may be reducible. Indeed, the irreducible components (which all occur with multiplicity one) make up a single L-packet of H indexed by $\tau_1 \boxtimes \tau_2$. Moreover, for any character χ,

$$(\tau_1 \otimes \chi) \boxtimes (\tau_2 \otimes \chi^{-1}) \cong \tau_1 \boxtimes \tau_2$$

as representations of H.

Consider the see-saw pair:



Suppose that on restriction to $O(2,2)$, $(\tau_1 \boxtimes \tau_2)^+ = \bigoplus_i \tau_i'$. Then we have:

$$\begin{aligned}
&\dim \mathrm{Hom}_{SO(2,2)}(\theta_{\psi,0}(\sigma^\epsilon), \tau_1 \boxtimes \tau_2)\\
&= \dim \mathrm{Hom}_{O(2,2)}(\theta_{\psi,0}(\sigma^\epsilon), (\tau_1 \boxtimes \tau_2)^+) \quad \text{(by Frobenius reciprocity)}\\
&= \sum_i \dim \mathrm{Hom}_{O(2,2)}(\theta_{\psi,0}(\sigma^\epsilon), \tau_i')\\
&= \sum_i \dim \mathrm{Hom}_{\tilde{SL}_2}(\theta_{\psi,0}(\tau_i') \otimes \omega_\psi, \sigma^\epsilon) \quad \text{(by see-saw identity)}\\
&= \dim \mathrm{Hom}_{\tilde{SL}_2}(\theta_0((\tau_1 \boxtimes \tau_2)^+) \otimes \omega_\psi, \sigma^\epsilon) \quad \text{(by Lemma 3.1(ii))}.
\end{aligned}$$

Now the theta correspondence from $GO(2,2)$ to GL_2 is well-understood (cf. [Ro2]). Indeed, one has:

LEMMA 4.1. *Let D be a quaternion algebra (possibly split) and consider the theta lifting between GL_2 and $GO(D,-N_D) \cong ((D^\times \times D^\times)/\Delta\mathbb{G}_m) \rtimes \mathbb{Z}/2\mathbb{Z}$. Let τ_i be irreducible infinite-dimensional representations of GL_2 and denote by $JL_D(\tau_i)$ the Jacquet-Langlands lift of τ_i to $D^\times$.*

(i) (Lifting to GL_2) If $\tau_1 \neq \tau_2^\vee$, then the induction $(\tau_1 \boxtimes \tau_2^\vee)^+$ of $\tau_1 \boxtimes \tau_2^\vee$ to $GO(D,-N_D)$ is irreducible and

$$\theta_0((JL_D(\tau_1) \boxtimes JL_D(\tau_2))^+) = 0.$$

On the other hand, of the two possible extensions of $\tau \boxtimes \tau^\vee$ to $GO(D,-N_D)$, exactly one of them, denoted by $(\tau \boxtimes \tau^\vee)^+$, participates in the theta correspondence and one has:

$$\theta_0((\tau \boxtimes \tau^\vee)^+) = \theta((\tau \boxtimes \tau^\vee)^+) = \tau.$$

(ii) (Lifting from GL_2) Similarly, we have

$$\theta_0(\tau) = \theta(\tau) = (JL_D(\tau) \boxtimes JL_D(\tau)^\vee)^+.$$

In particular, if D is non-split, then $\theta_0(\tau) = 0$ if τ is a principal series.

Moreover, one also has:

LEMMA 4.2. *Consider the theta lift from $\tilde{SL}_2$ to $SO(3,2)$. If σ is not equal to an even Weil representation or the principal series $\tilde{\pi}(|-|^{\pm 3/2})$, then*

$$\theta_{\psi,0}(\sigma) = \theta_\psi(\sigma).$$

PROOF. If σ is supercuspidal, this follows from a general result of Kudla. Now consider a (possibly reducible) principal series $\tilde{\pi}(\mu)$ of $\tilde{SL}_2$. Let ω_ψ denote the Weil representation of $\tilde{SL}_2 \times SO(3,2)$. An easy computation using the Schrodinger model shows that

$$\mathrm{Hom}_{\tilde{SL}_2}(\omega_\psi, \tilde{\pi}(\mu)) = I_P(\mu^{-1})^* \quad \text{(full linear dual)},$$

except possibly for $\mu = |-|^{3/2}$. If $\tilde{\pi}(\mu)$ is irreducible, so that $\mu \neq \chi_K|-|^{\pm 1/2}$ with χ_K a quadratic character, then we conclude that

$$\theta_{\psi,0}(\tilde{\pi}(\mu))^* = I_P(\mu^{-1})^*$$

Thus, if further $\mu \neq |-|^{\pm 3/2}$, we have

$$\theta_{\psi,0}(\tilde{\pi}(\mu)) = \theta_\psi(\tilde{\pi}(\mu)) = I_P(\mu^{-1}) = I_P(\mu).$$

It remains to consider the case when $\sigma = sp_{\chi_K}$ is the special representation associated to the quadratic character χ_K. From the above, we know that

$$\theta_{0,\psi}(sp_{\chi_K})^* \hookrightarrow I_P(\chi_K|-|^{-1/2})^*.$$

The latter degenerate principal series is of length 2 and thus we need to show

$$\theta_{0,\psi}(sp_{\chi_K}) \neq I_P(\chi_K|-|^{-1/2}).$$

Suppose not. Then we have a surjective equivariant map

$$\omega_\psi \longrightarrow sp_{\chi_K} \boxtimes I_P(\chi_K|-|^{-1/2}),$$

and thus an injection

$$sp^*_{\chi_K} \hookrightarrow \mathrm{Hom}_{SO(3,2)}(\omega_\psi, I_P(\chi_K|-|^{-1/2}).$$

An easy calculation, using a mixed model of the Weil representation, gives

$$\mathrm{Hom}_{SO(3,2)}(\omega_\psi, I_P(\mu)) = \tilde{\pi}(\mu^{-1})^*$$

except for $\mu = |-|^{-1/2}$. Thus if χ_K is nontrivial, we would have

$$sp^*_{\chi_K} \hookrightarrow \tilde{\pi}(\chi_K|-|^{1/2})^*$$

and deduce that there is a surjection

$$\tilde{\pi}(\chi_K|-|^{1/2}) \twoheadrightarrow sp_{\chi_K}$$

which is a contradiction. In the case when $\mu = |-|^{-1/2}$, one has a short exact sequence

$$0 \longrightarrow \tilde{\pi}(|-|^{1/2})^* \longrightarrow \mathrm{Hom}_{SO(3,2)}(\omega_\psi, I_P(|-|^{-1/2})) \longrightarrow V^* \longrightarrow 0$$

where

$$V^* \hookrightarrow \tilde{\pi}(|-|^{1/2})^*.$$

Considering smooth vectors, we thus have

$$0 \longrightarrow \tilde{\pi}(|-|^{1/2})^\vee \longrightarrow \mathrm{Hom}_{SO(3,2)}(\omega_\psi, I_P(|-|^{-1/2}))^\infty \longrightarrow \tilde{\pi}(|-|^{1/2})^\vee,$$

and so we would have a surjection

$$\tilde{\pi}(|-|^{1/2}) \longrightarrow sp$$

which is a contradiction. This completes the proof of the lemma. □

As a consequence of these two lemmas, we have:

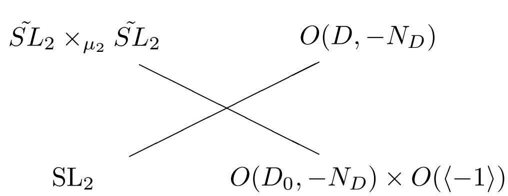
Corollary 4.3. *We have:*

$$\mathrm{Hom}_{SO(2,2)}(\eta^\epsilon(\pi), \tau_1 \boxtimes \tau_2) \neq 0 \Longrightarrow \tau_1 = \tau_2^\vee,$$

and

$$\dim \mathrm{Hom}_{SO(2,2)}(\eta^\epsilon(\pi), \tau \boxtimes \tau^\vee) = \dim \mathrm{Hom}_{\tilde{SL}_2}(\tau \otimes \omega_\psi, \sigma^\epsilon) = \dim \mathrm{Hom}_{SL_2}(\tau \otimes \sigma^{\epsilon\vee} \otimes \omega_\psi, \mathbb{C}).$$

Thus, our problem is transferred to that of studying the space of SL_2-invariant trilinear forms on $\tau \otimes \sigma^{\epsilon\vee} \otimes \omega_\psi$. For this, we consider the following see-saw pair:



Here, D is the unique (possibly split) quaternion algebra such that σ^ϵ is the theta lift from $SO(D_0, -N_D)$. Indeed, we know that

$$\theta_{\psi,0}(JL_D(\pi)) = \theta_\psi(JL_D(\pi)) = \sigma^\epsilon.$$

So by the see-saw identity, Lemma 3.1(ii) and Lemma 4.1(ii), we get:

$\dim \mathrm{Hom}_{SL_2}(\tau^\vee \otimes \sigma^\epsilon \otimes \omega_\psi^\vee, \mathbb{C}) = \dim \mathrm{Hom}_{SL_2}(\sigma^\epsilon \otimes \omega_\psi^\vee, \tau) = \dim \mathrm{Hom}_{PD^\times}(JL_D(\tau) \otimes JL_D(\tau)^\vee, JL_D(\pi))$.

By the main result of Prasad's thesis,

$$\dim \mathrm{Hom}_{PD^\times}(JL_D(\tau) \otimes JL_D(\tau)^\vee, JL_D(\pi)) \leq 1$$

and equality holds if and only if

$$\epsilon(1/2, \pi \otimes \tau \otimes \tau^\vee) = \epsilon.$$

So we have:

Proposition 4.4. *Let $\sigma^\epsilon \in \tilde{A}_\pi$ and let τ be an infinite-diimensional representation of $GL_2(F)$. Then*

$$\mathrm{Hom}_{SL_2}(\tau^\vee \otimes \sigma^\epsilon \otimes \omega_\psi^\vee, \mathbb{C}) \neq 0 \Longleftrightarrow \epsilon(1/2, \pi \otimes \tau \otimes \tau^\vee) = \epsilon.$$

Now suppose that τ and π (and hence σ^ϵ) are all unitary. Then

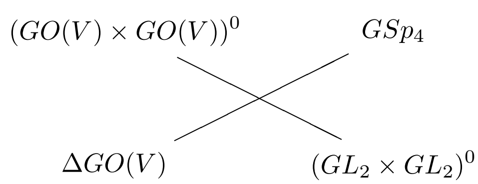
$$\mathrm{Hom}_{SL_2}(\tau \otimes \sigma^{\epsilon\vee} \otimes \omega_\psi, \mathbb{C}) \cong \mathrm{Hom}_{SL_2}(\tau^\vee \otimes \sigma^\epsilon \otimes \omega_\psi^\vee, \mathbb{C})$$

via the map $L \mapsto \overline{L}$. The main local theorem then follows from Cor. 4.3 and Prop. 4.4.

We conclude this section by describing another proof of the main local theorem, using the alternative construction of the Saito-Kurokawa representations given in Prop. 3.4. In [P2], D. Prasad studied the restriction problem (among other things) for the discrete series representations of $PGSp_4$ contained in certain tempered L-packets. These representations are theta lifts of

$$\begin{cases} (\pi \boxtimes St)^+ \text{ of } GO(2,2); \\ (JL(\pi) \boxtimes \mathbf{1_D})^+ \text{ of } GO(4) \end{cases}$$

with π supercuspidal. He made use of the following see-saw diagram:



Indeed, applying Cor. 3.3 to the representation $\sigma = (\pi_1 \boxtimes \pi_2)^+$ of $GO(V)$, one has

$$\begin{aligned} &\dim \mathrm{Hom}_H(\theta_0((\pi_1 \boxtimes \pi_2)^+), \tau_1 \boxtimes \tau_2) \\ = &\dim \mathrm{Hom}_{GO(V)}(\theta_0(\tau_1) \boxtimes \theta_0(\tau_2), (\pi_1 \boxtimes \pi_2)^+) \\ = &\dim \mathrm{Hom}_{GSO(V)}(\theta_0(\tau_1) \boxtimes \theta_0(\tau_2), \pi_1 \boxtimes \pi_2) \quad \text{(by Frobenius reciprocity).} \end{aligned}$$

Applying this to the special case $\pi_1 \boxtimes \pi_2 = \pi \boxtimes \mathbf{1}$ or $JL(\pi) \boxtimes \mathbf{1_D}$ and using Lemma 4.1, we obtain:

$$\dim \mathrm{Hom}_H(\theta_0(\pi \boxtimes \mathbf{1})^+, \tau_1 \boxtimes \tau_2) = \dim \mathrm{Hom}_{GL_2}(\tau_1 \boxtimes \tau_2, \pi) \cdot \dim \mathrm{Hom}_{GL_2}(\tau_1^\vee \boxtimes \tau_2^\vee, \mathbf{1}), \tag{4.1}$$

and

$$\begin{aligned} &\dim \mathrm{Hom}_H(\theta_0(JL(\pi) \boxtimes \mathbf{1_D})^+), \tau_1 \boxtimes \tau_2) \\ = &\dim \mathrm{Hom}_{D^\times}(JL(\tau_1) \boxtimes JL(\tau_2), \pi) \cdot \dim \mathrm{Hom}_{D^\times}(JL(\tau_1)^\vee \boxtimes JL(\tau_2)^\vee, \mathbf{1_D}). \end{aligned} \tag{4.2}$$

These two equations would prove the theorem if one knows that $\theta_0 = \theta$ on the left-hand-side. This is the case in (4.2), as well as for supercuspidal π in (4.1). However, we are not certain if it is the case when π is a principal series or a special representation in (4.1). Though these two remaining cases can be handled by some ad-hoc arguments, we shall not dwell on these here.

5. Consequences and Variants

In this section, we obtain some variants of the main local theorem for general forms of (SO_5, SO_4). Before coming to that, it is useful to restate Prop. 4.4 in the following form, which makes its dependence on the choice of the additive character ψ more transparent:

Theorem 5.1. *Let τ be an infinite dimensional representation of GL_2 and σ a representation of $\tilde{SL}_2$. Then for any nontrivial additive character ψ of F,*

$$\mathrm{Hom}_{SL_2}(\tau^\vee \otimes \sigma \otimes \omega_\psi^\vee, \mathbb{C}) \neq 0 \Longleftrightarrow \epsilon(1/2, Ad(\tau) \otimes Wd_\psi(\sigma)) = \epsilon_\psi(\sigma),$$

in which case the Hom space is 1-dimensional.

It is this result which is the key to all the restriction problems considered in this and the previous section.

Now we come to the restriction problem for arbitrary forms of (SO_5, SO_4). Since the argument is similar as in the split case, we shall be fairly brief. We do need, however, to introduce some more notations in order to state the theorems.

The only inner form of $SO(3,2)$ is the rank one group $SO(4,1)$. In [G], the Saito-Kurokawa packets of $SO(4,1)$ have been analyzed by means of theta lifting from $\tilde{SL}_2$, in analogy with the split case. We have the following analog of Lemma 4.2:

Lemma 5.2. *Consider the theta lift from $\tilde{SL}_2$ to $SO(4,1)$ and let σ be an irreducible unitary representation of $\tilde{SL}_2$. Then $\theta_{\psi,0}(\sigma) \neq 0$ iff σ is not an elementary Weil representation, in which case $\theta_{\psi,0}(\sigma) = \theta_\psi(\sigma)$ is irreducible.*

Fix an infinite-dimensional unitary representation π of PGL_2 with associated Waldspurger packet $\tilde{A}_\pi = \{\sigma^+, \sigma^-\}$. Then, following [G], set

$$\eta^{+-}(\pi) = \theta_\psi(\sigma^+) \quad \text{and} \quad \eta^{-+}(\pi) = \theta_\psi(\sigma^-).$$

The set $\{\eta^{+-}(\pi), \eta^{-+}(\pi)\}$ is the Saito-Kurokawa packet of $SO(4,1)$ attached to π. Note that it has 2 elements iff π is a discrete series but not the Steinberg representation. Indeed, if $\pi = St$, then $\eta^{+-}(\pi) = 0$ since σ^+ is the odd Weil representation ω_ψ^-.

Using the above lemma, the same argument as in the split case gives:

Theorem 5.3. *Let τ_1 and τ_2 be discrete series representation of GL_2 and let $SO(4)$ denote the anisotropic group $(D^\times \times D^\times)^0/\Delta\mathbb{G}_m$. Then*

$$\mathrm{Hom}_{SO(4)}(\eta^{\epsilon,-\epsilon}(\pi), JL(\tau_1) \boxtimes JL(\tau_2)) \neq 0 \Longrightarrow \tau_1 = \tau_2^\vee,$$

and

$$\mathrm{Hom}_{SO(4)}(\eta^{\epsilon,-\epsilon}(\pi), JL(\tau) \boxtimes JL(\tau)^\vee) \neq 0 \Longleftrightarrow \epsilon = \epsilon(1/2, \pi \otimes \tau \otimes \tau^\vee),$$

in which case the dimension of the Hom space is 1.

In the rest of this section, we consider the restriction of Saito-Kurokawa representations to $SO(3,1)$. The results here are slightly more intricate to state and we begin by introducing some notations for the representations of $SO(3,1)$.

Given any étale quadratic algebra K, there are two quadratic spaces of rank 4 and discriminant K. We denote them by:

$$V_K^+ = \mathbb{H} \oplus (K, N_{K/F}) \quad \text{and} \quad V_K^- = \mathbb{H} \oplus (K, \delta \cdot N_{K/F})$$

where $\mathbb{H}$ denote a hyperbolic plane and $\delta \in F^\times \smallsetminus N_{K/F}(K^\times)$. The associated orthogonal groups are isomorphic. In particular, we have:

$$GSO(V_K^\epsilon) \cong GL_2(K) \times F^\times / \Delta K^\times,$$

with $K^\times$ embeded diagonally via:

$$a \mapsto (diag(a,a), N_{K/F}(a)^{-1}).$$

A representation of $GSO(V_K^\epsilon)$ is thus of the form $\Sigma \boxtimes \chi$, where Σ is an irreducible representation of $GL_2(K)$ whose central character ω_Σ satisfies

$$\omega_\Sigma = \chi \circ N_{K/F}.$$

The subgroup $SO(V_K^\epsilon)$ is isomorphic to $GL_2(K)^0/F^\times$, where

$$GL_2(K)^0 = \{g \in GL_2(K) : \det(g) \in F^\times\}.$$

The embedding $GL_2(K)^0/F^\times \hookrightarrow GSO(V_K^\epsilon)$ is given by:

$$g \mapsto (g, \det(g)^{-1}).$$

An L-packet of $SO(V_K^\epsilon)$ is thus given by the constituents of the restriction of a representation of $GSO(V_K^\epsilon)$ (or equivalently, the restriction of a representation of $GL_2(K)/F^\times$).

We have an embedding of quadratic spaces

$$V_K^+ \hookrightarrow \mathbb{H}^2 \oplus \langle 1 \rangle$$

and thus an embedding

$$SO(V_K^+) \hookrightarrow SO(3,2).$$

On the other hand, V_K^- does not embed into $\mathbb{H}^2 \oplus \langle 1 \rangle$. Rather,

$$V_K^- \hookrightarrow \mathbb{H} \oplus (D_0, -N_D)$$

and so we have

$$SO(V_K^-) \hookrightarrow SO(4,1).$$

One may consider the theta correspondence for the similitude dual pair $GL_2^+ \times GO(V_K^\epsilon)$, which has been studied in [Co] and [Ro2]. Recall that if τ is an irreducible infinite-dimensional representation of GL_2, then the restriction of τ to GL_2^+ is reducible iff $\tau \otimes \chi_K \cong \tau$, in which case there are two constituents. We may label the two constituents by τ^+ and τ^-, so that τ^ϵ occurs in the theta correspondence with $GO(V_K^\epsilon)$ but not with $GO(V_K^{-\epsilon})$. On the other hand, if τ is irreducible when restricted to GL_2^+, then τ occurs in the theta correspondence with both $GO(V_K^\epsilon)$ and we simply set $\tau^+ = \tau^- = \tau|_{GL_2^+}$.

Now one has the following analog of Lemma 4.1:

LEMMA 5.4. *(i) Let τ be an irreducible infinite-dimensional unitary representation of GL_2. Then as a representation of $GSO(V_K^\epsilon)$,*

$$\theta_0(\tau^\epsilon) = \theta(\tau^\epsilon) = \Sigma_\tau := BC_K(\tau) \otimes (\omega_\tau \cdot \chi_K),$$

where $BC_K(\tau)$ is the base change of τ to $GL_2(K)$ and ω_τ is the central character of τ.

(ii) Let Σ be an infinite-dimensional unitary representation of $GO(V_K^\epsilon)$, then

$$\theta_0(\Sigma) \neq 0 \Longrightarrow \Sigma|_{GSO(V_K^\epsilon)} = \Sigma_\tau.$$

Moreover, of the two possible extensions of Σ_τ to $GO(V_K^\epsilon)$, exactly one of them, denoted by $\Sigma_\tau^\dagger$, participates in the theta correspondence and one has:

$$\theta_0(\Sigma_\tau^\dagger) = \theta(\Sigma_\tau^\dagger) = \tau^\epsilon.$$

A similar argument as in the split case now gives the following theorems:

THEOREM 5.5. *Consider the restriction of $\eta^\epsilon(\pi)$ from $SO(3,2)$ to $SO(V_K^+)$.*

(i) For an infinite dimensional unitary representation Σ of $GSO(V_K^+) = (GL_2(K) \times F^\times)/\Delta K^\times$, we have:

$$\mathrm{Hom}_{SO(V_K^+)}(\eta^\epsilon(\pi), \Sigma) \neq 0 \Longrightarrow \Sigma = \Sigma_\tau$$

for some infinite dimensional unitary representation τ of $GL_2(F)$.

(ii) If $\tau \otimes \chi_K \neq \tau$, then

$$\mathrm{Hom}_{SO(V_K^+)}(\eta^\epsilon(\pi), \Sigma_\tau) \neq 0 \Longleftrightarrow \epsilon_{\psi_K}(\sigma^\epsilon) = \epsilon(1/2, Ad(\tau) \otimes (\pi \otimes \chi_K))$$

or equivalently

$$\epsilon = \epsilon(1/2, (\pi \otimes \chi_K) \otimes \tau \otimes \tau^\vee) \cdot \left(\frac{\chi_K(-1) \cdot \epsilon(1/2, \pi \otimes \chi_K)}{\epsilon(1/2, \pi)} \right),$$

in which case the Hom space has dimension 1.

(iii) If $\tau \otimes \chi_K = \tau$, then

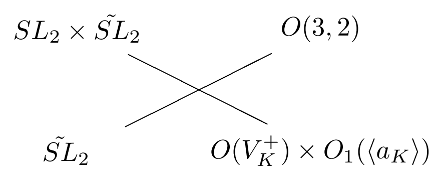
$$\mathrm{Hom}_{SO(V_K^+)}(\eta^-(\pi), \Sigma_\tau) = 0$$

whereas

$$\mathrm{Hom}_{SO(V_K^+)}(\eta^+(\pi), \Sigma_\tau) \neq 0 \Longleftrightarrow \epsilon(1/2, (\pi \otimes \chi_K) \otimes Ad(\tau)) \cdot \chi_K(-1) \cdot \epsilon(1/2, \pi) = 1,$$

in which case the Hom space has dimension 1.

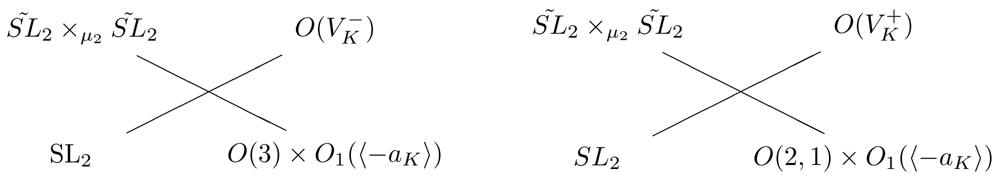
PROOF. We give a sketch of the proof, so as to illustrate why the extra complexity in (iii) occurs. Suppose that K corresponds to $a_K \in F^\times/F^{\times 2}$. By using the see-saw



and Lemma 5.4, one deduces (i) immediately. Moreover, if $\Sigma = \Sigma_\tau$, then

$$\mathrm{Hom}_{SO(V_K^+)}(\eta^\epsilon(\pi), \Sigma_\tau) \neq 0 \Longleftrightarrow \mathrm{Hom}_{SL_2}({\tau^+}^\vee \otimes \sigma^\epsilon \otimes \omega_{\psi_K}^\vee, \mathbb{C}) \neq 0.$$

If $\tau \otimes \chi_K \neq \tau$, then $\tau^+ = \tau$ and so (ii) follows from Thm. 5.1. Finally, if $\tau \otimes \chi_K = \tau$, then one cannot use Thm. 5.1 directly. Instead, consider the two companion see-saws



Since the theta lift of τ^+ to $GO(V_K^-)$ is zero, the first see-saw gives

$$\mathrm{Hom}_{SL_2}(\tau^{+\vee} \otimes \sigma^- \otimes \omega_{\psi_K}^\vee, \mathbb{C}) = 0$$

which implies the vanishing result of (iii). Similarly, the second see-saw allows one to conclude that

$$\mathrm{Hom}_{SL_2}(\tau^{-\vee} \otimes \sigma^+ \otimes \omega_{\psi_K}^\vee, \mathbb{C}) = 0,$$

so that

$$\mathrm{Hom}_{SL_2}(\tau^\vee \otimes \sigma^+ \otimes \omega_{\psi_K}^\vee, \mathbb{C}) = \mathrm{Hom}_{SL_2}(\tau^{+\vee} \otimes \sigma^+ \otimes \omega_{\psi_K}^\vee, \mathbb{C}).$$

Together with Thm. 5.1, this implies the second part of (iii). □

THEOREM 5.6. *Consider the restriction of* $\eta^{\epsilon,-\epsilon}(\pi)$ *from* $SO(4,1)$ *to* $SO(V_K^-)$.

(i) For an infinite dimensional unitary representation Σ *of* $GSO(V_K^-) = (GL_2(K) \times F^\times)/\Delta K^\times$, *we have:*

$$\mathrm{Hom}_{SO(V_K^-)}(\eta^{\epsilon,-\epsilon}(\pi), \Sigma) \neq 0 \Longrightarrow \Sigma = \Sigma_\tau$$

for some infinite dimensional unitary representation τ *of* $GL_2(F)$.

(ii) If $\tau \otimes \chi_K \neq \tau$, *then*

$$\mathrm{Hom}_{SO(V_K^-)}(\eta^{\epsilon,-\epsilon}(\pi), \Sigma_\tau) \neq 0 \Longleftrightarrow \epsilon_{\psi_K}(\sigma^\epsilon) = \epsilon(1/2, Ad(\tau) \otimes (\pi \otimes \chi_K))$$

or equivalently

$$\epsilon = \epsilon(1/2, (\pi \otimes \chi_K) \otimes \tau \otimes \tau^\vee) \cdot \left(\frac{\chi_K(-1) \cdot \epsilon(1/2, \pi \otimes \chi_K)}{\epsilon(1/2, \pi)} \right),$$

in which case the Hom space has dimension 1.

(iii) If $\tau \otimes \chi_K = \tau$, *then*

$$\mathrm{Hom}_{SO(V_K^-)}(\eta^{+-}(\pi), \Sigma_\tau) = 0$$

whereas

$$\mathrm{Hom}_{SO(V_K^-)}(\eta^{-+}(\pi), \Sigma_\tau) \neq 0 \Longleftrightarrow \epsilon(1/2, (\pi \otimes \chi_K) \otimes Ad(\tau)) \cdot \chi_K(-1) \cdot \epsilon(1/2, \pi) = -1,$$

in which case the Hom space has dimension 1.

Remarks: Consider the case when $\pi = St$ is the Steinberg representation. The representation $\eta^{+-}(\pi)$ is zero and so Thm. 5.6 had better predict that the space $\mathrm{Hom}_{SO(V_K^-)}(\eta^{+-}(\pi), \Sigma_\tau)$ is zero for any τ. Let us check that this is the case. If $\tau \neq \tau \otimes \chi_K$, then one knows that

$$\frac{\chi_K(-1) \cdot \epsilon(1/2, St \otimes \chi_K)}{\epsilon(1/2, \pi)} = -1 \quad \text{and} \quad \epsilon(1/2, (St \otimes \chi_K) \otimes \tau \otimes \tau^\vee) = 1.$$

Hence the RHS of the condition on epsilon factors in (ii) is -1, as required. On the other hand, if $\tau = \tau \otimes \chi_K$, then the desired vanishing of the above Hom space is affirmed by (iii).

We conclude this section with the following theorem which follows from Thm. 5.1 and the two companion see-saws in the proof of Thm. 5.5:

THEOREM 5.7. *Consider the representation Σ_τ of $GSO(V_K^\epsilon)$ and let π be an infinite dimensional representation of $SO(2,1) \cong PGL_2$. Then*

$$\dim \mathrm{Hom}_{PGL_2}(\Sigma_\tau, \pi) + \dim \mathrm{Hom}_{PD^\times}(\Sigma_\tau, JL_D(\pi)) = 1$$

and

$$\mathrm{Hom}_{PGL_2}(\Sigma_\tau, \pi) \neq 0 \Longleftrightarrow \epsilon(1/2, (\pi \otimes \chi_K) \otimes Ad(\tau)) \cdot \chi_K(-1) \cdot \epsilon(1/2, \pi) = 1.$$

This result is a special case of the extension of Prasad's thesis [P1] to the case of $GL_2(F)$-invariant forms on $GL_2(F) \times GL_2(K)$. Such an extension was given in [P2], but the epsilon factor condition was only shown for non-supercuspidal representations. In a recent paper [PSP], the complete extension was finally obtained by Prasad and Schulze-Pillot using a global-to-local argument, starting from the generalization of Jacquet's conjecture to an arbitrary étale cubic algebra.

6. Archimedean Restriction

In this section, assume that $F = \mathbb{R}$ or $\mathbb{C}$. We shall discuss the results of Savin [Sa] on the archimedean analog of our main theorem. Savin's paper, which has not been published before, appears as an appendix to this paper.

Suppose first that $\pi = \pi(\chi, \chi^{-1})$ is a unitary principal series of $PGL_2(F)$. The associated Saito-Kurokawa packet contains a single representation $\eta^+(\pi) = I_P(\chi)$. In this case, we know by [KR] that

$$\mathrm{Hom}_H(I_P(\chi), \tau \otimes \tau^\vee) \neq 0$$

for any irreducible representation τ of $GL_2(F)$. A nonzero element of this Hom space is given by the leading term in the Laurent expansion of the local zeta integral arising from the doubling Rankin-Selberg integral of Piatetski-Shapiro and Rallis (for the groups $SL_2 \times SL_2 \subset Sp_4$).

Henceforth, we focus on the case when $F = \mathbb{R}$ and $\pi = \pi_{2k}$ is the discrete series $(\mathfrak{sl}_2, O(2))$-module of extremal weights $\pm 2k$, with $k \geq 1$. The two representations in the Saito-Kurokawa packet are best described in terms of derived functor modules:

$$\eta^+(\pi_{2k)} = A_{\mathfrak{q}_{1,1}}(\lambda_k) \quad \text{and} \quad \eta^-(\pi_{2k}) = A_{\mathfrak{q}_{2,0}}(\lambda_k) \oplus A_{\mathfrak{q}_{0,2}}(\lambda_k).$$

Here $\mathfrak{q}_{1,1}$ (resp. $\mathfrak{q}_{2,0}$) is a θ-stable Siegel parabolic subalgebra whose Levi subalgebra corresponds to the group $U(1,1)$ (resp. $U(2,0)$) and $\lambda_k = det^{k-2}$. Note that $A_{\mathfrak{q}_{2,0}}(\lambda_k)$ and $A_{\mathfrak{q}_{0,2}}(\lambda_k)$ are irreducible $(\mathfrak{so}_5, SO(3) \times SO(2))$-modules but their sum extends to an irreducible $(\mathfrak{so}_5, S(O(3) \times O(2)))$-module.

Because $\eta^-(\pi_{2k})$ is a lowest/highest weight module, it is easy to determine its restriction to $SO(2,2)$ by K-type considerations. One has:

$$\eta^-(\pi_{2k}) = \bigoplus_{r \geq k+1} \pi_r \otimes \pi_r.$$

From this, the following proposition follows:

PROPOSITION 6.1. *We have:*

$$\mathrm{Hom}_{SO(2,2)}(\eta^-(\pi_{2k}), \tau \otimes \tau^\vee) \neq 0 \Leftrightarrow \epsilon(1/2, \pi_{2k} \otimes \tau \otimes \tau^\vee) = -1,$$

in which case the dimension of the Hom space is 1.

On the other hand, for $\eta^+(\pi_{2k})$, one has the following result of Savin [Sa]:

THEOREM 6.2. *Suppose that* $\Pi\otimes\Theta$ *occurs as a quotient of* $\eta^+(\pi_{2k})$*, where* Π *and* Θ *are* $(\mathfrak{sl}_2, O(2))$*-modules.*

(i) If $\Pi = \pi_r$ *with* $0 < r \leq k$*, then the possible wieghts of* Θ *are* $\pm r, \pm(r+2), \pm(r+4), \ldots$*, which are precisely the weights of* π_r*.*

(ii) If $\Pi = \pi_r$ *with* $k+1 \leq r$*, then there are no possible weights for* Θ*. In particular,* π_r *does not appear in the correspondence.*

As an immediate corollary of this and the case of principal series discussed at the beginning of this section, we have:

COROLLARY 6.3. *If* $F = \mathbb{R}$ *or* $\mathbb{C}$*, we have:*

$$\mathrm{Hom}_{SO(2,2)}(\eta^+(\pi), \tau\otimes\tau^\vee) \neq 0 \Longrightarrow \epsilon(1/2, \pi\otimes\tau\otimes\tau^\vee) = 1.$$

The converse holds if π *is a unitary principal series representation.*

We do not know how to show the converse in general and so the result is less complete than the non-archimedean case.

7. Proof of the Main Global Theorem

In this section, we shall investigate the analogous global restriction problem.

Suppose in the section that F is a number field with adele ring $\mathbb{A}$ and $\pi = \otimes\pi_v$ is a cuspidal representation of $PGL_2(\mathbb{A})$. As described in [G], there is a global Saito-Kurokawa packet associated to π. A representation in this packet has the form

$$\eta^{\underline{\epsilon}}(\pi) = \otimes_v \eta^{\epsilon_v}(\pi_v).$$

This representation occurs in the space of square-integrable automorphic forms of $PGSp_4$ iff

$$|\underline{\epsilon}| := \prod_v \epsilon_v = \epsilon(1/2, \pi).$$

We are interested in characterizing the cuspidal representations $\tau_1 \boxtimes \tau_2$ of $SO(2,2) = (GL_2\times GL_2)^0/\Delta\mathbb{G}_m$ such that the period integral

$$P_{H,\underline{\epsilon}} : (f, \varphi_1, \varphi_2) \mapsto \int_{SO(2,2)(F)\backslash SO(2,2)(\mathbb{A})} f(h)\cdot\varphi_1(h)\cdot\varphi_2(h)\, dh$$

defines a non-zero linear form on $\eta^{\underline{\epsilon}}(\pi)\otimes\tau_1\otimes\tau_2$.

THEOREM 7.1. *(i) If the linear form* $P_{H,\underline{\epsilon}}$ *is non-zero, then* $\tau_1 = \tau_2^\vee$*.*

(ii) Assume that $\tau_1 = \tau_2^\vee = \tau$*. There is at most one* $\underline{\epsilon}$ *for which the linear form* $P_{H,\underline{\epsilon}}$ *can be non-zero. This distinguished* $\underline{\epsilon}$ *is characterized by the requirement that*

$$\epsilon_v = \epsilon(1/2, \pi_v\otimes\tau_v\otimes\tau_v^\vee) \quad \textit{for all } v.$$

The associated representation occurs in the discrete spectrum iff $\epsilon(1/2, \pi\otimes Ad(\tau)) = 1$*.*

(iii) The distinguished representation in (ii) occurs in the discrete spectrum and the corresponding linear form $P_{H,\underline{\epsilon}}$ *is non-zero if and only if*

$$L(1/2, \pi\times Ad(\tau)) \neq 0.$$

Proof. Parts (i) and (ii) follow immediately from our main (local) theorem and the strong multiplicity-one theorem for GL_2. For (iii), note that the non-vanishing of $L(1/2, \pi \times Ad(\tau))$ implies by (ii) that the distinguished representation in (ii) occurs in the discrete spectrum. Thus, to prove (iii), we may assume that the distinguished representation in (ii) occurs in the discrete spectrum and show the equivalence of the non-vanishing of $P_{H,\underline{\epsilon}}$ and $L(1/2, \pi \times Ad(\tau))$.

In this case, the distinguished representation $\eta^{\underline{\epsilon}}(\pi)$ can be obtained as the global theta lift of a cuspidal representation σ of $\tilde{SL}_2$ in the global Waldspurger packet associated to π. By making use of the see-saw diagram

$$\begin{array}{ccc} SL_2 \times \tilde{SL}_2 & & O(3,2) \\ & \times & \\ \tilde{SL}_2 & & O(2,2) \times O(1) \end{array}$$

we deduce that the linear form $P_{H,\underline{\epsilon}}$ is non-zero iff

$$\int_{SL_2(F)\backslash SL_2(\mathbb{A})(F)\backslash SL_2(F)\backslash SL_2(\mathbb{A})(\mathbb{A})} \varphi(g) \cdot \overline{\varphi_\sigma(g)} \cdot \theta_\psi(\phi)(g)\, dg$$

for some $\varphi \in \tau$, $\varphi_\sigma \in \sigma$ and some theta function $\theta_\psi(\phi)$ in the Weil representation ω_ψ of $\tilde{SL}_2$.

Now there exists a quadratic field K such that

- σ possesses a nonzero ψ_K-Whittaker-Fourier coefficient.
- τ is not dihedral with respect to K.

Indeed, there are only finitely many K's with respect to which τ is dihedral whereas by results of Friedberg-Hoffstein [FH] and Waldspurger [W1], there are infinitely many K's such that σ has nonzero ψ_K-Fourier coefficient. For a quadratic K chosen as above, one then has (cf. [W1]):

$$L(1/2, \pi \otimes \chi_K) \neq 0 \quad \text{and} \quad \sigma = \Theta_{\psi_K}(\pi \otimes \chi_K).$$

Moreover, we have the see-saw diagram

$$\begin{array}{ccc} \tilde{SL}_2 \times_{\mu_2} \tilde{SL}_2 & & SO(V_K^+) \\ & \times & \\ \mathrm{SL}_2 & & SO(2,1) \times SO(\langle -a_K \rangle) \end{array}$$

and we may consider the global see-saw identity arising from the global theta lift with respect to the character ψ_K. One has the following lemma:

Lemma 7.2. *Consider the global theta lift from GL_2^+ to $GSO(V_K^+)$ with respect to ψ_K. If τ is a cuspidal representation of GL_2 which is not dihedral with respect to K, then the global theta lift $\Theta(\tau)$ is nonzero cuspidal and is equal to Σ_τ on $\Sigma_\tau = BC_K(\tau) \boxtimes (\omega_\tau \chi_K)$.*

One proves this lemma by computing the constant term and the non-trivial Whittaker-Fourier coefficient of the theta lift $\Theta(\tau)$. We omit the details.

Using the see-saw identity and the above lemma, we deduce that $P_{H,\underline{\epsilon}}$ is non-zero iff the global period integral of the cuspidal representation $BC(\tau)\otimes(\pi\otimes\chi_\tau^{-1})$ of $GL_2(K)\times GL_2(F)$ over the diagonal subgroup $GL_2(F)$ is non-zero.

Now Harris and Kudla has proved the Jacquet conjecture relating global trilinear period integral and the triple product L-function. In the recent paper [PSP], Prasad and Schulze-Pillot has extended the proof of Harris-Kudla [HK] to the case of $GL_2(F)$-period integral on $GL_2(E)$, where E is an étale cubic algebra. We consider the case $E = F \times K$. Then [PSP, Thm. 1.1] says that

$$L(1/2, \pi \otimes \chi_K) \cdot L(1/2, \pi \otimes Ad(\tau)) \neq 0$$

if and only if there is a quaternion algebra D (possible split) with

$$D^\times \hookrightarrow GL_2(K)$$

such that the cuspidal representation $BC(\tau) \otimes JL_D(\pi \otimes \chi_\tau^{-1})$ of $GL_2(K) \times D^\times$ has non-zero period integral over the diagonal subgroup $D^\times$.

However, Thm. 5.7 (applied to Σ_{τ_v} and $\pi_v \otimes \chi_{K_v}$) tells us that for each place v,

$$\mathrm{Hom}_{GL_2(F_v)}(BC(\tau_v), \pi_v \otimes \chi_{\tau_v}) \neq 0$$

whereas

$$\mathrm{Hom}_{D_v^\times}(BC(\tau_v), JL_{D_v}(\pi_v \otimes \chi_{\tau_v})) = 0,$$

where D_v is the quaternion division algebra here. This shows that in [PSP, Thm. 1.1] described above, the only possible non-vanishing period integral is the one over the split group GL_2.

Hence we conclude that $P_{H,\underline{\epsilon}}$ is nonzero if and only if

$$L(1/2, \pi \otimes \chi_K) \cdot L(1/2, \pi \times Ad(\tau)) \neq 0$$

or equivalently

$$L(1/2, \pi \times Ad(\tau)) \neq 0.$$

□

When the representations involved correspond to holomorphic modular forms of level 1, Ichino has given in [I] a refinement of part (iii) of the theorem by proving an exact formula expressing the value $L(1/2, \pi \otimes Ad(\tau))$ in terms of the period $P_{H,\underline{\epsilon}}$ evaluated at an explicit test vector.

8. Restricting from GSp_4 to Sp_4

We shall conclude the paper with a couple of miscellaneous questions concerning restrictions of Saito-Kurokawa representations. Ginzburg has raised the question of how the Saito-Kurokawa representations behave when restricted from GSp_4 to Sp_4. We shall answer this question in this section. Assume first that F is a p-adic field.

THEOREM 8.1. *If $\eta^\epsilon(\pi)$ is a Saito-Kurokawa representation, then $\eta^\epsilon(\pi)$ remains irreducible when restricted to Sp_4 unless $\pi = St_{\chi_K}$ with χ_K a* non-trival *quadratic character and $\epsilon = -$, in which case it is the sum of two irreducible representations.*

PROOF. Let us realize $\eta^\epsilon(\pi)$ as a theta lift from $GSO(2,2)$ or $GSO(4)$:

$$\eta^+(\pi) = \theta((\pi \boxtimes \mathbf{1})^+) \quad \text{and} \quad \eta^-(\pi) = \theta_D((JL(\pi) \boxtimes \mathbf{1_D})^+).$$

By Lemma 3.1, $\eta^+(\pi)$ is irreducible when restricted to Sp_4 iff $(\pi \boxtimes \mathbf{1})^+$ is irreducible when restricted to $O(2,2)$. But as a representation of $O(2,2)$,

$$(\pi \boxtimes \mathbf{1})^+ = ind_{SO(2,2)}^{O(2,2)}(\pi \boxtimes \mathbf{1})|_{SO(2,2)}.$$

This is irreducible iff as irreducible representations of $SO(2,2)$,

$$\pi \boxtimes \mathbf{1} \neq \mathbf{1} \boxtimes \pi.$$

Since this is always the case (as π is infinite-dimensional), we see that $\eta^+(\pi)$ is always irreducible when restricted to Sp_4.

Similarly, for $\eta^-(\pi)$, we need to examine when $(JL(\pi) \boxtimes \mathbf{1_D})^+$ is reducible when restricted to $O(4)$. If $\pi = St$ so that $JL(\pi) = \mathbf{1_D}$, then $(JL(\pi) \boxtimes \mathbf{1_D})^+$ is the trivial representation and thus remians irreducible when restricted to $O(4)$. Hence $\eta^-(St)$ is irreducible when restricted to Sp_4. Now assume that $JL(\pi)$ is non-trivial. Then as a representation of $O(4)$,

$$(JL(\pi) \boxtimes \mathbf{1_D})^+ = ind_{SO(4)}^{O(4)}(JL(\pi) \boxtimes \mathbf{1_D})|_{SO(4)}$$

which is irreducible iff as representations of $SO(4)$,

$$JL(\pi) \boxtimes \mathbf{1_D} \neq \mathbf{1_D} \boxtimes JL(\pi).$$

But this holds precisely when $JL(\pi)$ is *not* a 1-dimensional character of $D^\times$. This proves the theorem. □

In fact, one can deduce the theorem for $\eta^+(\pi)$ using the explicit description of $\eta^+(\pi)$ in Prop. 2.1. Indeed, the natural map $Sp_4 \longrightarrow PGSp_4$ induces a map on the Levi factors of the Siegel parabolics:

$$p : M' = GL_2 \longrightarrow GL_1 \times PGL_2$$

given by

$$p(g) = (\det(g), [g]).$$

From this, one sees that

$$I_P(\pi, 1/2)|_{Sp_4} = I_{P'}(\pi \cdot |\det|^{1/2})$$

which still has a unique irreducible quotient. Since $\eta^+(\pi) = J_P(\pi, 1/2)$, we conclude that $\eta^+(\pi)$ is irreducible when restricted to Sp_4.

How can we distinguish between the two irreducible constituents in the restriction of $\eta^-(St_{\chi_K})$? This can be done by examining the local analog of their Fourier coefficients. Recall that the $M(F)$-orbits of generic unitary characters of N are naturally parametrized by étale quadratic algebras, which are in turn classified by $F^\times/F^{\times 2}$. If E is a quadratic algebra, we let ψ_E denote a character in the orbit indexed by E. On the other hand, for the group Sp_4, the $M'(F)$-orbits of generic unitary characters of N' are parametrized by nondegenerate quadratic spaces of rank 2, which are indexed by their discriminants in $F^\times/F^{\times 2}$ and their Hasse-Witt invariants in $\{\pm 1\}$. In other words, when E is a quadratic field, the $M(F)$-orbit of ψ_E breaks up into two $M'(F)$-orbit. We denote representatives of these two orbits by $\psi_{E,+}$ and $\psi_{E,-}$.

Now the representation $\eta^-(St_{\chi_K})$ is a distinguished representation, in the sense that

$$\dim \mathrm{Hom}_N(\eta^-(St_{\chi_K}), \mathbb{C}_{\psi_E}) = \begin{cases} 1, & \text{if } E = K; \\ 0, & \text{if } E \neq K. \end{cases}$$

Since every infinite dimensional representation σ of Sp_4 must have non-zero $\mathrm{Hom}_{N'}(\sigma, \mathbb{C}_\psi)$ for some generic ψ, we can label the two constituents as follows:

$$\eta^-(St_{\chi_K}) = \Xi_K^+ \oplus \Xi_K^-$$

where

$$\dim \mathrm{Hom}_{N'}(\Xi_K^\epsilon, \mathbb{C}_{\psi_K, \epsilon'}) = \delta_{\epsilon\,\epsilon'}.$$

In fact, it is not difficult to see that $\Xi_K^\pm$ is the theta lift of the sign character of $O(V_K^\pm)$, where $V_K^\pm$ is the rank 2 quadratic space with discrimiant K and Hasse-Witt invariant ± 1.

Remarks: The archimedean situation is similar to the p-adic one. Namely, $\eta^+(\pi)$ remains irreducible when restricted to Sp_4, whereas if π is a discrete series representation, $\eta^-(\pi)$ decomposes into the sum of a highest weight module and a lowest weight module unless π has extremal weights ± 2.

Now we turn to the global situation so that F is now a number field. If

$$\eta^{\underline{\epsilon}}(\pi) \subset L^2_{disc}(PGSp_4(F)\backslash PGSp_4(\mathbb{A}))$$

is a Saito-Kurokawa representation associated to a cuspidal representation π of PGL_2, then we may restrict the automorphic functions in $\eta^{\underline{\epsilon}}(\pi)$ to $Sp_4(\mathbb{A})$. This gives a nonzero Sp_4-equivariant map

$$Res : \eta^{\underline{\epsilon}}(\pi) \longrightarrow L^2_{disc}(Sp_4(F)\backslash Sp_4(\mathbb{A})).$$

We have:

THEOREM 8.2. *The Sp_4-equivariant map Res is injective.*

PROOF. Clearly, if $\eta^{\underline{\epsilon}}(\pi)$ is irreducible as an abstract representation of $Sp_4(\mathbb{A})$, then the theorem is obvious. In general, let S be the finite set of places where $\pi_v = St_{\chi_{K_v}}$ for some quadratic field K_v and $\epsilon_v = -$. Then we know by the previous theorem that as an abstract representation of $Sp_4(\mathbb{A})$, $\eta^{\underline{\epsilon}}(\pi)$ is the sum of $2^{\#S}$ irreducible representations

$$\Xi_\alpha = \left(\bigotimes_{v \notin S} \eta^{\epsilon_v}(\pi_v)\right) \otimes \left(\bigotimes_{v \in S} \Xi_{K_v}^{\alpha_v}\right)$$

where $\alpha_v = \pm$. Moreover, these $2^{\#S}$ abstract representations can be distinguished by the abstract (N', ψ)-equivariant linear functionals they support.

Now choose a quadratic field E such that $\eta^{\underline{\epsilon}}(\pi)$ has a nonzero (N, ψ_E)-Fourier coefficient. In other words, the linear functional on $\eta^{\underline{\epsilon}}(\pi)$ given by

$$L_{\psi_E} : f \mapsto \int_{N(F)\backslash N(\mathbb{A})} f(n) \cdot \overline{\psi_E(n)}\, dn$$

is nonzero. Then there is a unique summand Ξ_{α_0} on which L_{ψ_E} is non-zero; namely for each $v \in S$, ψ_E has to lie in the $M'(F_v)$-orbit of $\psi_{E,\alpha_{0,v}}$. Now for any element

$$m = (\lambda, g) \in M(F) = GL_1(F) \times PGL_2(F),$$

the global Fourier coefficient $L_{m \cdot \psi_E}$ is also nonzero since

$$L_{m\cdot\psi_E}(f) = L_{\psi_E}(m^{-1} f).$$

Moreover, for each $v \in S$, the character $m \cdot \psi_E$ lies in the $M'(F_v)$-orbit of $\psi_{E,\alpha_0\bar{\lambda}}$ where $\bar{\lambda}$ is the image of λ in $F_v^\times / N_{E_v/F_v}(E_v^\times) \cong \{\pm 1\}$. Thus, to see that $Res(\Xi_\alpha) \neq 0$ for any α, it suffices to note that the natural map

$$F^\times \longrightarrow \prod_{v \in S} F_v^\times / N_{E_v/F_v}(E_v^\times)$$

is surjective, which follows since $F^\times$ is dense in $\prod_{v\in S} F_v^\times$. □

9. Fourier coefficients and Bessel models

In this final section, we address a question raised by D. Prasad, concerning the Fourier coefficients (or rather the local analogs) of Saito-Kurokawa representations.

We have seen in the previous section that each étale quadratic algebra E determines an $M(F)$-orbit of generic characters ψ_E of N. If $\eta = \eta^\epsilon(\pi) = \theta_\psi(\sigma^\epsilon)$, then we may consider the twisted Jacquet module η_{N,ψ_E}. This is naturally a representation for the stabilizer $M_{\psi_E}(F)$ of ψ_E in $M(F)$, and we are interested in determining this M_{ψ_E}-module.

In the first place, one knows from [W2] that

$$\eta_{N,\psi_E} \neq 0 \Longleftrightarrow \sigma^\epsilon_{U,\psi_E} \neq 0 \Longleftrightarrow \epsilon(1/2, \pi \otimes \chi_E) = \epsilon \cdot \chi_E(-1) \cdot \epsilon(1/2, \pi),$$

in which case η_{N,ψ_E} is 1-dimensional. Naturally, we assume that the above conditions hold.

The action of $M = GL_1 \times SO_3$ on $Hom(N, \mathbb{G}_a) \cong \mathbb{G}_a^3$ is given by the standard representation of $GL_1 \times SO_3$, so that GL_1 acts by scalar multiplication. If V_E is the line spanned by a vector whose norm defines E, then we have:

$$M_{\psi_E} = S(O(1) \times O(V_E) \times O(V_E^\perp)) \subset GL_1 \times SO_3.$$

Thus, $M_{\psi_E} \cong O(V_E^\perp)$ and there is a natural projection

$$\det : M_{\psi_E} \longrightarrow \{\pm 1\}$$

whose kernel is $SO(1) \times SO(V_E) \times SO(V_E^\perp)$.

Theorem 9.1. *The action of M_{ψ_E} on $\eta^\epsilon(\pi)_{N,\psi_E}$ factors through $O(V_E^\perp)/SO(V_E^\perp) \cong \{\pm 1\}$, which acts by $\epsilon_\psi(\sigma^\epsilon) = \epsilon \cdot \epsilon(1/2, \pi)$.*

Proof. This is proved by a standard computation which we will sketch. We realize the Weil representation ω_ψ of $\tilde{SL}_2 \times SO(V_5)$ using the mixed model relative to the decomposition

$$V_5 = X \oplus V_3 \oplus X^*$$

where X is a 1-dimensional isotropic space. The precise description of this mixed model can be found in [GG], where the action of $\tilde{SL}_2 \times P(X^*)$ is explicitly described. Here, $P = P(X^*)$ is the parabolic subgroup stabilizing X^* and is a Siegel parabolic. Its Levi factor is $M = SO(V_3) \times GL(X^*)$ and its unipotent radical is $N = V_3 \otimes X$. We shall freely use the formulas described in [GG].

Using the mixed model, one sees that as a representation of $\tilde{SL}_2 \times P$, ω_ψ sits in a short exacct sequence:

$$0 \longrightarrow ind_{(\tilde{B} \times GL_1)^0 SO(V_3) N}^{\tilde{SL}_2 \times P} C_c^\infty(V_3) \longrightarrow \omega_\psi \longrightarrow C_c^\infty(V_3) \longrightarrow 0.$$

Here, in the third nonzero term of the short exact sequence, N acts trivially and so this term is irrelevant for the computation of the twisted Jacquet module. In the first term of the short exact sequence, $(\tilde{B} \times GL_1)^0$ is the subgroup of $\tilde{B} \times GL_1$ consisiting of those elements of the form

$$\begin{pmatrix} t & * \\ & t^{-1} \end{pmatrix} \times t.$$

Moreover, $SO(V_3)$ acts on $C_c^\infty(V_3)$ geometrically and $n \in N$ acts by

$$(nf)(v) = \psi(\langle v, n\rangle) \cdot f(v).$$

In particular, we see that the natural map $C_c^\infty(V_3) \longrightarrow C_c^\infty(V_3)_{N,\psi_E}$ is given by evaluating functions at a nonzero vector in V_E.

This observation allows one to calculate the twisted Jacquet module $(\omega_\psi)_{N,\psi_E}$ as a representation of $\tilde{SL}_2 \times M_{\psi_E}$. One obtains:

$$(\omega_\psi)_{N,\psi_E} \cong ind_{(\tilde{Z}\times M_{\psi_E})^0 U}^{\tilde{SL}_2 \times M_{\psi_E}} \chi_\psi \boxtimes \psi_E$$

where

- $\tilde{Z}$ is the inverse image in $\tilde{SL}_2$ of the center Z of SL_2; it is a finite group of order 4,
- $(\tilde{Z} \times M_{\psi_E})^0$ is the index-2 subgroup of $\tilde{Z} \times M_{\psi_E}$ consisting of those elements of the form $(\det(m), \epsilon) \times m$.
- U is the unipotent radical of the Borel subgroup $\tilde{B}$ of $\tilde{SL}_2$,
- χ_ψ is the standard genuine character of $\tilde{Z}$; note that there are two genuine characters of $\tilde{Z}$,
- ψ_E is a character of a generic character of U in the orbit indexed by E.

By first inducing to $\tilde{Z}U \times M_{\psi_E}$ before going all the way to $\tilde{SL}_2 \times M_{\psi_E}$, one obtains:

$$(\omega_\psi)_{N,\psi_E} \cong ind_{\tilde{Z}U\times M_{\psi_E}}^{\tilde{SL}_2\times M_{\psi_E}} (\chi_\psi \otimes \psi_E) \boxtimes \mathbf{1}) \bigoplus ind_{\tilde{Z}U\times M_{\psi_E}}^{\tilde{SL}_2\times M_{\psi_E}} (sgn \cdot \chi_\psi \otimes \psi_E) \boxtimes sgn(\det))$$

where sgn denotes the nontrivial character of $\{\pm 1\}$. Thus, M_{ψ_E} acts trivially on the first summand and acts via the sign character in the second summand.

Now σ^ϵ occurs uniquely as a quotient of exactly one of the two summands above. It occurs in the first summand iff its central character is $\chi_\psi|_{\tilde{Z}}$, which in turn holds iff $\epsilon_\psi(\sigma^\epsilon) = \epsilon \cdot \epsilon(1/2, \pi) = 1$. Thus the action of M_{ψ_E} on $\eta_{N,\psi_E} = \theta_\psi(\sigma^\epsilon)_{N,\psi_E}$ factors through $\det(M_{\psi_E}) = \{\pm 1\}$ which acts by $\epsilon_\psi(\sigma^\epsilon) = \epsilon \cdot \epsilon(1/2, \pi)$, as desired. □

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Department of Mathematics, University of California, San Diego, 9500 Gilman Drive, La Jolla, CA 92093, U.S.A.

E-mail address: wgan@math.ucsd.edu

School of Mathematics, Ben Gurion University of the Negev, POB 653, Be'er Sheva 84105, Israel

E-mail address: ngur@cs.bgu.ac.il

Appendix A. RESTRICTING SMALL REPRESENTATIONS OF $Sp_4(\mathbb{R})$ TO $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$

By **GORDAN SAVIN**

1. Introduction

Much interest in the oscillator representation of $Sp_{2n}(\mathbb{R})$ lies in the fact that its restriction to Howe dual pairs yields correspondences of representations. On the other hand, the group $Sp_{2n}(\mathbb{R})$ also contains the dual pairs $Sp_{2k}(\mathbb{R}) \times Sp_{2(n-k)}(\mathbb{R})$. However, these dual pair are not interesting from the point of view of the oscillator representation. Indeed, the restriction of the oscillator representation to the dual pair $Sp_{2k}(\mathbb{R}) \times Sp_{2(n-k)}(\mathbb{R})$ is simply the tensor product of the corresponding oscillator representations. In particular, this shows that the oscillator representation is too small, and that we should consider larger representations of $Sp_{2n}(\mathbb{R})$, when restricting to $Sp_{2k}(\mathbb{R}) \times Sp_{2(n-k)}(\mathbb{R})$. This point of view has been taken in the recent work of David Ginzburg [Gi], as well as in the work of Lee and Loke ([LL] and its sequel dealing with $Sp(p,q)$). Finally, a rather general construction of small representations of p-adic groups has been given by Weissman in [We].

Following a suggestion of Wee Teck Gan, in this paper we consider the simplest possible case. More precisely, Adams and Johnson [AJ] constructed (Arthur) packets $\{V_k^{2,0}, V_k^{1,1}, V_k^{0,2}\}$ of representations of $Sp_4(\mathbb{R})$. A detailed description of these representations is given in Section 2. In the same section, we restrict $V_k^{2,0}$ and $V_k^{0,2}$ to $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$. Since $V_k^{2,0}$ and $V_k^{0,2}$ are highest and lowest weight representations, respectively, the restriction is discrete and rather easy to calculate. An important consequence, however, is that the matching of infinitesimal characters of the two $SL_2(\mathbb{R})$ holds for $V_k^{1,1}$ as well. In Section 3 we restrict $V_k^{1,1}$ to $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$. Using a result of Vogan [V] we can control the correspondence for highest and lowest weight representations of $SL_2(\mathbb{R})$ (Proposition A.3). Combined with the matching of infinitesimal characters, Proposition A.3 gives a rather complete picture of the restriction of $V_k^{1,1}$ (Corollary A.4).

2. Preliminaries

Let $\mathfrak{g} = sp_4(\mathbb{C})$ be the complexified Lie algebra of $Sp_4(\mathbb{R})$. We shall use the standard realization of the root system of the type C_2 in $\mathbb{R}^2$, such that $\pm(1,-1)$ are the compact roots. Following Adams and Johnson [AJ], for each integer $k \geq 0$, define an A-packet of $(\mathfrak{g}, K)$-modules ($K = GL_2(\mathbb{C})$),

$$\{V_k^{2,0}, V_k^{1,1}, V_k^{0,2}\}$$

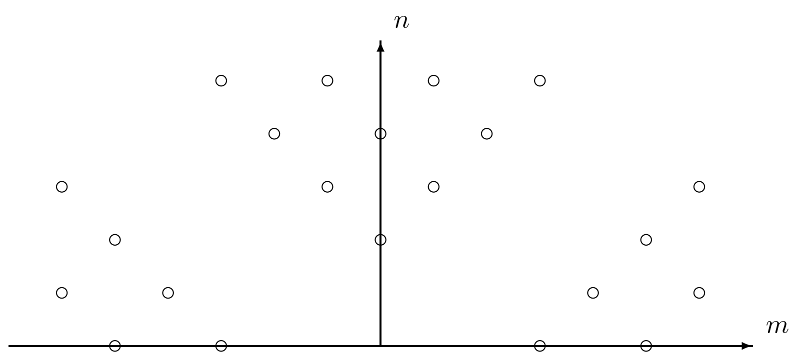
as follows. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$, be a θ-stable parabolic subalgebra such that $\mathfrak{l}_0 \cong u(p,q)$, where $\mathfrak{l}_0 = \mathfrak{l} \cap sp_4(\mathbb{R})$. Define $V_k^{p,q}$ to be the $A_{\mathfrak{q}}(\lambda)$-module, where

$$\lambda = \begin{cases} (k,k) \text{ if } (p,q) = (2,0) \\ (k,-k) \text{ if } (p,q) = (1,1) \\ (-k,-k) \text{ if } (p,q) = (0,2). \end{cases}$$

The multiplicities of K-types in $A_{\mathfrak{q}}(\lambda)$ are given as follows. A K-type will be denoted by $\Lambda_{a,b}$ where (a,b) with $a \geq b$ is the highest weight. Since the K-types of representations in the A-packet satisfy the congruence $a \equiv b \pmod 2$, it will be convenient to picture them using integer coordinates

$$\begin{cases} n = (a-b)/2 \\ m = (a+b)/2 \end{cases}$$

Then (this picture is modeled after $k = 0$):



Here the middle cone with vertex $(0, k+2)$ represents the K-types of $V_k^{1,1}$. The left and the right cones with vertices $(0, -k-3)$ and $(0, k+3)$ represent the K-types of $V_k^{0,2}$ and $V_k^{2,0}$, respectively. The restriction of the last two representations to $sl(2) \times sl(2)$ is easy to obtain:

Proposition A.1. *For any positive integer r, let D_r and D_{-r} be the representations of $sl(2)$ with the lowest weight r and the highest weight $-r$, respectively. Then*

$$\begin{cases} V_k^{2,0} = \oplus_{r \geq k+3} D_r \otimes D_r \\ V_k^{0,2} = \oplus_{r \geq k+3} D_{-r} \otimes D_{-r}. \end{cases}$$

Corollary A.2. *Let $\mathcal{J}_k$ be the annihilator of $V_k^{1,1}$ in the universal enveloping algebra of $sp(4)$. Let Ω_L and Ω_R be the Casimir operators of the two $sl(2)$. Then*

$$\Omega_L \equiv \Omega_R \pmod{\mathcal{J}_k}.$$

Proof. Note that all three modules in the packet have the same annihilator. Thus, in order to prove the congruence, it suffices to show that $\Omega_L = \Omega_R$ on $V_k^{2,0}$. This is follows from Proposition A.2. The corollary is proved. □

Case of $V_k^{1,1}$

Let Π a representation of the first $sl(2)$, and Θ a representation of the second $sl(2)$ such that $\Pi \otimes \Theta$ appears as a quotient of $V_k^{1,1}$. In this section we shall give an upper bound on Θ when Π is a highest or a lowest weight module.

Proposition A.3. *Let F_r denote the irreducible, finite dimensional representation with the highest weight r, and D_r be the holomorphic discrete series with weights $r, r+2, r+4 \ldots$.*

- *if $\Pi = F_r$, and $0 \leq r \leq k$, then the possible weights of Θ are $-r, -r+2, \ldots r$, which are precisely the weights of F_r.*
- *if $\Pi = F_r$, and $k+1 \leq r$, then there is no restriction on the weights of Θ, except they have the same parity as r.*
- *if $\Pi = D_r$, and $0 < r \leq k+2$, then the possible weights of Θ are $-r, -r-2, -r-4, \ldots$ which are precisely the weights of D_{-r}.*
- *if $\Pi = D_r$, and $k+3 \leq r$, then there are no possible weights of Θ. In particular, D_r does not appear in the correspondence.*

Moreover, in all cases the possible weights are of multiplicity one.

PROOF. The idea of the proof is as follows. Assume that r is the lowest weight of Π, and that s is a weight of Θ. Let $V(r,s)$ be the subspace of V_k such that the maximal compact subgroups of the two $sl(2)$ act by the indicated weights. Let E_1 be the weight raising member of the $sl(2)$-triple in the first $sl(2)$. Then

$$E_1 : V(r-2,s) \to V(r,s)$$

is injective [V; Lemma 3.4], but not surjective, since the image is contained in the kernel of the projection on $\Pi \otimes \Theta$. In particular, if for some s the map E_1 is bijective, then s cannot be a weight in Θ.

The apply this idea, we need to figure out which K-types of V_k contribute to $V(r,s)$. Note that the weights of $\Lambda_{a,b}$ are

$$(a,b),(a-1,b+1),\ldots(b,a).$$

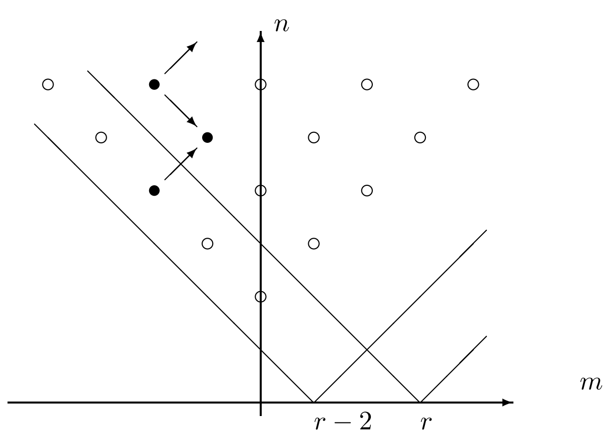
In particular, if $\Lambda_{a,b}$ contributes to $V(r,s)$, then for some integer l such that $0 \le l \le a-b = 2n$ we have

$$\begin{cases} a-l=r \\ b+l=s. \end{cases}$$

Summing up this two equations, and dividing by 2, this gives $m=p$ where $p=(r+s)/2$. Similarly, subtracting the two equations, and dividing by 2, gives $n-l=q$ where $q=(r-s)/2$. Since $|n-l| \le n$, and $q=r-p$, we see that $\Lambda_{a,b}$ contributes to $V(r,s)$ if and only

$$\begin{cases} m=p \text{ and} \\ |m-r| \le n. \end{cases}$$

Note that the second condition is independent of s. The graph of $|m-r|=n$ is $\vee$-shaped with vertex at r. If $k+3 \le r$ then we have the following picture.

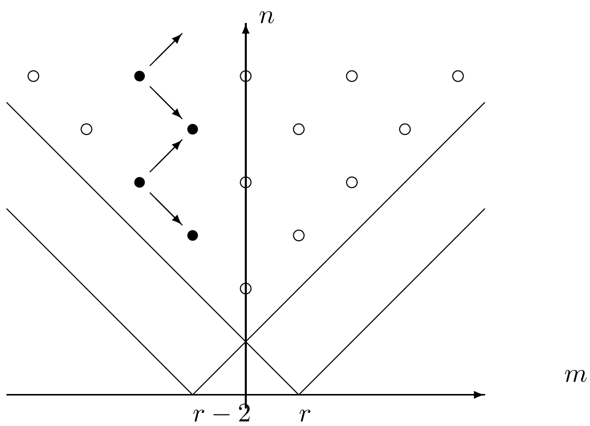


Here the black dots on the lines $m=p-1$ and $m=p$ represent the K-types which contribute to $V(r-2,s)$ and $V(r,s)$, respectively. The arrows represent the action of E_1. Indeed, by a variant of Clebsh-Gordan,

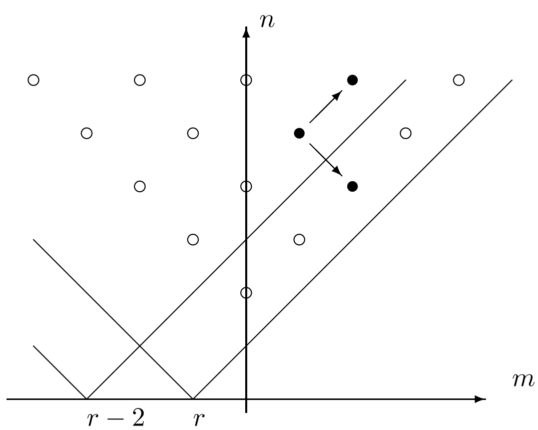
$$\mathfrak{p}^+ \otimes \Lambda_{a,b} = \Lambda_{a+2,b} \oplus \Lambda_{a+1,b+1} \oplus \Lambda_{a,b+2}.$$

In particular, if a K-type corresponds to a point (m,n), then acting by E_1 on it will end up in K-types parameterized by $(m+1,n+1)$ and $(m+1,n-1)$. It follows that E_1 maps the contribution to $V(r-2,s)$ at the point (m,n) to the contribution to $V(r,s)$ at the points $(m+1,n+1)$ and

$(m+1, n-1)$, as claimed. Since E_1 is injection, it restricts to an isomorphism between $V(r-2, s)$ and $V(r, s)$ (we have a non-degenerate upper-triangular system of equations). In particular, there is no weight s appearing here. Next, consider the case $-k \leq r < k+3$. Then



Here we cannot conclude that E_1 is a a bijection unless the line $m = p$ is right of the n axis. This means that $s \leq -r$. Finally, consider the case $r \leq -k-1$. Then



Here we can never conclude that E_1 is a bijection, and we cannot derive any restrictions on the type s.

Clearly, we can perform analogous calculations if Π is a highest weight module. Proposition is proved. □

We summarize our results with the following corollary. It is interesting to note that the representations appearing in the restriction of $V_k^{2,0}$ and $V_k^{0,2}$ are precisely those that we have eliminated for $V_k^{1,1}$.

Corollary A.4. *Let* $\Pi_L \otimes \Pi_R$ *be an irreducible* $sl(2) \times sl(2)$ *quotient of* $V_k^{1,1}$. *Then* $\Pi_L \cong \Pi_R$, *unless* $\Pi_L \cong D_r$ *or* D_{-r} *with* $r = 1, \ldots k+2$. *In that case* $\Pi_R \cong D_{-r}$ *or* D_r, *respectively. Finally,* Π_L *can never be isomorphic to* D_r *or* D_{-r} *with* $r \geq k+3$.

Proof. Note that the correspondence preserves the parity of the weights for the two $sl(2)$. In particular, the statement for irreducible principal series representations follows from Corollary A.2. Other statements follow from Proposition A.3. □

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Department of Mathematics, University of Utah, Salt Lake City, UT 84112, U.S.A.

E-mail address: savin@math.utah.edu