Non-tempered A-packets of G_2: Liftings from $\widetilde{SL}_2$

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§1. Introduction

J. Arthur has given a conjectural description of the discrete spectrum $L^2_{disc}(G(F)\backslash G(\mathbb{A}))$ of a reductive linear algebraic group G defined over a number field F. His conjectures, which are quite precise and delicate, have turned out to be very influential in the theory of automorphic forms. Let us recall these conjectures briefly, assuming for simplicity that G is split and simply-connected, so that the dual group has no center. The references are of course [A1] and [A2].

(1.1) Arthur's Conjectures. According to Arthur, the discrete spectrum possesses a decomposition

$$L^2_{disc}(G(F)\backslash G(\mathbb{A})) = \widehat{\bigoplus}_{\psi} L^2_{\psi},$$

where the Hilbert space direct sum runs over equivalence classes of A-parameters ψ, i.e. admissible maps

$$\psi : L_F \times SL_2(\mathbb{C}) \longrightarrow \hat{G}$$

where L_F denotes the conjectural Langlands group of F and $\hat{G}$ is the complex dual group of G. For any ψ, the space L^2_{ψ} is a direct sum of nearly equivalent representations, which we now describe.

(1.2) Local A-packets. The global A-parameter ψ gives rise to a local A-parameter

$$\psi_\nu : L_{F_\nu} \times SL_2(\mathbb{C}) \longrightarrow \hat{G}$$

for each place ν of F. Denote by S_{ψ_ν} the group of components of $Cent_{\hat{G}}(Im(\psi_\nu))$. To each irreducible representation η_ν of S_{ψ_ν}, Arthur conjecturally attached a unitarizable admissible (possibly reducible, possibly zero) representation π_{η_ν} of $G(F_\nu)$. The set

$$A_{\psi_\nu} = \left\{ \pi_{\eta_\nu} \, : \, \eta_\nu \in \widehat{S}_{\psi_\nu} \right\}$$

is called a local A-packet. It is required that

- for almost all ν, π_{η_ν} is irreducible and unramified if η_ν is the trivial character 1_ν. Its Satake parameter is

$$s_{\psi_\nu} = \psi_\nu \left(Fr_\nu \times \begin{pmatrix} q_\nu^{1/2} & \\ & q_\nu^{-1/2} \end{pmatrix} \right),$$

  where Fr_ν is a Frobenius element at ν and q_ν is the number of elements of the residue field at ν;

- the distribution

$$\sum_{\eta_\nu} \epsilon_{\eta_\nu} dim(\eta_\nu) Tr(\pi_{\eta_\nu}), \quad \text{for explicitly determined } \epsilon_\nu \in \{\pm 1\},$$

  is stable, and certain identities involving transfer of character distributions to endoscopic groups of $G(F_\nu)$ should hold.

These requirements may not characterize the set A_{ψ_ν} but they come pretty close; for more details, see [A2].

(1.3) Global A-packets and multiplicity formula. With the local packets A_{ψ_ν} at hand, we may define the global A-packet by:

$$A_\psi = \{\pi = \otimes_\nu \pi_{\eta_\nu} : \pi_{\eta\nu} \in A_{\psi_\nu}, \eta_\nu = 1_\nu \text{ for almost all } \nu\}.$$

It is a set of nearly equivalent representations of $G(\mathbb{A})$ and basically, L^2_ψ is the sum of the elements of A_ψ with some multiplicities. This multiplicity is precisely given as follows. Arthur attached to ψ a quadratic character ϵ_ψ of $S_\psi = Cent_{\hat{G}}(Im(\psi))$. Now if $\pi = \otimes_\nu \pi_{\eta_\nu} \in A_\psi$, set

$$m(\pi) = \frac{1}{\# S_\psi} \cdot \left(\sum_{s \in S_\psi} \epsilon_\psi(s) \cdot \eta(s) \right)$$

with $\eta = \otimes_\nu \eta_\nu$. Then Arthur conjectures that there is a $G(\mathbb{A})$-equivariant embedding

$$\iota_\psi : \bigoplus_{\pi \in A_\psi} m(\pi)\pi \hookrightarrow L^2_{disc}(G(F)\backslash G(\mathbb{A})).$$

The image of this embedding is the subspace L^2_ψ.

(1.4) Tempered and non-tempered parameters. An A-parameter ψ is called *tempered* if ψ is trivial when restricted to $SL_2(\mathbb{C})$. In this case, the representations in A_ψ are conjectured to be tempered. A non-tempered A-paramter, on the other hand, tends to factor through a subgroup $\hat{H} \subset \hat{G}$ with H an endoscopic group of G. The representations corresponding to such a parameter are considered more degenerate. One might thus hope to construct the packets A_{ψ_ν} and the embedding $L^2_\psi \hookrightarrow L^2_{disc}$ by lifting from a group smaller than G.

(1.5) The parameters of G_2. The main purpose of this paper is to carry out such a construction for a family of non-tempered A-parameters of the split exceptional group G of type G_2. Let us describe the A-parameters of G_2 in greater detail. Recall that the complex dual group of G is $G_2(\mathbb{C})$, and there are 4 conjugacy classes of non-trivial homomorphisms $SL_2(\mathbb{C}) \longrightarrow G_2(\mathbb{C})$ corresponding to the 4 non-trivial unipotent conjugacy classes in $G_2(\mathbb{C})$:

$$\mathcal{O}_{long} < \mathcal{O}_{short} < \mathcal{O}_{subreg} < \mathcal{O}_{reg}.$$

There are thus 4 families of non-tempered A-parameters for G_2. We list them below:

- $\psi|_{SL_2}$ corresponds to the regular orbit. The corresponding local and global A-packets are singletons consisting of the trivial representation of $G(F_\nu)$ and $G(\mathbb{A})$ respectively.

- $\psi|_{SL_2}$ corresponds to the subregular orbit. The detailed study of these parameters was conducted in [GGJ]. In particular, the local A-packets were defined using the results of [HMS] and [V], and the embedding of $L^2_\psi \hookrightarrow L^2_{disc}$ was constructed. These A-packets are (cubic) unipotent, and they provide the first example of a family of representations whose cuspidal multiplicities are unbounded. The multiplicity formula was established in [G].

- $\psi|_{SL_2}$ gives the short root SL_2 in G_2. This is the case considered in this paper and we shall give a detailed discussion below.

- $\psi|_{SL_2}$ gives the long root SL_2 in G_2. This is an ongoing project, which we hope to complete in the near future.

(1.6) The short root parameter. Let us return to the case of interest in this paper. The centralizer of the short root SL_2 is the long root SL_2; so the parameter factors through $SL_2 \times SL_2$:

$$\psi : L_F \times SL_2(\mathbb{C}) \xrightarrow{\phi\times id} SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \xrightarrow{\quad\quad i \quad\quad} G_2(\mathbb{C})$$

The centralizer S_ψ of the image of ψ is finite if and only if ϕ is an irreducible representation of L_F. Conjecturally, there is a cuspidal representation τ of $PGL_2(\mathbb{A})$ corresponding to such a ϕ. Thus we shall write ϕ_τ for ϕ. In this case, the global component group is $S_\tau = \mathbb{Z}/2\mathbb{Z}$.

The local component groups $S_{\tau,v}$ are given by

$$S_{\tau,v} = \begin{cases} \mathbb{Z}/2\mathbb{Z} \text{ if } \phi_{\tau,v} \text{ is irreducible;} \\ 1 \text{ if } \phi_{\tau,v} \text{ is reducible.} \end{cases}$$

Note that the condition that $\phi_{\tau,v}$ be irreducible is equivalent to τ_v being a discrete series representation of $PGL_2(F_v)$. Thus, we deduce that $A_{\tau,v}$ has the form

$$A_{\tau,v} = \begin{cases} \{\pi^+_{\tau,v}, \pi^-_{\tau,v}\}, \text{ if } \tau_v \text{ is discrete series;} \\ \{\pi^+_{\tau,v}\}, \text{ if } \tau_v \text{ is not.} \end{cases}$$

Here, $\pi^+_{\tau,v}$ is the representation indexed by the trivial character of $S_{\tau,v}$.

In particular, the global A-packet A_τ has 2^S elements where S is number of places where τ_v belongs to the discrete series. What is the multiplicity with which each of these representations occurs in $L^2_{\psi_\tau}$? For this, we need to know the quadratic character ϵ_{ψ_τ}. It turns out that ϵ_{ψ_τ} is the non-trivial character of $S_\psi \cong \mathbb{Z}/2\mathbb{Z}$ if and only if $\epsilon(\tau, 1/2) = -1$. From this, one deduces that if $\pi = \otimes_v \pi^{\epsilon_v}_{\tau,v}$, with $\epsilon_v = \pm$, then

$$m(\pi) = \begin{cases} 1, \text{ if } \epsilon(\tau, 1/2) = \epsilon_\pi := \prod_v \epsilon_v; \\ 0, \text{ if } \epsilon(\tau, 1/2) = -\epsilon_\pi. \end{cases}$$

In particular, the formal structure of this family of A-packets is identical to that of the Saito-Kurakawa packets of $PGSp_4$ (cf. [G3]).

In this paper, we shall give a natural construction of the packets $\mathcal{A}_{\tau,v}$ and the subspace $L^2_{\psi_\tau}$ of the discrete spectrum. We shall also prove that the space $L^2_{\psi_\tau}$ we construct is a full near equivalence class, thereby justifying the claim that our packets are the correct ones.

(1.7) The work of Rallis-Schiffmann. For almost all v, the local A-packet is a singleton and we know what the representation $\pi^+_{\tau,v}$ has to be: it has Satake parameter $s_{\psi_{\tau,v}}$ and is the unramified constituent of the induced representation

$$Ind_{P_2}^{G_2}\delta_{P_2}^{2/3}\tau_v,$$

where P_2 is the Heisenberg parabolic subgroup of G_2. How can one construct the packets $A_{\tau,v}$ in general?

In the paper [RS], Rallis and Schiffmann constructed a lifting of cuspidal automorphic forms from the metaplectic group $\widetilde{SL}_2$ to G_2 by exploiting the fact that $SL_2 \times G_2$ is a subgroup of $SL_2 \times O_7$, which is a classical dual pair in Sp_{14}. The lifting is then defined using the theta kernel furnished by the Weil representation $\omega_\psi^{(7)}$ of $\widetilde{Sp}_{14}$ (which depends on the choice of an additive character ψ).

The surprising discovery of Rallis-Schiffmann is that, despite restricting from O_7 to the smaller group G_2, one still obtains a correspondence of representations. More precisely, if σ is an irreducible cuspidal representation of $\widetilde{SL}_2(\mathbb{A})$, let $V(\sigma)$ be the theta lift of σ; it is a non-zero subspace of the space of automorphic forms on G_2. Then the main results of Rallis-Schiffmann are:

- $V(\sigma)$ is contained in the space of cusp forms if and only if the theta lift (associated to ψ) of σ to SO_3 (studied by Waldspurger) is zero.
- The cuspidal representations obtained as lifts from $\widetilde{SL}_2$ are non-generic and can be characterized as those having a non-zero period with respect to some quasi-split SU_3 (which is a subgroup of G_2).
- The local correspondence of *unramified* representations is precisely determined. In particular, when $V(\sigma)$ is cuspidal, the local components of each irreducible constituent of $V(\sigma)$ are determined for *almost all* places v, in terms of the local components of σ.
- As a consequence of the unramified correspondence, the irreducible cuspidal representations contained in $V(\sigma)$ are CAP with respect to the Heisenberg parabolic or the Borel subgroup of G_2. This gives the first construction of CAP representations of G_2. In particular, for certain σ's, $V(\sigma)$ contains cuspidal representations which are nearly equivalent to a constituent of $Ind_{P_2}^{G_2}\delta_{P_2}^{2/3}\tau$ for a cuspidal representation τ of PGL_2.

The results of Rallis-Schiffmann indicate that the local A-packets $A_{\tau,v}$ can be constructed by theta lifting from $\tilde{SL}_2$ to G_2.

(1.8) The results of this paper. In this paper, we give a complete analysis of the theta correspondence arising from $\tilde{SL}_2 \times G_2$, in the style of Waldspurger's analysis of the Shimura correspondence. In particular, we complete the study initiated in [RS] by giving a precise determination of the representation $V(\sigma)$. The first step in this is the complete determination of the *local* theta correspondence. More precisely, if v is a p-adic place of F and σ_v an irreducible representation of $\widetilde{SL_2}(F_v)$, the maximal σ_v-isotypic quotient of $\omega_{\psi_v}^{(7)}$ can be expressed as $\sigma_v \otimes \theta(\sigma_v)$, where $\theta(\sigma_v)$ is a smooth representation of $G_2(F_v)$. Let $\Theta(\sigma_v)$ be the maximal semisimple quotient of $\theta(\sigma_v)$. In the archimedean case, one has an analogous (but slightly different) definition for $\Theta(\sigma_v)$, with σ_v unitary. Our main local result is:

(1.9) Theorem $\Theta(\sigma_v)$ *can be completely determined for any* σ_v *(to the extent that classification of representations of* $G_2(F_v)$ *is known). It turns out that* $\Theta(\sigma_v)$ *is irreducible except when* $\sigma_v = \omega_{\psi_v}^{\pm}$ *(the even and odd Weil representations of* $\tilde{SL}_2(F_v)$ *associated to* ψ_v*). In these two exceptional cases,* $\Theta(\sigma_v)$ *is the sum of 2 unipotent representations.*

A precise statement of the results is given in Theorems 9.1, 10.10 and 10.13 for p-adic fields, $\mathbb{R}$ and $\mathbb{C}$ respectively. We remark that over $\mathbb{R}$, the result was already shown by Li-Schwermer for most discrete series representations.

(1.10) Definition of local A-packets. Using this Theorem, we may define the local packet $A_{\tau,v}$ as follows. After the work of Waldspurger [W1, W2] on the Shimura correspondence, there is a natural grouping of irreducible admissible genuine representations of $\tilde{SL}_2$ in packets of either one or two elements. These packets are indexed by the infinite-dimensional representations τ_v of PGL_2, and contains a distinguished element $\sigma_{\tau_v}^+$. It contains another element $\sigma_{\tau_v}^-$ iff τ_v is discrete series. We set

$$\begin{cases} \pi_{\tau,v}^+ = \Theta(\sigma_{\tau_v}^+) \\ \pi_{\tau,v}^- = \Theta(\sigma_{\tau_v}^-). \end{cases}$$

(1.11) Reducibility. Let us mention a surprising aspect of our definition here. Namely, if τ_v is the Steinberg representation, then $\sigma_{\tau_v}^+$ is the odd Weil representation associated to ψ_v, and thus $\pi_{\tau,v}^+$ is reducible. Indeed,

$$\pi_{\tau,v}^+ = \pi_{\tau,v}^{++} \oplus \text{ (a supercuspidal unipotent representation)},$$

with $\pi_{\tau,v}^{++} = J_{P_2}(St, 1/2)$ non-supercuspidal.

To the best of our knowledge, this is the first instance of such a phenomenon for split p-adic groups. There are various local justifications for our definition; the details can be found in Section 11. However, the strongest justification is global, as we explain next.

(1.12) Global results. We now turn to the global situation. For any cuspidal σ on $\widetilde{SL}_2$, one can show that $V(\sigma)$ is contained in the space of square-integrable automorphic forms. Thus $V(\sigma)$ is semisimple and is a non-zero summand of $\Theta(\sigma) := \otimes_v \Theta(\sigma_v)$. Our precise local result immediately shows that $V(\sigma) \cong \Theta(\sigma)$, whenever σ_v is not the even or odd Weil representations associated to ψ_v for any place v, since $\Theta(\sigma)$ is irreducible then. However, when $\Theta(\sigma)$ is reducible, there are more than one possibilities for $V(\sigma)$. The determination of $V(\sigma)$ in this case is easily the trickiest part of the paper. In any case, our main global result (Theorem 13.1) says:

(1.13) Theorem *If $\sigma \subset \mathcal{A}_{00}$, i.e. σ is not an irreducible summand of any Weil representation, then $V(\sigma) \cong \Theta(\sigma)$.*

In fact, one can define the theta lift of any square-integrable automorphic representation of $\widetilde{SL}_2$ by a regularization of the theta integral. Thus, one can speak of the regularized theta lift of the orthogonal complement of $\mathcal{A}_{00}$, which consists of the Weil representations of $\widetilde{SL}_2$. One can show that the space of automorphic forms of G_2 thus obtained is precisely equal to that constructed in [GGJ], by restriction of the minimal representation of the various quasi-split $Spin_8$'s attached to étale quadratic algebras.

(1.14) Global A-packets. Our global theorem, together with the deep results of Waldspurger on the automorphic forms of $\tilde{SL}_2$, allows us to define the embedding

$$\iota_{\psi_\tau} : \oplus_{\pi \in A_\tau} m(\pi)\pi \hookrightarrow L^2_{disc}(G_2(F)\backslash G_2(\mathbb{A}))$$

as required by Arthur's conjectures. We denote the (closure of the) image of ι_{ψ_τ} by $L^2_{\psi_\tau}$. From this, one has:

(1.15) Corollary *If π_0 is an irreducible constituent of $\pi \in \mathcal{A}_\tau$, and $m_{disc}(\pi_0)$ denotes the multiplicity of π_0 in L^2_{disc}, then*

$$m_{disc}(\pi_0) \geq m(\pi).$$

This provides some global justification for our definition of the local packets, especially when $m(\pi) = 1$. When $m(\pi) = 0$, however, it says nothing that we don't already know. We would thus like to strengthen the inequality of the corollary to an equality.

(1.16) SL_3 period. Let us highlight another corollary of the global theorem before moving on. It pertains to the question of whether there are cuspidal representations of G_2 with non-zero SL_3-period. Such cuspidal representations should be very scarce, but can be obtained by restriction of the minimal representation of split $Spin_8$ [GGJ]. It wasn't known previously if other SL_3-distinguished cuspidal representations exist. Now, as a consequence of our global theorem, we know that they do. Indeed, when $\sigma \subset \mathcal{A}_{00}$ is such that its theta lift to SO_3 is non-zero, we know from [RS] that $V(\sigma)$ is not totally contained in $\mathcal{A}_{cusp}(G_2)$. However, this does not exclude the possibility that $V(\sigma) \cap \mathcal{A}_{cusp}(G_2)$ is non-zero. In [RS, Pg. 823], Rallis-Schiffmann remarked that they do not know if such a possibility can actually happen. Our global theorem implies that it does, and the cuspidal representations thus obtained are SL_3-distinguished.

(1.17) Multiplicity formula and near equivalence classes. Now let us return to the question of showing equality in Cor. 1.15. We shall show the following statement:

(1.18) Theorem *Suppose that $\rho \subset L^2_{disc}$ is nearly equivalent to the representations in $\mathcal{A}_\tau$. Then ρ is not orthogonal to the image of $L^2_{\psi_\tau}$. In particular, ρ is isomorphic (as an abstract representation) to a representation in $\mathcal{A}_\tau$.*

This theorem has several consequences:

(1.19) Corollary *(i) The projection of* $L^2_{\psi_\tau}$ *to* L^2_{cusp} *is a full near equivalence class in the space of cusp forms.*

(ii) If π_0 *is an irreducible constituent of* $\pi \in \mathcal{A}_\tau$*, then*

$$m_{disc}(\pi_0) = m(\pi),$$

except possibly in the following case: $\pi_0 = \pi_\tau^{++} := \otimes_v \pi_{\tau,v}^{++}$ and $L(\tau, 1/2) \neq 0$*. In the exceptional case, we have* $m(\pi) = 1$ *and*

$$m_{disc}(\pi_0) = \text{ 1 or 2.}$$

Here, $\pi_{\tau,v}^{++}$ *is simply equal to* $\pi_{\tau_v}^{+}$ *if the latter is irreducible, and is defined in (1.11) otherwise.*

Of course, in the exceptional case of (ii), we expect $m_{disc}(\pi_0)$ to be equal to 1. The reason for the ambiguity is that if $L(\tau, 1/2) \neq 0$, then π_τ^{++} occurs in the residual spectrum, and we don't know how to rule out the possibility that it also occurs in the cuspidal spectrum.

In any case, the corollary provides conclusive justification that the local packets defined here are the correct ones. Indeed, if any local representation ρ_{v_0} is to belong to $A_{\tau_{v_0}}$, then ρ_{v_0} should occur as a local component of a global representation ρ which is nearly equivalent to the representations in A_τ and which occurs in the discrete spectrum. The above result shows that our definition of A_{τ_v} already captures all such local representations.

(1.20) A Rankin-Selberg integral. The proof of Theorem 1.18 is given in §16, and depends on a certain Rankin-Selberg integral, whose analysis is given in §15. The use of this Rankin-Selberg integral is quite amusing. More precisely, if $\pi \subset \mathcal{A}_{cusp}(G_2)$ and $\varphi \in \pi$, we consider the integral

$$J_K(\varphi, f, s) = \int_{SU_3^K(F)\backslash SU_3^K(\mathbb{A})} \varphi(g) \cdot E_K(f, s, g)\, dg$$

where $E_K(f, s, g)$ is an Eisenstein series on SU_3^K. This integral defines a meromorphic function of $s \in \mathbb{C}$. On unfolding, we obtain

$$J_K(\varphi, f, s) = \int_{T_K(\mathbb{A})\backslash SU_3^K(\mathbb{A})} f_s(g) \cdot \varphi_{\Psi_K}(g)\, dg$$

where T_K is a maximal torus of SU_3^K and φ_{Ψ_K} is a certain Fourier coefficient of φ.

In general, one does not have a local uniqueness theorem for the local functionals corresponding to the global Fourier coefficient φ_{Ψ_K}. Hence, the value of this Rankin-Selberg integral is unclear. For our applications, however, we are interested in the case when π is nearly equivalent to the representations in $\mathcal{A}_\tau$. Even in this case, one does not have local uniqueness for *half* of the places of F. However, we shall see that for almost all v, there is a distinguished local functional which intervenes in the global Fourier coefficient φ_{Ψ_K}. This implies that the Fourier coefficient φ_{Ψ_K} is almost Eulerian. Computing the local factors at almost all places, we obtain

$$J_K(f, \varphi, s) = d_S(f, \varphi, s) \cdot L^S(s)$$

where $L^S(s)$ is a certain partial L-function and d_S is the bad factor associated to a finite set S of places. Using this, one shows that π has non-zero SU_3^K-period, which leads to Thm. 1.18.

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§2. Preliminaries

Let F be a non-archimedean local field of characteristic zero. In this section, we briefly recall the structure of the maximal parabolic subgroups of special orthogonal groups as well as the definition of the metaplectic group $\tilde{SL}_2(F)$. In fact, most of the material in this section holds over any local or global field.

(2.1) Special orthogonal groups and their maximal parabolics. Let (V, q) be a quadratic space over F, with associated symmetric bilinear form $b_q(x,y) = q(x+y) - q(x) - q(y)$. In this paper, we shall only consider quadratic spaces of odd dimension. Hence we assume that $dim(V) = m$ is odd. If $SO(V, q)$ denotes the corresponding special orthogonal group, we recall the structure of the maximal parabolic subgroups of $SO(V, q)$. Our exposition follows [MVW, Chap. 1] closely.

Suppose we have a decomposition

$$V = X \oplus U \oplus X^* \tag{1}$$

where (U, q_U) is nondegenerate, X is isotropic with dual space X^* and $b_q : X \times X^* \to F$ agrees with the natural pairing. The stabilizer of X^* in $SO(V, q)$ is a maximal parabolic subgroup $P(X^*) = M(X^*) \cdot N(X^*)$. Here,

$$M(X^*) \cong GL(X^*) \times SO(U, q_U)$$

is a Levi subgroup. The unipotent radical $N(X^*)$ is at most 2-step nilpotent. An element $n \in N(X^*)$ fixes X^* pointwise and for $u \in U$, $e[n](u) = n(u) - u$ lies in X^*. The map

$$e : N(X^*) \to Hom(U, X^*)$$

is surjective and its kernel is the commutator subgroup $Z(X^*)$ of $N(X^*)$. There is thus a short exact sequence:

$$1 \longrightarrow Z(X^*) \longrightarrow N(X^*) \xrightarrow{e} Hom(U, X^*) \longrightarrow 1.$$

(2.2) The commutator $Z(X^*)$. The group $Z(X^*)$ can be naturally identified with

$$\{S \in Hom(X, X^*) : S^* = -S\} \cong \{\text{alternating forms on } X\}.$$

Here, $S^* : X \to X^*$ is characterized by

$$\langle x, S^*(y) \rangle = \langle S(x), y \rangle.$$

The element $z(S) \in Z(X^*)$ acts as identity on $X^* \oplus U$ and for $x \in X$,

$$z(S)(x) = x + S(x).$$

Under this identification, the adjoint action of $M(X^*)$ on $Z(X^*)$ is the natural one, with $SO(U, q_U)$ acting trivially.

(2.3) The section n. We define a section $n : Hom(U, X^*) \to N(X^*)$ of the morphism e as follows. For $T \in Hom(U, X^*)$, we have $T^* \in Hom(X, U^*)$ naturally associated to T. Identifying U and U^* using the quadratic form q_U (or rather its associated symmetric bilinear form), we regard T^* as an element of $Hom(X, U)$. Hence, we have:

$$b_q(T^*(x), u) = b_q(x, T(u)) \quad \text{for } x \in X \text{ and } u \in U.$$

We now define $n(T) \in N(X^*)$ by its action on V:

$$\begin{aligned} &x^* \mapsto x^* \\ &u \mapsto u + T(u) \\ &x \mapsto x - T^*(x) - \frac{1}{2} T \circ T^*(x). \end{aligned}$$

The section n satisfies:

$$n(S) \cdot n(T) = n(S+T) \cdot z(\frac{1}{2}(TS^* - ST^*)) \quad \text{and} \quad n(T)^{-1} = n(-T).$$

Hence, n is not a group morphism. However, it is equivariant with respect to the natural action of $M(X^*)$ on $Hom(U, X^*)$ and the adjoint action of $M(X^*)$ on $N(X^*)$.

(2.4) Weil index. We now recall the notion of the Weil index of a quadratic form. To a non-trivial additive character ψ of F, Weil has associated an 8th root of unity $\gamma(\psi)$. If a quadratic form q has orthogonal decomposition $a_1 x_1^2 + ... + a_m x_m^2$, then the Weil index of q is

$$\gamma(\psi \circ q) = \prod_{i=1}^{m} \gamma(\psi_{a_i})$$

where ψ_a is the character $x \mapsto \psi(ax)$. Also, for $a \in F^\times$, we set

$$\gamma(a, \psi) = \gamma(\psi_a)/\gamma(\psi).$$

We refer the reader to [K2, I.4] for the basic properties of Weil indices.

In this paper, we shall exclusively be concerned with the split quadratic spaces

$$(V_m, q_m) = \langle 1 \rangle \oplus \mathbb{H}^{(m-1)/2}$$

where $\mathbb{H}$ denotes a hyperbolic plane (i.e. split quadratic space of rank 2) and m is an odd positive integer ≤ 7.

(2.5) The group $\tilde{SL}(2)$. For simplicity, we shall introduce the following notations for elements of $SL_2(F)$:

$$t(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \qquad n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \qquad w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let $(-,-)_F$ be the Hilbert symbol of F. We consider the 2-cocycle

$$c : SL_2(F) \times SL_2(F) \longrightarrow \{\pm 1\}$$

defined by Rao [R]. Explicitly, it is given by [K2, Chapter 1, Lemma 6.2]:

$$c(g_1, g_2) = (x(g_1), x(g_2))_F(-x(g_1)x(g_2), x(g_1g_2))_F$$

where

$$x\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} c \text{ if } c \neq 0, \\ d \text{ if } c = 0. \end{cases}$$

In particular, if $t(a)$ denotes the diagonal matrix $diag(a, a^{-1})$, then

$$c(t(a_1), t(a_2)) = (a_1, a_2)_F.$$

The group $\tilde{SL}_2(F)$ is a topological central extension of $SL_2(F)$ defined as follows. As a set, it is $SL_2(F) \times \{\pm 1\}$, and the group law is given by:

$$(g_1, \epsilon_1) \cdot (g_2, \epsilon_2) = (g_1, g_2, \epsilon_1 \epsilon_2 c(g_1, g_2)).$$

Observe that it has a natural projection onto $SL_2(F)$.

If H is a subgroup of SL_2, then we shall let $\tilde{H}$ be its inverse image in $\tilde{SL}_2(F)$. If $g \in SL_2(F)$, we shall use the same symbol to denote the element $(g, 1) \in \tilde{SL}_2(F)$.

(2.6) The character χ_ψ. Let T be the diagonal torus in SL_2. Then it is easy to see that $\tilde{T}$ is still abelian. We define a character χ_ψ of $\tilde{T}$ by:

$$\chi_\psi(t(a), \epsilon) = \epsilon \cdot \gamma(a, \psi)^{-1}.$$

This character plays an important role in the general theory.

(2.7) The character χ_q. Let $det(q)$ be the determinant of a quadratic form q regarded as an element of $F^\times/F^{\times 2}$. More precisely, if q has orthogonal decomposition $a_1x_1^2 + ..., a_mx_m^2$, then $det(q) = a_1a_2...a_m$. We define a character χ_q of $F^\times$ by:

$$\chi_q(a) = (a, (-1)^{m(m-1)/2}det(q))_F.$$

For the quadratic space (V_m, q_m), the charcater χ_{q_m} is trivial.

§3. The Weil Representation of $\tilde{SL}(W) \times SO(V, q)$

(3.1) The standard symplectic space. Let $(W, \langle -, - \rangle)$ be a 2-dimensional symplectic space. We shall assume that W comes equipped with a basis $\{f_1, f_2\}$ with $\langle f_1, f_2 \rangle = 1$. This basis allows us to identify $Sp(W)(F)$ with $SL_2(F)$ and thus we have a 2-fold cover $\tilde{SL}(W)$ of $SL(W)$. We let $\tilde{SL}(W)$ act on W via its projection to $SL(W)$.

(3.2) The Heisenberg group. Let (V, q) be a quadratic space of odd dimension m. Then $W \otimes V$ is naturally equipped with an alternating form $(-, -) = \langle -, - \rangle \otimes b_q$. Thus we can consider the Heisenberg group $H(W \otimes V)$ associated to $W \otimes V$. As a set, it is $(W \otimes V) \oplus F$, and the group law is given by:

$$(x_1, t_1) \cdot (x_2, t_2) = (x_1 + x_2, t_1 + t_2 + \frac{1}{2}(x_1, x_2)).$$

The center is $Z_{H(W \otimes V)} = \{(0, t); t \in F\}$. For $x \in W \otimes V$, we shall let $h(x) = (x, 0) \in H(W \otimes V)$. Of course, the map h is not a group morphism.

Now the group $SL(W) \times SO(V, q)$ acts naturally on $H(W \otimes V)$ as group automorphisms:

$$g \cdot (x, t) = (g \cdot x, t).$$

Thus we may form the semi-direct product $(SL(W) \times SO(V, q)) \ltimes H(W \otimes V)$.

(3.3) Schrodinger model. We can now describe a model for the Weil representation $\omega_{\psi,q}$ of this semi-direct product, associated to a non-trivial additive character ψ of F and the isomorphism class of the quadratic space (V, q). This model is realized on the space $S(f_2 \otimes V)$ of Schwarz functions on $f_2 \otimes V$. The center of $H(W \otimes V)$ acts by the character ψ and the actions of other elements are given by:

$$\begin{cases} \omega_{\psi,q}(n(b))f(x) = \psi(n \cdot q(x))f(x), \\ \omega_{\psi,q}(t(a), \epsilon)f(x) = |a|^{m/2}\chi_\psi(t(a), \epsilon)\chi_q(a)f(ax), \\ \omega_{\psi,q}(w_0)f(x) = \gamma(\psi \circ q) \int_V \psi(b_q(x, y))f(y)dy \\ \omega_{\psi,q}(g)f(x) = f(g^{-1} \cdot x) \quad \text{for } g \in SO(V, q) \\ \omega_{\psi,q}(h(f_1 \otimes v))f(x) = \psi(-b_q(x, v))f(x) \quad \text{for } v \in V, \\ \omega_{\psi,q}(h(f_2 \otimes v))f(x) = f(x + v) \end{cases}$$

The action of w_0 given above is essentially a Fourier transform, and the Haar measure dy is chosen to be self-dual for the Fourier transform in question. In the sequel, the same comment applies to other such choices of measure.

The above realization of the representation $\omega_{\psi,q}$ will be called the Schrodinger model. We shall now introduce other models which will be very useful. These are collectively called the mixed models.

(3.4) Mixed models. Suppose we have a decomposition

$$V = X \oplus U \oplus X^*$$

as in (1). We shall define a linear isomorphism

$$\mathcal{F}_{X^*} : S(f_2 \otimes V) \longrightarrow S(W \otimes X) \otimes S(f_2 \otimes U).$$

To simplify notations, we shall write (y, x) for the element $f_1 \otimes y + f_2 \otimes x \in W \otimes X$. Also, we shall regard an element of $S(W \otimes X) \otimes S(f_2 \otimes U)$ as a Schwarz function on $W \otimes X$ taking values in $S(f_2 \otimes U)$. Observe that $S(f_2 \otimes U)$ affords the representation ω_{ψ,q_U} of the group $(\tilde{SL}(W) \times SO(U, q_U)) \ltimes H(W \otimes U)$.

The linear map $\mathcal{F}_{X^*}$ is a partial Fourier transform defined by:

$$\mathcal{F}_{X^*}(f)(y, x)(u) = \int_{X^*} f(x + u + x^*) \cdot \psi(\langle y, x^* \rangle) dx^*.$$

Using $\mathcal{F}_{X^*}$, we obtain a representation of our semi-direct product on $S(W \otimes X) \otimes S(f_2 \otimes U)$ by transport of structure. We denote this representation by the same symbol $\omega_{\psi,q}$.

(3.5) Formulas. We now want to write down some formulas for the action of $\omega_{\psi,q}$ in this model. It turns out that one has rather clean formulas for the action of the subgroup

$$(\tilde{SL}(W) \times P(X^*)) \ltimes H(W \otimes V).$$

To describe these, we introduce the following notations:

- Let $\tilde{SL}(W) \times M(X^*)$ act on $W \otimes X$ via its projection onto $SL(W) \times GL(X^*)$, followed by the natural action of the latter.
- Extend the action ω_{ψ,q_U} of $\tilde{SL}(W) \times SO(U, q_U)$ on $S(f_2 \otimes U)$ to

  $$\tilde{SL}(W) \times M(X^*) = (\tilde{SL}(W) \times SO(U, q_U)) \times GL(X^*)$$

  by letting $GL(X^*)$ act by $|det_{X^*}|$. Denote the extended action by ω_{ψ,q_U} also.

Then we have the following formula for the action of $\tilde{SL}(W) \times M(X^*)$ in the mixed model:

- For $g \in \tilde{SL}(W) \times M(X^*)$,

  $$\omega_{\psi,q}(g)\phi(y, x) = \omega_{\psi,q_U}(g)(\phi(g^{-1}(y, x))).$$

The following formulas give the action of $N(X^*)$ on the mixed model:

- For $z(S) \in Z(X^*)$,

  $$\omega_{\psi,q}(z(S))\phi(y, x) = \psi(\langle S(y), x \rangle)\phi(y, x).$$

- For $n(T) \in N(X^*)$,

$$\omega_{\psi,q}(n(T))\phi(y,x) = \omega_{\psi,q_U}(h(f_1 \otimes T^*(y) + f_2 \otimes T^*(x)))(\phi(y,x)).$$

Finally, we would like to describe the action of $H(W \otimes V)$ on the mixed model. There is a natural surjective map

$$H(W \otimes (X \oplus X^*)) \times H(W \otimes U) \longrightarrow H(W \otimes V).$$

When regarded as a representation of this direct product, $S(W \otimes X) \otimes S(f_2 \otimes U)$ is simply the tensor product of the Heisenberg representation of each factor. In particular, we note the following special instances:

- for $u \in W \otimes U$,

$$\omega_{\psi,q}(h(u))\phi(y,x) = \omega_{\psi,q_U}(h(u))(\phi(y,x));$$

- for $x^*, y^* \in X^*$,

$$\omega_{\psi,q}(h(f_1 \otimes x^* + f_2 \otimes y^*))\phi(y,x) = \psi(\langle y^*, y\rangle - \langle x^*, x\rangle) \cdot \phi(y,x);$$

- for $x', y' \in X$,

$$\omega_{\psi,q}(h(f_1 \otimes y' + f_2 \otimes x'))\phi(y,x) = \phi(y+y', x+x').$$

(3.6) Fourier-Jacobi functor. Suppose that π is a smooth representation of the semi-direct product $SL(W) \ltimes H(W \otimes V) = SL(W) \ltimes H$, where (V,q) is a rank 1 quadratic space. Set

$$FJ_\psi(\pi) = Hom_H(\omega_{\psi,q}, \pi).$$

This is a smooth module for $\tilde{SL}(W)$ via:

$$(gT)(v) = \pi(g)(T(\omega_{\psi,q}(g^{-1})v))$$

with $g \in \tilde{SL}(W)$, $T \in FJ_\psi(\pi)$ and $v \in \omega_{\psi,q}$. We call FJ_ψ the Fourier-Jacobi functor.

§4. The Octonion Algebra

We shall now specialize to the case of interest. Let $\mathbb{O}$ be the split octonion algebra over F. A good way to visualize $\mathbb{O}$ is to use the so-called Zorn's model. In this model, an element of $\mathbb{O}$ is denoted by

$$\begin{pmatrix} a & x \\ x^* & d \end{pmatrix} \quad \text{with } a, d \in F,\ x \in V \text{ and } x^* \in V^*.$$

Here, V is a 3-dimensional vector space; we give it a basis $\{e_1, e_2, e_3\}$ and let $\{e_1^*, e_2^*, e_3^*\}$ denote the associated dual basis of V^*. The multiplication on $\mathbb{O}$ is given by:

$$\begin{pmatrix} a & x \\ x^* & b \end{pmatrix} \cdot \begin{pmatrix} \alpha & y \\ y^* & \beta \end{pmatrix} = \begin{pmatrix} a\alpha + \langle x, y^* \rangle & ay + \beta x - x^* \wedge y^* \\ \alpha x^* + by^* + x \wedge y & b\beta + \langle y, x^* \rangle \end{pmatrix}.$$

Here $e_i \wedge e_j = \pm e_k^*$ and $e_i^* \wedge e_j^* = \pm e_k$ if (ijk) is a permutation of (123) with sign $\pm$.

There is a linear trace form Tr and a quadratic norm form N on $\mathbb{O}$ given by:

$$Tr \begin{pmatrix} a & x \\ y^* & d \end{pmatrix} = a + d \qquad N \begin{pmatrix} a & x \\ y^* & d \end{pmatrix} = ad - \langle y^*, x \rangle.$$

Let V_7 denote the vector space of trace zero elements of $\mathbb{O}$ and equip it with the quadratic form $q_7 = -N$. So $(V_7, q_7) = \langle 1 \rangle \oplus \mathbb{H}^3$. Indeed, (V_7, q_7) comes equipped with a Witt basis:

$$V_7 = \langle e_1, e_2, e_3 \rangle \oplus \langle e_0 \rangle \oplus \langle e_1^*, e_2^*, e_3^* \rangle,$$

where $e_0 = diag(1, -1)$ is such that $q_7(e_0) = 1$ and $b_{q_7}(e_i, e_j^*) = \delta_{ij}$.

(4.1) Simple roots for $SO(V_7)$. Let us begin by fixing a maximal split torus of $SO(V_7)$ and a system of simple roots. The maximal parabolic subgroup of $SO(V_7)$ stabilizing the maximal isotropic subspace V has Levi subgroup $GL(V)$. We identify $GL(V)$ with GL_3 using the basis $\{e_1, e_2, e_3\}$ and the diagonal torus is then a maximal split torus of $SO(V_7)$. Following [RS, Pg. 813], we choose the simple roots of $SO(V_7)$ to be the characters:

$$\begin{cases} \delta_1 : diag(t_1, t_2, t_3) \mapsto 1/(t_1 t_3) \\ \delta_2 : diag(t_1, t_2, t_3) \mapsto t_1/t_2 \\ \delta_3 : diag(t_1, t_2, t_3) \mapsto t_2. \end{cases}$$

The highest root is then

$$\delta_0 : diag(t_1, t_2, t_3) \mapsto t_1/t_3.$$

(4.2) The maximal parabolic $P_7(X^*)$ of $SO(V_7)$. We shall need two maximal parabolic subgroups of $SO(V_7)$. For the first, consider the decomposition

$$V_7 = X \oplus V_5 \oplus X^*$$

where

$$X = \langle e_3 \rangle, \quad X^* = \langle e_3^* \rangle, \quad V_5 = \langle e_1, e_2, e_0, e_2^*, e_1^* \rangle.$$

Then the corresponding parabolic $P_7(X^*) = M_7(X^*) \cdot N_7(X^*)$ is such that

$$N_7(X^*) \cong Hom(V_5, X^*) \cong V_5 \otimes X^*$$

so that $Z_7(X^*)$ is trivial. Recall that we have a section $n_7 : V_5 \otimes X^* \to N_7(X^*)$ which is a group isomorphism in this case.

(4.3) The maximal parabolic $P_5(Y)$ of $SO(V_5)$. It will also be necessary to consider the decomposition of V_5 given by:

$$V_5 = Y \oplus V_1 \oplus Y^*$$

where

$$Y = \langle e_1, e_2 \rangle, \quad V_1 = \langle e_0 \rangle, \quad Y^* = \langle e_1^*, e_2^* \rangle.$$

The associated maximal parabolic subgroup of $SO(V_5)$ is $P_5(Y) = M_5(Y)N_5(Y)$ with

$$\begin{aligned} &M_5(Y) \cong GL(Y) \\ &Z_5(Y) \cong \{S \in Hom(Y^*, Y) : S^* = -S\} \\ &N_5(Y)/Z_5(Y) \cong Hom(V_1, Y) \cong V_1^* \otimes Y \cong V_1 \otimes Y. \end{aligned}$$

Recall that we have a section $n_5 : V_1 \otimes Y \hookrightarrow N_5(Y)$ which is $M_5(Y)$-equivariant. Also, the action of $M_5(Y)$ on $Z_5(Y)$ is via det_Y.

The unipotent radical $N_5(Y)$ is a Heisenberg group. We fix an isomorphism $Z_5(Y) \cong \mathbb{G}_a$ using the basis element $e_1 \wedge e_2 \in Z_5(Y)$. Also, we equip Y with the alternating form given by $\langle e_1, e_2 \rangle = 1$. Then we have an explicit isomorphism

$$N_5(Y) \to H(V_1 \otimes Y) = (V_1 \otimes Y) \oplus F$$

given by:

$$n_5(T) \cdot z_5(S) \mapsto (T, S/e_1 \wedge e_2).$$

Note that $(GL(X^*) \times P_5(Y)) \ltimes N_7(X^*)$ is simply the non-maximal parabolic subgroup of $SO(V_7)$ which stabilizes the flag $X^* \subset X^* \oplus Y$.

(4.4) The maximal parabolic $P_7(X'^*)$ of $SO(V_7)$. The second maximal parabolic subgroup of $SO(V_7)$ that we will need is associated to the decomposition

$$V_7 = X' \oplus V_3 \oplus X'^*$$

where

$$X' = \langle e_3, e_1^* \rangle, \quad V_3 = \langle e_2, e_0, e_2^* \rangle, \quad X'^* = \langle e_1, e_3^* \rangle.$$

The parabolic $P_7(X'^*)$ is such that

$$M_7(X'^*) \cong GL(X'^*) \times SO(V_3)$$
$$Z_7(X'^*) \cong \{S \in Hom(X', X'^*) : S^* = -S\}$$
$$N_7(X'^*)/Z_7(X'^*) \cong Hom(V_3, X'^*) \cong V_3 \otimes X'^*.$$

Note that the action of $M_7(X'^*)$ on $Z_7(X'^*)$ is via *det* of $GL(X'^*)$ and we have a section $n'_7 : V_3 \otimes X'^* \hookrightarrow N_7(X'^*)$ which is $M_7(X'^*)$-equivariant.

The unipotent radical $N_7(X'^*)$ is a Heisenberg group. If we fix an isomorphism $Z_7(X'^*) \cong \mathbb{G}_a$ using the basis element $e_1 \wedge e_3^*$ and equip X'^* with the alternating form $\langle e_1, e_3^* \rangle = 1$, then $N_7(X'^*) \cong H(V_3 \otimes X'^*)$ via the isomorphism

$$n'_7(T) \cdot z'_7(S) \mapsto (T, S/e_1 \wedge e_3^*) \in (V_3 \otimes X'^*) \oplus F.$$

§5. The Group G_2

The automorphism group of the octonion algebra $\mathbb{O}$ is the split exceptional group of type G_2. Since G_2 is connected and fixes the norm form, it is a subgroup of $SO(V_7)$. In this section, we shall describe the embedding $G_2 \hookrightarrow SO(V_7)$ in some detail.

(5.1) The subgroup $SL(V)$ and simple roots. The group $SL(V) \cong SL_3$ is a subgroup of G_2. Its diagonal torus is thus a maximal split torus for G_2. As a system of simple roots, we may choose:

$$\begin{cases} \alpha : diag(t_1, t_2, t_3) \mapsto t_2 \\ \beta : diag(t_1, t_2, t_3) \mapsto t_1/t_2. \end{cases}$$

In particular, α is short and β is long. For any root γ, we write U_γ for the corresponding root subgroup. The root subgroups contained in $SL(V)$ are precisely the ones corresponding to long roots. We shall later describe the embeddings of the short root subgroups.

(5.2) The maximal parabolic P_1. We now describe two maximal parabolic subgroups of G_2. Let $P_1 = L_1U_1$ be the stabilizer of the line $X^* = \langle e_3^* \rangle$, so that $P_1 \hookrightarrow P_7(X^*)$. In fact, the embedding factors through the non-maximal parabolic subgroup $(GL(X^*) \times P_5(Y)) \ltimes N_7(X^*)$. More precisely,

$$L_1 \hookrightarrow GL(X^*) \times M_5(Y) \cong GL(X^*) \times GL(Y)$$

is the subgroup consisting of the elements $(det_Y(g), g)$. In particular, $L_1 \cong GL(Y)$ by projection onto the second factor. Via this isomorphism, we have an identification of L_1 with $GL_2(F)$, which is well-defined up to inner automorphisms.

The unipotent radical U_1 is generated by all positive root subgroups except U_β, and is contained in $N_5(Y) \ltimes N_7(X^*)$. The center of U_1 is

$$Z_1 = U_{3\alpha+2\beta} \times U_{3\alpha+\beta}.$$

As a subgroup of $N_7(X^*) = n_7(V_5 \otimes X^*)$,

$$Z_1 = n_7(Y \otimes X^*).$$

Let $Z = [U_1, U_1]$. It is a 3-dimensional abelian unipotent group equal to $U_{2\alpha+\beta} \times Z_1$. The root subgroup $U_{2\alpha+\beta}$ embeds into $Z_5(Y) \times n_7(V_1 \otimes X^*)$. If we fix the basis $e_1 \wedge e_2$ of $Z_5(Y)$, then $U_{2\alpha+\beta}$ consists of elements of the form

$$(n_5(\lambda e_1 \wedge e_2), n_7(\lambda \cdot e_0 \otimes e_3^*)), \quad \lambda \in F.$$

Similarly,

$$\begin{cases} U_{\alpha+\beta} \hookrightarrow n_5(V_1 \otimes e_1) \times n_7(e_2^* \otimes X^*) \\ U_\alpha \hookrightarrow n_5(V_1 \otimes e_2) \times n_7(e_1^* \otimes X^*) \end{cases}$$

consist of elements of the form:

$$\begin{cases} (n_5(\lambda \cdot e_0 \otimes e_1), n_7(\lambda \cdot e_2^* \otimes e_3^*)) \\ (n_5(\lambda \cdot e_0 \otimes e_2), n_7(-\lambda \cdot e_1^* \otimes e_3^*)) \end{cases}$$

respectively. Observe that the natural projection map

$$P_1 = L_1 \cdot U_1 \hookrightarrow P_7(X^*) \to P_7(X^*)/N_7(X^*) \cong M_7(X^*) \to SO(V_5)$$

induces an isomorphism of P_1/Z_1 with $P_5(Y)$. Thus we obtain an identification of $P_1/Z_1 = L_1 \cdot U_1/Z_1$ with $GL(Y) \ltimes H(V_1 \otimes Y)$ where Y is given the alternating form $\langle e_1, e_2 \rangle = 1$.

(5.3) The Heisenberg parabolic P_2. The other maximal parabolic subgroup $P_2 = L_2U_2$ of G_2 is defined to be the stabilizer of X'^*. It is thus contained in $P_7(X'^*)$. The Levi subgroup L_2 embeds into $M_7(X'^*) \cong GL(X'^*) \times SO(V_3)$ in a diagonal fashion. It is isomorphic to $GL(X'^*)$ via the first projection, giving us an identification of L_2 with $GL_2(F)$ (well-defined up to inner automorphisms). Via the second projection, we obtain an action of L_2 on V_3.

The unipotent radical U_2 is a Heisenberg group, whose center $Z_2 = U_{3\alpha+2\beta}$ is equal to $Z_7(X'^*)$. Let us describe how the other root subgroups in U_2 embeds into $N_7(X'^*)$:

$$\begin{cases} U_{3\alpha+\beta} = n_7'(Fe_2 \otimes e_3^*) \\ U_{2\alpha+\beta} = n_7'(F \cdot (e_0 \otimes e_3^* - e_2 \otimes e_1)) \\ U_{\alpha+\beta} = n_7'(F \cdot (-e_0 \otimes e_1 - e_2^* \otimes e_3^*) \\ U_\beta = n_7'(Fe_2^* \otimes e_1). \end{cases}$$

We shall write X_γ for the above basis element of U_γ, and $x_\gamma : \mathbb{G}_a \to U_\gamma$ for the associated épinglage. Thus the vector group U_2/Z_2 has basis $\{X_\beta, X_{\alpha+\beta}, X_{2\alpha+\beta}, X_{3\alpha+\beta}\}$.

We may restrict the alternating form on $N_7(X'^*)/Z_2$ (cf. 4.4) to the subspace U_2/Z_2. Then the resulting form is given by:

$$\langle X_\beta, X_{3\alpha+\beta} \rangle = 1 \quad \text{and} \quad \langle X_{\alpha+\beta}, X_{2\alpha+\beta} \rangle = -3.$$

(5.4) The representation of L_2 on U_2/Z_2. Recall that we have fixed an identification of L_2 with $GL_2(F)$. We may identify the vector space U_2/Z_2 with the space of binary cubic forms as follows:

$$aX_\beta + bX_{\alpha+\beta} + cX_{2\alpha+\beta} + dX_{3\alpha+\beta} \mapsto aX^3 + 3bX^2Y + 3cXY^2 + dY^3.$$

Then the action of

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in L_2$$

on a binary cubic form $F(X,Y)$ is given by:

$$(g \cdot F)(X,Y) = \det(g)^{-1} \cdot F(AX + CY, BX + DY).$$

Thus the adjoint action of L_2 on U_2/Z_2 is isomorphic to $det^{-1} \otimes Sym^3(std)$. Note that the action of $g \in L_2$ multiplies the alternating form $\langle -, - \rangle$ by $det(g)$.

(5.5) The characters of U_2/Z_2. The group of unitary characters of U_2 can be identified as an L_2-module with the dual space $\mathrm{Hom}(U_2, \mathbb{G}_a)$, via composition with ψ. Using the alternating form on U_2/Z_2, one may identify $Hom(U_2, \mathbb{G}_a)$ with U_2/Z_2 and the action of L_2 on the latter is dual to that described above. In any case, the L_2-orbits on $Hom(U_2, \mathbb{G}_a)$ are naturally parametrized by cubic algebras over F (cf. [GGS]). Thus it makes sense to say that a character of U_2 corresponds to a cubic algebra E. We shall say that a character is degenerate if the corresponding cubic algebra is non-separable.

(5.6) The subgroups SU_3. The subgroup $SL(V) \subset G_2$ is actually the stabilizer in G_2 of the vector e_0. More generally, for any $a \in F^{\times}$, the stabilizer of a vector $v \in V_7$ with $q_7(v) = a$ is isomorphic to the quasi-split SU_3 determined by the class of a in $F^{\times}/F^{\times 2}$.

In particular, we consider the stabilizer SU_3^a of the vector $ae_2 + e_2^*$. Then $SU_3^a \cap P_2$ is a Borel subgroup of SU_3^a. If N_a is the unipotent radical of this Borel subgroup, then the center of N_a is equal to Z_2, so that we have an injection:

$$N_a/Z_2 \hookrightarrow U_2/Z_2.$$

Using the above description of the root subgroups in U_2 and their action on V_7, it is not difficult to check that the 2-dimensional subspace N_a/Z_2 is spanned by the two vectors:

$$X_\beta + aX_{2\alpha+\beta} \quad \text{and} \quad X_{\alpha+\beta} + aX_{3\alpha+\beta}.$$

Now consider the space of those characters of U_2 which restricts to the trivial character on N_a. Regarded as elements of $U_2/Z_2 \cong Hom(U_2, \mathbb{G}_a)$, it is easy to see that these form a 2-dimensional subspace spanned by the two elements:

$$X_\beta - \frac{a}{3}X_{2\alpha+\beta} \quad \text{and} \quad -\frac{1}{3}X_{\alpha+\beta} + aX_{3\alpha+\beta}$$

which correspond to the binary cubic forms $X^3 - aXY^2$ and $-X^2Y + aY^3$ respectively. In particular, we have shown:

(5.7) Lemma *The characters of U_2 which restricts to the trivial character of N_a are either degenerate or correspond to the étale cubic algebra $F \times K_a$, where K_a is the étale quadratic algebra determined by a.*

§6. Representations of $\tilde{SL}(W)$.

Since G_2 is a subgroup of $SO(V_7)$, we may restrict the representation ω_{ψ,q_7} to the subgroup $\tilde{SL}(W)\times G_2$. The main purpose of this paper is to understand this restriction. Before going into that, let us recall the classification of the irreducible genuine representations of $\tilde{SL}(W)\cong\tilde{SL}_2(F)$ and recall some fundamental results of Waldspurger [W1,2] which lead to a definition of local packets for $\tilde{SL}(W)$. Henceforth, we fix the non-trivial additive character ψ of the non-archimedean local field F.

(6.1) The Weil representations. Let χ be a quadratic character of $F^\times$ (possibly trivial). Then χ corresponds to an element $a_\chi\in F^\times/F^{\times 2}$. We may consider the rank 1 quadratic space $q_\chi=\langle a_\chi\rangle$ and the associated Weil representation $\omega_{\psi,q_{a_\chi}}$ of $\tilde{SL}(W)$, which we shall simply denote by ω_χ, suppressing the mention of ψ. As a representation of $\tilde{SL}_2(F)$, ω_χ is reducible; in fact, it is the direct sum of two irreducible representations:

$$\omega_\chi=\omega_\chi^+\oplus\omega_\chi^-,$$

where ω_χ^- is supercuspidal and ω_χ^+ is not.

(6.2) The principal series. Given a character μ of $F^\times$, we regard μ as a character of $\tilde{T}$ by:

$$\mu(t(a),\epsilon)=\mu(a).$$

Recall that we have a character χ_ψ of $\tilde{T}$ defined earlier. Now let B be the Borel subgroup of upper triangular matrices in SL_2. Then the modulus character of $\tilde{B}$ is

$$\delta_{\tilde{B}}:(t(a),\epsilon)\mapsto|a|^2.$$

Now we may form the induced representation

$$\tilde{\pi}(\mu)=Ind_{\tilde{B}}^{\tilde{SL}_2}\chi_\psi\cdot\delta_{\tilde{B}}^{1/2}\cdot\mu.$$

We have:

(6.3) Proposition *(i) $\tilde{\pi}(\mu)$ is irreducible if and only if $\mu^2\neq|-|^{\pm1}$, in which case $\tilde{\pi}(\mu)\cong\tilde{\pi}(\mu^{-1})$.*

(ii) If $\mu=\chi\cdot|-|^{1/2}$ where χ is a quadratic character, then we have a short exact sequence:

$$0\longrightarrow sp_\chi\longrightarrow\tilde{\pi}(\mu)\longrightarrow\omega_\chi^+\longrightarrow 0.$$

We call sp_χ the special representation associated to χ.

(iii) If $\mu=\chi\cdot|-|^{-1/2}$, then we have a short exact sequence,

$$0\longrightarrow\omega_\chi^+\longrightarrow\tilde{\pi}(\mu)\longrightarrow sp_\chi\longrightarrow 0.$$

The proposition gives all the non-supercuspidal genuine representations of $\tilde{SL}_2(F)$. The other irreducible representations of $\tilde{SL}_2(F)$ are all supercuspidal, including the ω_χ^-'s introduced above.

(6.4) Whittaker functionals. For any $a \in F^\times$, let ψ_a be the character of N defined by:

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mapsto \psi(ax).$$

It is known that

$$\dim(\sigma_{N,\psi_a}) \leq 1.$$

We say that σ has a ψ_a-Whittaker functional if $\sigma_{N,\psi_a} \neq 0$. It is also known that any genuine σ has a ψ_a-Whittaker functional for some a. We let

$$\widehat{F}(\sigma) = \{a \in F^\times : \ \sigma \text{ has a } \psi_a\text{-Whittaker functional}\}.$$

Clearly, $\widehat{F}(\sigma)$ is a union of square classes. The following proposition gives the set $\widehat{F}(\sigma)$ for some representations σ.

(6.5) Proposition *(i) If $\sigma = \tilde{\pi}(\mu)$, then $\widehat{F}(\sigma) = F^\times$.*

(ii) If $\sigma = sp_\chi$, then $\widehat{F}(\sigma) = F^\times \smallsetminus a_\chi F^{\times 2}$.

(iii) If $\sigma = \omega_\chi^+$ or ω_χ^-, then $\widehat{F}(\sigma) = a_\chi F^{\times 2}$.

(6.6) Lifting to $SO(V_m)$. In the remaining of this section, we shall recall some results of Waldspurger [W1,2] concerning the lifting of representations of $\tilde{SL}_2(F)$ to various orthogonal groups.

Consider the representation $\omega^{(m)}$ of $\tilde{SL}(W) \times SO(V_m)$. The maximal σ-isotypic quotient of $\omega^{(m)}$ can be expressed as $\sigma \otimes \theta^{(m)}(\sigma)$ for some smooth representation $\theta^{(m)}(\sigma)$ of $SO(V_m)$. The following proposition gives some general results about $\theta^{(m)}(\sigma)$:

(6.7) Proposition *(i) If $\theta^{(m)}(\sigma)$ is non-zero, then it has a unique irreducible quotient $\Theta^{(m)}(\sigma)$.*

(ii) $\theta^{(m)}(\sigma)$ is non-zero if $m \geq 5$.

(iii) If σ is supercuspidal, then $\theta^{(m)}(\sigma)$ is either zero or irreducible. It is supercuspidal if and only if $\theta^{(m')}(\sigma) = 0$ for all $m' < m$.

We now specialize to the cases $m \leq 7$. Note that $SO(V_3)$ is isomorphic to $PGL_2(F)$ and in the following, we shall use the usual convention for representations of $PGL_2(F)$. In particular, St will stand for the Steinberg representation, St_χ the twist of St by the quadratic character χ, and $\pi(\mu_1, \mu_2)$ for a principal series representation.

Here is the first result of Waldspurger that we need:

(6.8) Proposition (Lifting to $SO(V_3)$)
Assume that σ is not equal to ω_χ^+ for any χ.

(i) The representation $\theta^{(3)}(\sigma)$ is non-zero if and only if $1 \in \widehat{F}(\sigma)$.

(ii) The map $\sigma \mapsto \Theta^{(3)}(\sigma)$ defines a bijection between

$$\{\sigma \text{ with } 1 \in \widehat{F}(\sigma)\} \leftrightarrow \{\text{infinite dim. representations of } SO(V_3)\}$$

(iii) The bijection is described more explicitly by the following table. Here we have identified $SO(V_3)$ with $PGL_2(F)$.

σ	$\tilde{\pi}(\mu)$	ω_1^-	sp_χ $(\chi \neq 1)$	*supercuspidal* $(\neq \omega_1^-)$
$\Theta^{(3)}(\sigma)$	$\pi(\mu, \mu^{-1})$	St	St_χ	*supercuspidal*

(6.9) Proposition (Lifting to $SO(V_5)$)
Assume that σ is supercuspidal.

(ii) $\theta^{(5)}(\sigma)$ is supercuspidal if and only if $1 \notin \widehat{F}(\sigma)$.

(iii) If $1 \in \widehat{F}(\sigma)$, then $\theta^{(5)}(\sigma)_{N_5(Y)} = 0$.

(iv) If $1 \notin \widehat{F}(\sigma)$, then $\theta^{(5)}(\sigma)_{Z_5(Y)} = 0$.

(v) Regard $\omega^{(5)}$ as a representation of

$$(\tilde{SL}(W) \times M_5(Y)_{ss}) \ltimes N_5(Y) \cong (\tilde{SL}(W) \times SL(Y)) \ltimes N_5(Y).$$

Then $FJ_\psi(\omega^{(5)})$ is a smooth representation of $\tilde{SL}(W) \times \tilde{SL}(Y)$ which can be described as follows. Let $C_c^\infty(\tilde{SL}_2, \epsilon)$ be the space of Schwarz functions f on $\tilde{SL}_2(F)$ such that

$$f(\epsilon g) = \epsilon \cdot f(g)$$

with ϵ in the kernel of the projection to $SL_2(F)$. Then

$$FJ_\psi(\omega^{(5)}) \cong C_c^\infty(\tilde{SL}_2, \epsilon).$$

In particular, as a representation of $M_5(Y)_{ss} \ltimes N_5(Y)$, we have

$$FJ_\psi(\theta^{(5)}(\sigma)) \cong \sigma^\vee.$$

(6.10) Lifting to quaternions. We come now to another result of Waldspurger. Let (V_3', q_3') be the quadratic space realized on the trace zero elements of the quaternion division algebra over F with $q_3' = -N$. One has the corresponding Weil representation $\tilde{SL}_2(F) \times SO(V_3')$. Let $\theta'_{(3)}(\sigma)$ be defined as before. Then we have:

(6.11) Proposition *Assume that σ is not equal to ω_χ^+.*

(i) $\theta'_{(3)}(\sigma)$ is non-zero if and only if $1 \notin \widehat{F}(\sigma)$. In this case, σ is either supercuspidal or equal to sp_1. Further, $\theta'_{(3)}(\sigma)$ has a unique irreducible quotient $\Theta'_{(3)}(\sigma)$, and is in fact irreducible if σ is supercuspidal.

(ii) The map $\sigma \mapsto \Theta'_{(3)}(\sigma)$ is a bijection

$$\{\textit{irreducible } \sigma \textit{ with } 1 \notin \widehat{F}(\sigma)\} \leftrightarrow \{\textit{irreducible representations of } SO(V_3')\}.$$

(iii) The bijection is described more explicitly by the following table.

σ	ω_χ^- $(\chi \neq 1)$	sp_1	*supercuspidal*
$\Theta'_{(3)}(\sigma)$	$\chi \circ N$	1	$\dim > 1$

(6.12) Waldspurger lift and packets for $\tilde{SL}_2(F)$. Using the above results, Waldspurger defined a map Wd_ψ from the set of irreducible genuine representations of $\tilde{SL}_2(F)$, which are not equal to ω_χ^+ for any χ, to the set of infinite dimensional representations of $PGL_2(F)$. More precisely, for a given σ, exactly one of $\Theta^{(3)}(\sigma)$ or $\Theta'_{(3)}(\sigma)$ is non-zero, and we set

$$Wd_\psi(\sigma) = \begin{cases} \Theta^{(3)}(\sigma), \text{ if } \Theta^{(3)}(\sigma) \neq 0; \\ JL(\Theta'_{(3)}(\sigma)), \text{ if } \Theta^{(3)}(\sigma) = 0, \end{cases}$$

where JL denotes the Jacquet-Langlands lift.

As a result, one obtains a partition of the set of irreducible genuine representations of $\tilde{SL}_2(F)$, not equal to ω_χ^+ for any χ, indexed by the irreducible infinite-dimensional representations τ of $PGL_2(F)$. Each equivalence class contains 2 or 1 elements, depending on whether τ is discrete series or not. We denote by $\tilde{A}_\tau$ the equivalence class associated to τ.

More precisely, $\tilde{A}_\tau$ contains a distinguished element $\sigma(\tau)^+$ characterized by

$$\Theta^{(3)}(\sigma(\tau)^+) = \tau.$$

If τ is discrete series, let $\sigma(\tau)^-$ be the other element of $\tilde{A}_\tau$ so that

$$\Theta'_{(3)}(\sigma(\tau)^-) = JL(\tau).$$

§7. Representations of G_2

In this section, we shall recall some results from the representation theory of G_2.

(7.1) Langlands quotients. Recall that we have two maximal parabolic subgroups P_1 and P_2 of G_2. We have already fixed an isomorphism of their Levi subgroup L_1 and L_2 with $GL_2(F)$ (up to inner automorphisms):

$$L_1 \cong GL(Y) \quad \text{and} \quad L_2 \cong GL(X'^*).$$

Let τ be a tempered representation of $GL_2(F)$ and $s > 0$. Then the induced representations

$$I_{P_i}(\tau, s) = Ind_{P_i}^{G_2} \delta_{P_i}^{1/2} \cdot \tau \cdot |det|^s$$

has a unique irreducible quotient $J_{P_i}(\tau, s)$. The reducibility points of these induced representations are known by [M, Thm. 3.1 and Thm. 5.3].

(7.2) Degenerate principal series. Consider now the induced representation

$$I_{P_1}(\mu) = Ind_{P_1}^{G_2} \delta_{P_1}^{1/2} (\mu \circ det),$$

where μ is a character of $F^\times$. The following was shown in [M, Thm. 3.1 and Props. 4.1, 4.3, 4.4]:

(7.3) Lemma *Assume that $|\mu| = |-|^s$ with $Re(s) \geq 0$. Then $I_{P_1}(\mu)$ is irreducible unless $\mu^2 = |-|$ or $\mu = |-|^{5/2}$. For these exceptional cases, we have the following non-split exact sequences:*

(i) If $\mu = \chi|-|^{1/2}$, with $\chi \neq 1$ a quadratic character, then we have:

$$0 \longrightarrow J_{P_2}(St_\chi, 1/2) \longrightarrow I_{P_1}(\mu) \longrightarrow J_{P_1}(\pi(1,\chi), 1) \longrightarrow 0$$

(ii) If $\mu = |-|^{1/2}$, then we have:

$$0 \longrightarrow J_{P_1}(St, 1/2) \longrightarrow I_{P_1}(\mu) \longrightarrow J_{P_1}(\pi(1,1), 1) \oplus J_{P_2}(St, 1/2) \longrightarrow 0$$

(iii) If $\mu = |-|^{5/2}$,

$$0 \longrightarrow J_{P_2}(St, 3/2) \longrightarrow I_{P_1}(\mu) \longrightarrow 1 \longrightarrow 0$$

(7.4) U_2/Z_2-spectrum.

Recall that the L_2-orbits of characters of U_2 can be naturally parametrized by cubic algebras over F. The U_2/Z_2-spectrum of a smooth representation π of G_2 is the set of those cubic algebras E such that the corresponding twisted Jacquet module π_{U_2,ψ_E} is non-zero. In this paper, we shall only look at the cubic algebras of the form $F \times K$ where K is an étale quadratic algebra. We set

$$\widehat{F}(\pi) = \{a \in F^\times : \ K_a = F(\sqrt{a}) \text{ is in the } U_2/Z_2\text{-spectrum of } \pi\}.$$

Clearly, $\widehat{F}(\pi)$ is a union of square classes. Further, if $\bar{\pi}$ is the conjugate of π, then $\widehat{F}(\bar{\pi}) = \widehat{F}(\pi)$.

Recall that we have the subgroups $Z_1 \subset Z$ of G_2, where Z_1 is the center of U_1 and $Z = [U_1, U_1]$. It is easy to see:

(7.5) Proposition *(i) π is a non-generic supercuspidal representation if and only if $\pi_Z = 0$. In this case, the U_2/Z_2-spectrum of π consists only of étale cubic algebras.*

(ii) If $\pi_{Z_1} = 0$, then $\widehat{F}(\pi)$ is empty.

(7.6) SU_3-invariant forms. Recall that for each $a \in F^\times/F^{\times 2}$, G_2 contains the quasi-split group SU_3^a as a subgroup. We set

$$\widetilde{F}(\pi) = \{a \in F^\times : \ \pi \text{ has a non-zero } SU_3^a\text{-invariant linear functional}\}.$$

Observe that $\widetilde{F}(\bar{\pi}) = \widetilde{F}(\pi)$. Now we have the following crucial observation:

(7.7) Proposition *Let π be non-generic supercuspidal. Then we have the inclusion:*

$$\widetilde{F}(\pi) \subset \widehat{F}(\pi).$$

In particular, if $\pi_{Z_1} = 0$, then $\widetilde{F}(\pi)$ is empty.

Proof. Let N_a be the unipotent radical of the Borel subgroup $SU_3^a \cap P_2$ of SU_3^a. By Lemma 5.7, the characters of U_2 which restricts to the trivial character on N_a either correspond to non-étale cubic algebras or the étale cubic algebra $F \times K_a$. Hence, if $a \notin \widehat{F}(\pi)$, then $\pi_{N_a} = 0$. Thus, when regarded as a representation of SU_3^a, π has no irreducible subquotients whose cuspidal support is along the Borel subgroup. In particular, the trivial representation of SU_3^a is not a quotient of π. The proposition is proved. ∎

(7.8) Fourier-Jacobi functor for G_2. Recall that we have introduced a Fourier-Jacobi functor in (3.6). In (5.2), we have fixed an isomorphism $L_1^{ss} \cdot U_1/Z_1 \cong SL(Y) \cdot H(V_1 \otimes Y^*)$; in particular, an identification of Z/Z_1 with F. Thus we may regard ψ as a character of Z/Z_1. Now if π is a smooth representation of G_2, then $\pi_{Z,\psi}$ is a smooth representation of $L_1^{ss} \cdot U_1/Z_1$. Thus we may apply the Fourier-Jacobi functor to $\pi_{Z,\psi}$ to obtain a smooth representation of $\tilde{L}_1^{ss}$. We denote this $\tilde{L}_1^{ss}$-module simply by $FJ_\psi(\pi)$, and call FJ_ψ the Fourier-Jacobi functor for G_2.

(7.9) Some unipotent representations. Finally, we recall the results of [HMS] concerning the restriction of the (unique) unitarizable minimal representation Π_K of the quasi-split $Spin^K(8)$ to the subgroup G_2, where K is an étale quadratic algebra. The representation Π_K is trivial on the center of $Spin_8^K$ and can be extended to a representation of SO_8^K. Any such extension will be called a minimal representation of SO_8^K and each has the same restriction to G_2.. Now we have:

(7.10) Proposition *(i) When $K = F \times F$,*

$$\Pi_K = J_{P_1}(\pi(1,1),1) \oplus 2 \cdot J_{P_2}(St, 1/2) \oplus \pi_\epsilon$$

where π_ϵ is supercuspidal.

(ii) When K is a quadratic field, with associated quadratic character χ,

$$\Pi_K = J_{P_1}(\pi(1,\chi),1) \oplus \pi(\chi)$$

where $\pi(\chi)$ is supercuspidal.

§8. Jacquet Modules

In this section, we compute the Jacquet modules of $\omega^{(7)}$ with respect to various unipotent subgroups. It is for the purpose of these computations that we gave such precise descriptions of the various models of the Weil representation. The first result gives the Jacquet modules of $\omega^{(7)}$ with respect to the unipotent subgroup $N(F)$ of $\tilde{SL}(W)$.

(8.1) Proposition *(i) As $\tilde{T} \times G_2$-modules, $\omega_N^{(7)}$ sits in the exact sequence:*

$$0 \longrightarrow |-|^{7/2}\chi_\psi \cdot Ind_{\tilde{T}\times P_1}^{\tilde{T}\times G_2} S(F^\times) \longrightarrow \omega_N^{(7)} \longrightarrow |-|^{7/2}\chi_\psi \longrightarrow 0.$$

Here $\tilde{T}$ acts on $S(F^\times)$ by

$$(t(a), \epsilon)\phi(x) = \phi(ax)$$

whereas $L_1 \cong GL_2(F)$ acts by

$$g \cdot \phi(x) = \phi(det(g)^{-1}x).$$

(ii) As a representation of G_2,

$$\omega_{N,\psi_a}^{(7)} \cong ind_{SU_3^a}^{G_2} 1.$$

PROOF. This is easily proved using the Schrodinger model for $\omega^{(7)}$. We omit the details. ∎

(8.2) Corollary *Let σ be an irreducible representation of $\tilde{SL}(W)$ and π of G_2. Suppose that $\sigma\otimes\pi^\vee$ is a non-zero irreducible quotient of $\omega^{(7)}$. If $b \in \widehat{F}(\sigma)$, then $b \in \widehat{F}(\pi)$. In particular, $b \in \widehat{F}(\pi)$.*

Now we come to Jacquet modules for G_2.

(8.3) Proposition *(i) As an $\tilde{SL}(W) \times L_1$-module, $\omega_Z^{(7)}$ sits in the exact sequence:*

$$0 \longrightarrow Q \longrightarrow \omega_Z^{(7)} \longrightarrow R \longrightarrow 0.$$

Here,

$$Q \cong Ind_{\tilde{B}\times L_1}^{\tilde{SL}(W)\times L_1} (S(F^\times) \otimes S(Y^*))$$

where the action of $\tilde{B} \times L_1$ factors through the quotient $\tilde{T} \times L_1$. In particular, Q does not contain supercuspidal representations of $\tilde{SL}(W)$ as subquotients. Further,

$$R \cong \omega_{Z_5(Y)}^{(5)}.$$

(ii) As a representation of $\tilde{SL}(W) \times \tilde{L}_1^{ss}$, $FJ_\psi(\omega^{(7)})$ sits in the exact sequence:

$$0 \longrightarrow Q \longrightarrow FJ_\psi(\omega^{(7)}) \longrightarrow R \longrightarrow 0.$$

with

$$Q \cong ind_{\tilde{N}}^{\tilde{SL}(W)}(sgn \otimes \psi_{\frac{1}{4}}) \otimes \omega_{\psi}^{(1)\vee}$$

and

$$R \cong C_c^{\infty}(\tilde{SL}_2, \epsilon).$$

Here, $\tilde{N} = \langle \pm 1 \rangle \times N$ *and* sgn *is the non-trivial character of* $\langle \pm 1 \rangle$*.*

(8.4) Proposition *(i) As an* $\tilde{SL}(W) \times L_1$*-module,* $\omega_{U_1}^{(7)}$ *sits in the short exact sequence:*

$$0 \longrightarrow |det|^{5/2} \cdot Ind_{\tilde{B} \times L_1}^{\tilde{SL}(W) \times L_1} \delta_{\tilde{B}}^{1/2} \cdot \chi_{\psi} \cdot S(F^{\times}) \longrightarrow \omega_{U_1}^{(7)} \longrightarrow |det| \cdot \omega_{N_5(Y)}^{(5)} \longrightarrow 0.$$

The action of $\tilde{B} \times L_1$ *factors through* $\tilde{T} \times L_1$*. In particular,* $\omega_{U_1}^{(7)}$ *does not contain supercuspidal representations of* $\tilde{SL}(W)$ *as subquotients.*

(ii) As an $\tilde{SL}(W) \times L_2$*-module,* $\omega_{U_2}^{(7)}$ *sits in the short exact sequence:*

$$0 \longrightarrow |det| \cdot ind_{\tilde{B} \times B_{L_2}}^{\tilde{SL}(W) \times L_2} S(F^{\times}) \longrightarrow \omega_{U_2}^{(7)} \longrightarrow |det| \cdot \omega^{(3)} \longrightarrow 0$$

where $B_{L_2} = L_2 \cap P_1$ *is the stabilizer of* $\langle e_3^* \rangle$ *in* L_2*.*

(8.5) Computation of $\omega_{Z_1}^{(7)}$. The rest of this section is devoted to the proofs of Props. 8.3 and 8.4. The first step is the computation of $\omega_{Z_1}^{(7)}$. Using the mixed model relative to the decomposition $V_7 = X \oplus V_5 \oplus X^*$, the representation $\omega^{(7)}$ can be realized on $S(W \otimes X) \otimes \tau$ where τ is any model for the representation $\omega^{(5)}$ of $M_7(X^*)$. Regarding $\omega^{(7)}$ as a representation of $\tilde{SL}(W) \times P_7(X^*)$, we have an exact sequence

$$0 \longrightarrow V' \longrightarrow \omega^{(7)} \longrightarrow V'' \longrightarrow 0$$

where V' consists of τ-valued functions which vanishes at 0, so that V'' is realized on the space of τ.

Clearly, $N_7(X^*)$ acts trivially on V'', and by evaluation at 0, we have an isomorphism $V'' \cong \omega^{(5)}$ of $\tilde{SL}(W) \times M_7(X^*)$-modules. On the other hand, we have:

(8.6) Lemma *Set* $G = \tilde{SL}(W) \times M_7(X^*)$ *and let* H *be the stabilizer in* G *of the vector* $f_1 \otimes e_3 \in W \otimes X$*. Then* $H = (\tilde{B} \times GL(X^*))_0 \times SO(V_5)$ *where*

$$(\tilde{B} \times GL(X^*))_0 = \{((t(a)n(x), \epsilon), a) : a \in F^{\times}, x \in F\}.$$

We have an isomorphism

$$V' \longrightarrow ind_{H \ltimes N_7(X^*)}^{G \ltimes N_7(X^*)} \tau.$$

Here, the action of H *on* τ *is given by the restriction of* $\omega^{(5)}$*, whereas* $n_7(v \otimes e_3^*) \in N_7(X^*)$ *acts by* $\omega^{(5)}(h(f_1 \otimes v))$*.*

PROOF. The isomorphism $\varphi \mapsto \Phi$ is given by:

$$\Phi(n_7(v \otimes e_3^*) \cdot g) = \omega^{(5)}(h(f_1 \otimes v))\omega^{(5)}(g) \left(\varphi(g^{-1}(f_1 \otimes e_3)\right),$$

with $v \in V_5$ and $g \in G$. The lemma is proved. ∎

(8.7) Z_1-coinvariants. Recall that $Z_1 = n_7(Y \otimes X^*)$ so that the normalizer of Z_1 in $M_7(X^*) \cong GL(X^*) \times SO(V_5)$ is the subgroup $GL(X^*) \times P_5(Y)$. Restricting V' to the subgroup

$$G' \ltimes N_7(X^*) = (\tilde{SL}(W) \times GL(X^*) \times P_5(Y)) \ltimes N_7(X^*) \subset G \ltimes N_7(X^*),$$

the lemma implies that $\omega_{Z_1}^{(7)}$ sits in the exact sequence

$$0 \longrightarrow ind_{H' \ltimes N_7(X^*)}^{G' \ltimes N_7(X^*)} \tau_{Z_1} \longrightarrow \omega_{Z_1}^{(7)} \longrightarrow \omega^{(5)} \longrightarrow 0 \tag{2}$$

where $H' = H \cap G'$.

To compute τ_{Z_1}, we use the mixed model relative to the decomposition $V_5 = Y \oplus V_1 \oplus Y^*$, realized on the space $S(W \otimes Y^*) \otimes S(f_2 \otimes V_1)$. Then it is easy to see that the projection $\tau \mapsto \tau_{Z_1}$ is given by restriction of functions to $f_1 \otimes Y^*$. Thus

$$\tau_{Z_1} \cong S(f_1 \otimes Y^*) \otimes S(f_2 \otimes V_1).$$

(8.8) Computation of $\omega_Z^{(7)}$. We are now ready to prove Prop. 8.3(i). To compute $\omega_Z^{(7)}$, we need to know the action of $U_{2\alpha+\beta}$ on τ_{Z_1}. Recall from 5.2 that $U_{2\alpha+\beta}$ embeds in a diagonal fashion into $Z_5(Y) \times n_7(V_1 \otimes X^*)$; so we need to know the action of the latter two groups. It is easy to see that $Z_5(Y)$ acts trivially on τ_{Z_1} whereas $n_7(\lambda e_0 \otimes e_3^*) \in n_7(V_1 \otimes X^*)$ acts by

$$\begin{aligned}
&(n_7(\lambda e_0 \otimes e_3*)\phi)(f_1 \otimes y^*)(\mu \cdot f_2 \otimes e_0) \\
=&h(\lambda \cdot f_1 \otimes e_0)(\phi(f_1 \otimes y^*))(\mu \cdot f_2 \otimes e_0) \\
=&\psi(-2\lambda\mu) \cdot \phi(f_1 \otimes y^*)(\mu \cdot f_2 \otimes e_0)
\end{aligned} \tag{3}$$

From this, we deduce that the projection $\tau_{Z_1} \to \tau_Z$ is given by evaluation of $\phi \in S(f_1 \otimes Y^*) \otimes S(f_2 \otimes V_1)$ at $0 \in f_2 \otimes V_1$. Thus $\tau_Z \cong S(f_1 \otimes Y^*)$ and we have an exact sequence of $\tilde{SL}(W) \times P_1/Z$-modules

$$0 \longrightarrow ind_{(\tilde{B} \times L_1)_0 \ltimes U_1/Z}^{\tilde{SL}(W) \times P_1/Z_1} S(f_1 \otimes Y^*) \longrightarrow \omega_Z^{(7)} \longrightarrow |det| \cdot \omega_{Z_5(Y)}^{(5)} \longrightarrow 0.$$

In the above exact sequence, P_1/Z acts on $\omega_{Z_5(Y)}^{(5)}$ via the isomorphism $P_1/Z \cong M_5(Y) \ltimes N_5(Y)/Z_5(Y)$ (cf. 5.2), whereas the action of $(\tilde{B} \times L_1)_0 \ltimes U_1/Z$ on $S(f_1 \otimes Y^*)$ can be described as follows:

- $N \subset \tilde{B}$ acts trivially.
- an element of $(\tilde{T} \times L_1)_0$ has the form $l(g, \epsilon) = ((t(det(g)), \epsilon), g)$ with $g \in L_1$ and this acts by:

$$(l(g, \epsilon)\phi)(f_1 \otimes y^*) = |det(g)|^{5/2} \cdot \chi_\psi(t(det(g)), \epsilon) \cdot \phi(det(g)^{-1} f_1 \otimes g^{-1} y^*).$$

- Via the natural embedding

$$U_1/Z \hookrightarrow n_5(V_1 \otimes Y) \times n_7(Y^* \otimes X^*)$$

and the second projection, we have an isomorphism of U_1/Z_1 with $n_7(Y^* \otimes X^*)$. If $u \in U_1/Z$ corresponds to $n_7(u^* \otimes e_3^*) \in n_7(Y^* \otimes X^*)$, then

$$(u\phi)(f_1 \otimes y^*) = \phi(f_1 \otimes (y^* + u^*)).$$

At this point, we have proven Prop. 8.3(i).

(8.9) Computation of $\omega_{U_1}^{(7)}$. From the description of the U_1/Z-action above, we see that the projection $S(f_1 \otimes Y^*) \to S(f_1 \otimes Y^*)_{U_1} \cong \mathbb{C}$ is given by integration over Y^*. Thus by taking U_1-coinvariants, we obtain an exact sequence

$$0 \longrightarrow ind_{(\tilde{B}\times L_1)_0}^{\tilde{SL}(W)\times L_1}\mathbb{C} \longrightarrow \omega_{U_1}^{(7)} \longrightarrow |det| \cdot \omega_{N_5(Y)}^{(5)} \longrightarrow 0$$

where $(\tilde{B} \times L_1)_0$ acts on $\mathbb{C}$ by

$$g \mapsto |det(g)|^{7/2} \cdot \chi_\psi(t(det(g)), \epsilon),$$

and L_1 acts on $\omega_{N_5(Y)}^{(5)}$ via the isomorphism $L_1 \cong GL(Y) \cong M_5(Y)$. This gives Prop. 8.4(i).

(8.10) Computation of $\omega_{Z,\psi}^{(7)}$. Recall from 5.2 that we have fixed an isomorphism

$$L_1^{ss} \ltimes U_1/Z_1 \to SL(Y) \ltimes H(Y \otimes V_1).$$

In particular, we have an isomorphism of Z/Z_1 with F. Thus we may regard ψ as a character of Z/Z_1. From equation (2) above, we see that

$$0 \longrightarrow ind_{\tilde{N}\times L_1^{ss}\ltimes U_1/Z_1}^{\tilde{SL}(W)\times L_1^{ss}\ltimes U_1/Z_1}\tau_{Z,\psi} \longrightarrow \omega_{Z,\psi}^{(7)} \longrightarrow \omega_{Z_5(Y),\psi}^{(5)} \longrightarrow 0,$$

and from equation (3), we see that the projection $\tau_{Z_1} \to \tau_{Z,\psi}$ is given by evaluating the function $\phi \in S(f_1 \otimes Y^*) \otimes S(f_2 \otimes V_1)$ at $-\frac{1}{2}f_2 \otimes e_0 \in f_2 \otimes V_1$. Thus $\tau_{Z,\psi}$ can be realized on $S(f_1 \otimes Y^*)$.

The action of $\tilde{N} = N \times \langle\pm 1\rangle$ on $S(f_1 \otimes Y^*)$ is via the character $\psi_{\frac{1}{4}} \otimes sgn$, with sgn the non-trivial character of $\langle\pm 1\rangle$. Thus the first term of the above exact sequence is isomorphic to

$$(ind_{\tilde{N}}^{\tilde{SL}(W)}\psi_{1/4} \otimes sgn) \otimes S(f_1 \otimes Y^*)$$

as a representation of $\tilde{SL}(W) \times (L_1^{ss} \ltimes U_1/Z)$. To prove Prop. 8.3(ii), it remains to identify the action of $L_1^{ss} \ltimes U_1/Z_1 \cong SL(Y) \ltimes H(Y \otimes V_1)$ on $S(f_1 \otimes Y^*)$. In particular, it suffices to show:

(8.11) Lemma *As a representation of $\tilde{L}_1^{ss} \cong \tilde{SL}(Y)$, we have:*

$$FJ_\psi(S(f_1 \otimes Y^*)) \cong \omega_\psi^{(1)\vee}.$$

Proof. First let us write down the action of U_α and $U_{\alpha+\beta}$ on $S(f_1 \otimes Y^*)$. A short computation shows that for $\phi \in S(f_1 \otimes Y^*)$,

$$\begin{cases} (n_5(e_0 \otimes y)\phi)(f_1 \otimes y_0^*) = \psi(\langle y, y_0^* \rangle)\phi(f_1 \otimes y_0^*), \ y \in Y; \\ (n_7(e_3^* \otimes y^*)\phi)(f_1 \otimes y_0^*) = \phi(f_1 \otimes (y_0^* + y^*)), \ y^* \in Y^*. \end{cases}$$

Let us write $y^* = (a, b)$ if $y^* = -ae_1^* + be_2^*$. Since

$$U_\alpha = \{(u_\alpha(\lambda) = n_5(\lambda \cdot e_0 \otimes e_2), n_7(-\lambda \cdot e_3^* \otimes e_1^*) : \lambda \in F\}.$$

we see that

$$(u_\alpha(\lambda)\phi)(a, b) = \psi(\lambda b) \cdot \phi(a + \lambda, b).$$

Similarly, if

$$u_{\alpha+\beta}(\mu) = (n_5(\mu \cdot e_0 \otimes e_1), n_7(\mu \cdot e_3^* \otimes e_2^*)) \in U_{\alpha+\beta},$$

then

$$(u_{\alpha+\beta}(\mu)\phi)(a, b) = \psi(-\mu a) \cdot \phi(a, b + \mu).$$

Moreover, the action of $L_1^{ss} \cong SL(Y)$ on ϕ is geometric, i.e.

$$(l\phi)(f_1 \otimes y^*) = \phi(f_1 \otimes l^{-1}y^*).$$

Under the isomorphism $U_1/Z \cong H(V_1 \otimes Y)$, $u_\alpha(\lambda) \mapsto h(\lambda \cdot e_0 \otimes e_2)$ and $u_{\alpha+\beta}(\mu) \mapsto h(\mu \cdot e_0 \otimes e_1)$. Thus from the above formulas, we have:

Observation: As a representation of $H(V_1 \otimes Y)$, $S(f_1 \otimes Y^*)$ is isomorphic to the tensor product of 2 copies of the unique irreducible representation with central character $\psi_{1/2}$.

In any case, let us define an automorphism of $S(f_1 \otimes Y^*)$ using the partial Fourier transform:

$$\phi \mapsto \widehat{\phi}(a, b) = \int_F \phi(a, b') \cdot \psi(bb')db'$$

where db' is self-dual with respect to this Fourier transform, i.e.

$$\widehat{\widehat{\phi}}(a, b) = \phi(a, -b).$$

Then by transport of structure, we obtain a new action of $SL(Y) \ltimes H(Y \otimes V_1)$ given by:

$$\begin{cases} h(\lambda \cdot e_0 \otimes e_2)\widehat{\phi}(a, b) = \widehat{\phi}(a + \lambda, b + \lambda) \\ h(\mu \cdot e_0 \otimes e_1)\widehat{\phi}(a, b) = \psi(-\mu(a + b)) \cdot \widehat{\phi}(a, b) \\ n(x)\widehat{\phi}(a, b) = \psi(abx) \cdot \widehat{\phi}(a, b) \\ t(r)\widehat{\phi}(a, b) = |r| \cdot \widehat{\psi}(ra, rb) \\ w_0\widehat{\phi}(a, b) = \int \int \widehat{\phi}(\alpha, \beta) \cdot \psi(a\beta + b\alpha)d\alpha d\beta \end{cases}$$

Finally, let us set $f(a,b) = \widehat{\phi}(a+b, a-b)$. Then the action of $SL(Y) \ltimes H(V_1 \otimes Y)$ becomes:

$$\begin{cases} h(\lambda \cdot e_0 \otimes e_2) f(a,b) = f(a+\lambda, b) \\ h(\mu \cdot e_0 \otimes e_1) f(a,b) = \psi(-2a\mu) \cdot f(a,b) \\ n(x) f(a,b) = \psi(a^2 x) \cdot \psi(-b^2 x) \cdot f(a,b) \\ t(r) f(a,b) = |r| \cdot f(ra, rb) \\ w_0 f(a,b) = \int \int f(x,y) \cdot \psi(2ax) \cdot \psi(-2by) dx dy \end{cases}$$

From these formulas, the assertion of the lemma is now evident. This proves Prop. 8.3(ii). ∎

It remains to prove Prop. 8.4(ii). This requires the mixed model relative to the decomposition $V_7 = X' \oplus V_3 \oplus X'^*$ and the explicit description of U_2 given in 5.3. We leave it as an exercise for the reader.

§9. The Main Local Theorem

In this section, we shall prove our main local theorem, which determines the theta lifting from $\widetilde{SL}_2(F)$ to $G_2(F)$ completely for a p-adic field F. In the following section, we shall deal with the case when F is archimedean.

(9.1) Theorem *Let σ be an irreducible genuine representation of $\widetilde{SL}_2(F)$ and write the maximal σ-isotypic quotient of ω_{ψ,q_7} as $\sigma \otimes \theta(\sigma)$. Then $\theta(\sigma)$ is a non-zero admissibile representation of $G_2(F)$. Further, we have:*

(a) (Principal series) If $\sigma = \tilde{\pi}(\mu)$ is an irreducible principal series (so that $\mu^2 \neq |-|^{\pm 1}$) with $\mu \neq |-|^{5/2}$, then

$$\theta(\sigma) \cong I_{P_1}(\mu^{-1}).$$

In particular, $\theta(\sigma)$ is irreducible, unless $\sigma = \tilde{\pi}(|-|^{-5/2}) \cong \tilde{\pi}(|-|^{5/2})$, in which case $\theta(\sigma)$ has the trivial representation of G_2 as its unique irreducible quotient.

(b) (Special representations) If $\sigma = sp_\chi$, then

$$\theta(\sigma) \text{ has unique irreducible quotient} \cong \begin{cases} J_{P_2}(St_\chi, 1/2) & \text{if } \chi \neq 1; \\ J_{P_1}(St, 1/2) & \text{if } \chi = 1. \end{cases}$$

(c) (Weil representations) If $\sigma = \omega_\chi^+$ where χ is a quadratic character of $F^\times$, then

$$\theta(\sigma) = \begin{cases} J_{P_1}(\pi(1,\chi), 1) & \text{if } \chi \neq 1; \\ J_{P_1}(\pi(1,1), 1) \oplus J_{P_2}(St, 1/2) & \text{if } \chi = 1. \end{cases}$$

If $\sigma = \omega_\chi^-$, then

$$\theta(\sigma) = \begin{cases} \pi(\chi) & \text{if } \chi \neq 1; \\ J_{P_2}(St, 1/2) \oplus \pi_\epsilon & \text{if } \chi = 1. \end{cases}$$

Here $\pi(\chi)$ and π_ϵ were defined in Prop. 7.10.

(d) (Supercuspidals) Suppose that σ is supercuspidal and $\sigma \neq \omega_\chi^-$ for any χ. If $\theta^{(3)}(\sigma) \neq 0$, then

$$\theta(\sigma) = J_{P_2}(\theta^{(3)}(\sigma), 1/2) \quad \text{and} \quad FJ_\psi(\theta(\sigma)) \cong \sigma^\vee \oplus \omega_1^\vee.$$

If $\theta^{(3)}(\sigma) = 0$, then $\theta(\sigma)$ is an irreducible non-generic supercuspidal representation such that

$$\begin{cases} FJ_\psi(\theta(\sigma)) \cong \sigma^\vee, \\ \widetilde{F}(\theta(\sigma)) = \widehat{F}(\theta(\sigma)) = \widehat{F}(\sigma). \end{cases}$$

In particular, if $\theta(\sigma) \cong \theta(\sigma')$, then $\sigma \cong \sigma'$.

(9.2) Proof of Theorem. The rest of the section is devoted to the proof of the theorem. Observe that $\theta(\sigma)$ is actually a smooth representation for $SO(V_7)$, though we are only interested in its restriction to G_2. Since we are in the stable range, it is known that $\theta(\sigma)$ is non-zero. Let $\Theta(\sigma)$ be the maximal semisimple G_2-quotient of $\theta(\sigma)$.

(9.3) Principal series and special representations. Assume that $\mu \neq |-|^{5/2}$ (but may be equal to $|-|^{\pm 1/2}$). We have:

Claim:

$$\mathrm{Hom}_{\tilde{SL}(W)}(\omega^{(7)}, I_{\tilde{B}}(\mu)) \cong I_{P_1}(\mu^{-1})^*.$$

Let us assume the claim for a moment. It implies that if $\sigma \hookrightarrow I_{\tilde{B}}(\mu)$, then $\theta(\sigma)$ is a non-zero quotient of $I_{P_1}(\mu^{-1})$. This has the following consequences:

- it immediately implies (a) and (b).
- it gives:

$$\begin{cases} \Theta(\omega_\chi^+) = J_{P_1}(\pi(1,\chi),1) \text{ if } \chi \neq 1; \\ \Theta(\omega_1^+) \subset J_{P_1}(\pi(1,1),1) \oplus J_{P_2}(St,1/2). \end{cases} \tag{4}$$

Now we need to justify the claim. Using Frobenius reciprocity and Prop. 8.1, we deduce that

$$\mathrm{Hom}_{\tilde{SL}(W)}(\omega^{(7)}, I_{\tilde{B}}(\mu)) \cong Hom_T(Ind_{T\times P_1}^{T\times G_2} S(F^\times), |-|^{-5/2}\cdot\mu).$$

If we write the maximal $|-|^{-5/2}\cdot\mu$-isotypic quotient of $Ind_{T\times P_1}^{T\times G_2} S(F^\times)$ as $(|-|^{-5/2}\cdot\mu)\otimes V$ for some smooth representation V of G_2, then

$$V^* \cong Hom_T(Ind_{T\times P_1}^{T\times G_2} S(F^\times), |-|^{-5/2}\cdot\mu).$$

Here V^* is the full linear dual of V, as opposed to the contragredient. Hence, we need to show that

$$V \cong I_{P_1}(\mu^{-1}).$$

For this, we note the following lemma:

(9.4) Lemma *Suppose that G and H are two connected reductive groups over F, and $P = M\cdot N$ is a parabolic subgroup of G. Suppose that W is a smooth representation of $H\times M$, and τ is an irreducible smooth representation of H. Write the maximal τ-isotypic quotient of W as $\tau\otimes V$ for some smooth representation V of M. Then the maximal τ-isotypic quotient of $I_P^G W$ is equal to $\tau\otimes I_P^G V$. Here, I_P^G denotes normalized induction.*

PROOF. The proof is divided into a number of steps:

- (General nonesense). Suppose that one has a $H \times M$-equivariant surjection $p : W \to \tau \otimes V$. Consider the two functors from the category $Rep(M)$ of smooth representations of M to the category of sets:

$$Hom_{H\times M}(W, \tau \otimes -) \quad \text{and} \quad Hom_M(V, -).$$

The map p gives a natural transformation of functors

$$p^* : Hom_M(V, -) \to Hom_{H\times M}(W, \tau \otimes -).$$

To say that $\tau \otimes V$ is the maximal τ-isotypic quotient of W via the projection p is equivalent to saying that p^* is an isomorphism of functors.

- (What we need to show). Now let's go back to the lemma. We are given a $H \times M$-equivariant surjection

$$p : W \to \tau \otimes V$$

which exhibits $\tau \otimes V$ as the maximal τ-isotypic quotient of W. By the exactness of the induction functor, we have a $H \times G$-equivariant surjection

$$I_P^G(p) : I_P^G W \to \tau \otimes I_P^G V.$$

Hence, by the general nonesense above, we need to show that the natural transformation

$$I_P^G(p)^* : Hom_G(I_P^G(V), -) \to Hom_{H\times G}(I_P^G(W), \tau \otimes -)$$

of functors on $Rep(G)$ is an isomorphism.

- (A result of Bernstein). We recall a surprising Frobenius reciprocity type result of Bernstein; this is the only non-trivial ingredient in the proof of the lemma. Let $\bar{P}$ be a parabolic subgroup of G opposite to P, and let

$$R_{\bar{P}} : Rep(G) \to Rep(M)$$

be the normalized Jacquet functor with respect to the unipotent radical of $\bar{P}$. Bernstein showed that $R_{\bar{P}}$ is the right-adjoint functor of I_P^G:

$$Hom_G(I_P^G W, U) \cong Hom_M(W, R_{\bar{P}}(U)), \quad \text{for } W \in Rep(M) \text{ and } U \in Rep(G).$$

Moreover, the above identity is functorial in W and U. More precisely, there is a natural isomorphism of the following two functors on $Rep(M)^{op} \times Rep(G)$:

$$(W, U) \mapsto Hom_G(I_P^G W, U) \quad \text{and} \quad (W, U) \mapsto Hom_M(W, R_{\bar{P}}(U)).$$

A complete proof of this result has been given by Bushnell in [B].

- (The proof of the lemma). The above result implies that there is a commutative diagram:

$$\begin{array}{ccc} Hom_G(I_P^G(V), -) & \longrightarrow & Hom_M(V, R_{\bar{P}}(-)) \\ \downarrow & & \downarrow \\ Hom_{H\times G}(I_P^G(W), \tau\otimes -) & \longrightarrow & Hom_{H\times M}(W, \tau\otimes R_{\bar{P}}(-)) \end{array}$$

where the two horizontal arrows are isomorphisms of functors on $Rep(G)$. Since the right vertical arrow is an isomorphism by assumption, the left vertical arrow is an isomorphism also. The lemma is proved.

∎

(9.5) Supercuspidal representations. We now consider the case when σ is supercuspidal. In this case, Prop. 8.3(ii) shows that:

$$FJ_\psi(\theta(\sigma)) = \begin{cases} \sigma^\vee \text{ if } \theta^{(3)}(\sigma) = 0 \\ (\sigma^\vee \oplus \omega_1^\vee) \text{ otherwise.} \end{cases}$$

This implies that if η is a character of U_2/Z_2 which corresponds to $F \times K_a$, with $a \notin F^{\times 2}$, then

$$\dim \theta(\sigma)_{U_2,\eta} = \dim \sigma_{\psi_a} = \text{ 0 or 1.}$$

On the other hand, if $a \in F^{\times 2}$, then

$$\dim \theta(\sigma)_{U_2,\eta} = \begin{cases} 0 \text{ if } \theta^{(3)}(\sigma) = 0; \\ 3 \text{ if } \theta^{(3)}(\sigma) \neq 0. \end{cases}$$

Assume first that $\theta^{(3)}(\sigma) = 0$, so that $\theta^{(5)}(\sigma)$ is supercuspidal. The above already shows that

$$\widehat{\theta(\sigma)} = \widehat{F}(\sigma).$$

Moreover, Prop. 8.3(i) and Prop. 6.9(iv) implies that $\theta(\sigma)_Z = 0$, so that $\theta(\sigma)$ is non-generic and supercuspidal. Further, the irreducibility of $FJ_\psi(\theta(\sigma))$ implies that $\theta(\sigma)_{Z,\psi}$ is an irreducible $L_1^{ss} \ltimes U_1/Z_1$-module. Hence, $\theta(\sigma)$ has a unique irreducible summand π such that $\pi_{Z_1} \neq 0$. We claim that there is no other summand. Indeed, if τ is another summand, so that $\tau_{Z_1} = 0$, then Prop. 7.5(ii) and Prop. 7.7 lead to a contradiction with Cor. 8.2. This shows that $\theta(\sigma)$ is an irreducible non-generic supercuspidal representation. Finally, Prop. 7.7 and Cor. 8.2 give:

$$\widehat{F}(\sigma) \subset \widetilde{F}(\theta(\sigma)) \subset \widehat{F}(\theta(\sigma)) = \widehat{F}(\sigma),$$

so that equality holds throughout.

Suppose now that $\theta^{(3)}(\sigma) \neq 0$, in which case it is supercuspidal. Prop. 8.4(ii) implies that $\theta(\sigma)$ contains $J_{P_2}(\theta^{(3)}(\sigma), 1/2)$ as a subquotient, and any other irreducible subquotient is supercuspidal. Thus $\theta(\sigma)$ is semisimple, and we shall show that it is irreducible if $\sigma \neq \omega_1^-$. To see this, note that since $\sigma \neq \omega_1^-$, there is a $b \in \widehat{F}(\sigma)$ with $b \notin F^{\times 2}$. By Cor. 8.2 and Prop. 7.7, $b \in \widehat{F}(\tau^\vee)$ for any summand τ of $\theta(\sigma)$. Since any summand τ satisfies $\bar{\tau} \cong \tau^\vee$, we deduce that $b \in \widehat{F}(\tau)$. Hence if $\theta(\sigma)$ has more than one summand, then

$$\dim \theta(\sigma)_{U_2,\eta} > 1,$$

where η corresponds to $F \times K_b$. But we have observed that

$$\dim \theta(\sigma)_{U_2,\eta} \leq 1$$

since $b \notin F^{\times 2}$. With this contradiction, (d) is proved.

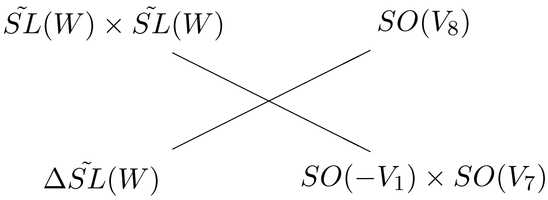
(9.6) Weil representations and see-saw pair. Finally, we need to prove (c). From equation (4), we already have some information concerning (c). To deduce (c) completely, we shall use a see-saw pair argument. Let $V_8 = \mathbb{H}^4$ be the split quadratic space of rank 8. Let $\mathbb{W} = W \otimes V_8$, giving rise to the dual pair $SL(W) \times SO(V_8)$ in the metaplectic group $\tilde{Sp}(\mathbb{W})$. On the other hand, we have the decomposition $\mathbb{W} = \mathbb{W}_1 \oplus \mathbb{W}_2$, where

$$\mathbb{W}_1 = W \otimes (-V_1) \quad \text{and} \quad \mathbb{W}_2 = W \otimes V_7.$$

This gives the dual pair $(\tilde{SL}(W) \times \tilde{SL}(W)) \times (SO(-V_1) \times SO(V_7))$ which contains as a subgroup the commuting pair

$$(\tilde{SL}(W) \times \tilde{SL}(W)) \times G_2.$$

Observe that we have the following see-saw diagram:



Let Π_ψ be the Weil representation of $\tilde{Sp}(\mathbb{W})$ determined by ψ. We shall determine the $\Delta\tilde{SL}(W)$-coinvariants of Π_ψ, as a representation of G_2, in two different ways. On one hand, it is known that $(\Pi_\psi)_{\Delta\tilde{SL}(W)}$ is a unitarizable minimal representation of $SO(V_8)$. As we saw in (7.9), the restriction of this to G_2 is known. On the other hand, when restricted to the subgroup $(\tilde{SL}(W) \times SO(-V_1)) \times (\tilde{SL}(W) \times SO(V_7))$,

$$\Pi_\psi = \omega_1^\vee \otimes \omega^{(7)}.$$

This implies that

$$(\Pi_\psi)_{\Delta\tilde{SL}(W)} \cong \theta(\omega_1^+) \oplus \theta(\omega_1^-).$$

Thus, we deduce that, when $\chi = 1$,

(5) $$\theta(\omega_\chi^+) \oplus \theta(\omega_\chi^-) = \text{the restriction of the minimal rep. of } SO(V_8) \text{ to } G_2.$$

For a general quadratic χ, this continues to hold, using the quasi-split SO_8^χ associated to χ in place of $SO(V_8)$. Since the argument is the same, we omit the details.

As we have seen in the previous subsection,

$$\theta(\omega_\chi^-) \text{ is supercuspidal if } \chi \neq 1;$$

and

$$\theta(\omega_1^-) = J_{P_2}(St, 1/2) \oplus (\text{supercuspidal stuff})$$

Together with equations (4) and (5), these two statements give (c).

The Theorem is proved completely. ∎

§10. Archimedean Correspondences

In this section, we shall consider the analog of Theorem 9.1 when F is an archimedean local field. We fix the additive character ψ of F so that

$$\psi(x) = \begin{cases} exp(2\pi i x), & \text{if } F = \mathbb{R}; \\ exp(2\pi i(x+\overline{x})), & \text{if } F = \mathbb{C}. \end{cases}$$

Then we have the Weil representation $\omega_\psi^{(7)}$ of $\tilde{SL}(W) \times SO(V_7)$ associated to ψ.

If σ is an irreducible genuine representation of $\tilde{SL}(W)$, let $\Theta^{(7)}(\sigma)$ be the unique irreducible representation of $SO(V_7)$ such that $\sigma \otimes \Theta^{(7)}(\sigma)$ is a non-zero quotient of $\omega^{(7)}$. The representation $\Theta^{(7)}(\sigma)$ has been determined by the work of various people, as we shall recall below. What we want to determine in this section is the restriction of $\Theta^{(7)}(\sigma)$ to the subgroup $G_2(F)$. We denote this restriction by $\Theta(\sigma)$. In fact, because of the nature of our global applications, we shall be less ambitious and only determine $\Theta(\sigma)$ when σ is unitary.

(10.1) Unitary dual of $\tilde{SL}_2(\mathbb{R})$. We begin by listing the irreducible unitary representations of $\tilde{SL}_2(\mathbb{R})$. As in (6.2), for any character μ of $\mathbb{R}^\times$, one has a principal series representation $\tilde{\pi}(\mu)$. Note that this parametrization depends on the character χ_ψ defined in (2.6). The unitary representations of $\tilde{SL}_2(\mathbb{R})$ are given by:

- the representation $\tilde{\pi}(\mu)$ for each unitary character μ. These are the tempered principal series.
- the representation $\tilde{\pi}(\mu)$ where $\mu = sgn^\epsilon \cdot |-|^s$, with $\epsilon \in \{0,1\}$ and $0 < s < 1/2$. These are the complementary series.
- for each half integer $k \geq 1/2$, there is a lowest weight module $\tilde{\pi}_k^+$ with K-types $k + 2\mathbb{N}$, and a highest weight module $\tilde{\pi}_k^-$ with K-types $-k - 2\mathbb{N}$. For $k \geq 3/2$, the representations $\tilde{\pi}_k^\pm$ are discrete series representations. Moreover,

$$\begin{cases} \dim \mathrm{Hom}_{N(\mathbb{R})}(\tilde{\pi}_k^+, \mathbb{C}_\psi) = 1 \\ \dim \mathrm{Hom}_{N(\mathbb{R})}(\tilde{\pi}_k^-, \mathbb{C}_\psi) = 0. \end{cases}$$

(10.2) Weil representations. As in (6.1), $\tilde{SL}_2(\mathbb{R})$ has even and odd Weil representations associated to rank one quadratic spaces. These are unitary, and in terms of the above description of the unitary dual, we have:

$$\begin{cases} \tilde{\pi}_{1/2}^+ = \omega_\psi^+ \\ \tilde{\pi}_{1/2}^- = \omega_{\overline{\psi}}^+ \end{cases} \quad \text{and} \quad \begin{cases} \tilde{\pi}_{3/2}^+ = \omega_\psi^- \\ \tilde{\pi}_{3/2}^- = \omega_{\overline{\psi}}^-. \end{cases}$$

(10.3) Waldspurger lift and packets of $\tilde{SL}_2(\mathbb{R})$. As in (6.12), one has a Waldspurger map Wd_ψ from the unitary dual of $\tilde{SL}_2(\mathbb{R})$ minus the even Weil representations to the set of infinite-dimensional unitary representations of $PGL_2(\mathbb{R})$. If τ_k denotes the discrete series representation of $PGL_2(\mathbb{R})$ of extremal weight $\pm k$, then Wd_ψ is given by:

$$\begin{cases} Wd_\psi(\tilde{\pi}(\mu)) = \pi(\mu, \mu^{-1}); \\ Wd_\psi(\tilde{\pi}_k^{\pm}) = \tau_{2k-1}. \end{cases}$$

In particular, every infinite-dimensional unitary representation τ of $PGL_2(\mathbb{R})$ determines a packet

$$\tilde{A}_\tau = Wd_\psi^{-1}(\tau)$$

of unitary representations of $\tilde{SL}(\mathbb{R})$. This contains one element σ_τ^+ unless $\tau = \tau_{2k-1}$ is discrete series, in which case the elements of $\tilde{A}_{\tau_{2k-1}}$ are

$$\sigma_\tau^+ = \tilde{\pi}_k^+ \qquad \text{and} \qquad \sigma_\tau^- = \tilde{\pi}_k^-.$$

(10.4) Degenerate principal series of $G_2(\mathbb{R})$. As in (7.2), each character μ of $\mathbb{R}^\times$ gives rise to a degenerate principal series $I_{P_1}(\mu)$ of $G_2(\mathbb{R})$. The main fact we need about these representations is the following irreducibility result proved by C. Jantzen [J, Pg. 85]:

(10.5) Proposition *Assume that μ is either unitary or of the form $(sgn)^\epsilon \cdot |-|^s$ with $\epsilon \in \{0,1\}$ and $0 < s < 1/2$. Then $I_{P_1}(\mu)$ is irreducible.*

(10.6) Derived functor modules. We also need to introduce some derived functor modules $A_{\mathfrak{q}}(\lambda)$. To be more precise, let us fix a maximal compact subgroup K of $G_2(\mathbb{R})$, with corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Here $\mathfrak{g}$ denotes the complexified Lie algebra of G_2. Fix a maximal torus T of K, and let $\mathfrak{t}_0$ be its (real) Lie algebra. Given any element $x \in \mathfrak{t}_0$, one can associate a parabolic subalgebra

$$\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u} \subset \mathfrak{g}$$

by the requirement that $\mathfrak{q}$ is the sum of non-negative eigenspaces of $ad(x)$. The unique Levi subalgebra $\mathfrak{l}$ containing $\mathfrak{t}$ is then defined over $\mathbb{R}$ and is the complexified Lie algebra of a connected Lie subgroup $L \subset G_2(\mathbb{R})$. We simply call L the Levi factor of $\mathfrak{q}$.

We shall only be interested in those x whose associated $\mathfrak{q}$ is a maximal parabolic whose nilpotent radical is 3-step nilpotent. There are basically 2 different $\mathfrak{q}$'s which arise in this way. Let's call them $\mathfrak{q}^+$ and $\mathfrak{q}^-$, depending on whether the Levi factor is $U(1,1)$ or $U(2)$. For each of these two $\mathfrak{q}$'s, we choose (separately) a set of simple roots $\{\alpha, \beta\}$ of $\mathfrak{g}$ so that $\mathfrak{l}$ contains the root spaces of the long simple roots $\pm\beta$.

Let λ be a (unitary) character of L. Its differential $d\lambda$ is an element of $i\mathfrak{t}_0$ which is of the form $i \cdot k(2\alpha + \beta)$ for some integer k. Denote this character of L by λ_k. In the terminology of [KV], such λ_k's are said to be

$$\begin{cases} \textit{fair}, \text{ if } k \geq -2; \\ \textit{weakly good}, \text{ if } k \geq -1; \\ \textit{good}, \text{ if } k \geq 0. \end{cases}$$

In the following, we shall only look at those λ_k's which are fair. For a fair λ_k, one can define a derived functor module [KV]

$$\pi_k^{\pm} = \mathcal{A}_{\mathfrak{q}^{\pm}}(\lambda_k).$$

The main fact we need about these representations is:

(10.7) Proposition *Assume that $k \geq -1$. Then*

(i) $\pi_k^{\pm}$ is an irreducible unitary representation of $G_2(\mathbb{R})$ with infinitesimal character $\lambda_k + \rho$ and minimal K-type $\lambda_k + 2\rho(\mathfrak{u} \cap \mathfrak{p})$.

(ii) the representation π_k^- is a discrete series representation when $k \geq 0$ and is a limit of discrete series when $k = -1$. Further,

$$\pi_k^+ \cong J_{P_2}(\tau_{2k+6}, 1/2),$$

where τ_{2k+6} is the discrete series representation of $PGL_2(\mathbb{R})$ with extremal weights $\pm(2k+6)$.

Proof. (i) When $k \geq 0$, this is a standard fact in the theory of cohomological induction [KV]. When $k = -1$, the only issue is the non-vanishing of $\pi_k^{\pm}$. This can be easily proven by observing that the weight $\lambda_k + 2\rho(\mathfrak{u} \cap \mathfrak{p})$ is K-dominant and contributes non-trivially to the Blattner-type formula for the K-types of $\pi_k^{\pm}$.

(ii) The first statement follows since the Levi factor L of $\mathfrak{q}^-$ is compact. The second statement was shown by Vogan [V, Thm. 10.9(f)]. ∎

(10.8) Unipotent representations of $G_2(\mathbb{R})$. The 5 special unipotent representations of $G_2(\mathbb{R})$ associated to the subregular unipotent class was described in [V, Thm. 18.5]. As in (7.9), they can be obtained by restriction from the minimal representation Π_K of the two quasi-split $Spin_8^K$'s. More precisely, we have the following result of Vogan [V, Thm. 18.10]:

(10.9) Proposition *(i) When $K = \mathbb{R} \times \mathbb{R}$,*

$$\Pi_K = J_{P_1}(\pi(1,1),1) \oplus 2 \cdot J_{P_2}(St, 1/2) \oplus \pi_\epsilon$$

where π_ϵ is tempered.

(ii) When $K = \mathbb{C}$,

$$\Pi_K = J_{P_1}(\pi(1, sgn), 1) \oplus J_{P_1}(St, 1/2).$$

(iii) We have:

$$\begin{cases} \pi_{-2}^+ = J_{P_2}(St, 1/2) \oplus \pi_\epsilon \\ \pi_{-2}^- = J_{P_1}(St, 1/2). \end{cases}$$

These 5 representations can be easily distinguished from each other because the minimal K-type of each is not a K-type of the others.

We can now state the main local theorem when $F = \mathbb{R}$:

(10.10) Theorem *Let σ be an irreducible unitary representation of $\tilde{SL}_2(\mathbb{R})$, and let $\Theta(\sigma)$ be the restriction to $G_2(\mathbb{R})$ of thete lift of σ to $SO(V_7)$ with respect to $\omega_\psi^{(7)}$. Then $\Theta(\sigma)$ is irreducible except when $\sigma = \omega_\psi^\pm$. More precisely, we have:*

(a) (Principal series) If $\sigma = \tilde{\pi}(\mu)$ is a tempered principal series or a complementary series, then

$$\Theta(\tilde{\pi}(\mu)) = I_{P_1}(\mu).$$

(b) (Weil representations) If σ is a Weil representation, then

$$\begin{cases} \Theta(\omega_\psi^+) = J_{P_1}(\pi(1,1),1) \oplus J_{P_2}(St,1/2); \\ \Theta(\omega_\psi^-) = J_{P_2}(St,1/2) \oplus \pi_\epsilon \end{cases}$$

and

$$\begin{cases} \Theta(\omega_{\overline{\psi}}^+) = J_{P_1}(\pi(1,sgn),1); \\ \Theta(\omega_{\overline{\psi}}^-) = J_{P_1}(St,1/2). \end{cases}$$

(c) (Discrete series) If σ is a discrete series representation $\tilde{\pi}_k^\pm$ (so that $k \geq 3/2$, then

$$\Theta(\tilde{\pi}_k^\pm) = \pi_{k-7/2}^\pm.$$

Proof. The theta correspondence from $\tilde{SL}_2(\mathbb{R})$ to $SO(V_7)$ was completely determined by [LTZ] and [RS2]. From their results, one sees that if $\sigma = \tilde{\pi}(\mu)$, then the representation $\Theta^{(7)}(\sigma)$ can be described as follows. Recall that $SO(V_7)$ has a maximal parabolic subgroup P_7 stabilizing an isotropic line, with Levi factor $\mathbb{R}^\times \times SO(V_5)$. Then we have:

$$\Theta^{(7)}(\tilde{\pi}(\mu)) = Ind_{P_7}^{SO(V_7)} \delta_{P_7}^{1/2} \cdot \mu,$$

where μ is regarded as a character of P_7 in the obvious way. Since G_2 acts transitively on the set of isotropic lines in V_7, we deduce that

$$\Theta(\tilde{\pi}(\mu)) = I_{P_1}(\mu),$$

which is irreducible by Prop. 10.5. This proves (a).

Now we consider the lift of $\tilde{\pi}_k^\pm$ for $k \geq 1/2$. When $k \geq 7/2$, the result was shown by Li-Schwermer [LS, Props. 3.4 and 3.8]. Let us explain how to extend it to the case $k < 7/2$. From [LS, Lemma 3.2] and the proof of [LS, Prop. 3.4], we already know the infinitesimal character and K-type structure of $\Theta(\tilde{\pi}_k^\pm)$, which is admissible and has Gelfand-Kirillov dimension 5. When $k \geq 7/2$, one appeals to a well-known theorem of Vogan-Zuckerman [VZ] to conclude that $\Theta(\tilde{\pi}_k^\pm) \cong \pi_{k-7/2}^\pm$. Unfortunately, when we are not in the good range, there is no general theorem that identifies $\Theta(\tilde{\pi}_k^\pm)$ as an $A_{\mathfrak{q}}(\lambda)$.

To overcome this for $k = 5/2$, we go through Vogan's classification of the unitary dual of $G_2(\mathbb{R})$ [V] and list down all the non-generic unitary representations with this infinitesimal character. It turns out that there are only 2 such, namely the representations $\pi_{5/2}^\pm$. An inspection shows that

$\Theta(\tilde{\pi}_{5/2}^{\pm})$ contains the minimal K-type of $\pi_{5/2}^{\pm}$, but not the minmal K-type of the other. This proves (c) for $k \geq 5/2$.

Finally, when $k = \pm 1/2$ or $\pm 3/2$, the representation $\Theta(\tilde{\pi}_k^{\pm})$ has the same infinitesimal character as the 5 special unipotent representations. Hence, $\Theta(\tilde{\pi}_k^{\pm})$ is a linear combination of these unipotent representations. Since the unipotent representations can be characterized by their minimal K-types, and we know the K-type structure of $\Theta(\tilde{\pi}_k^{\pm})$, it is easy to prove that (b) holds. Observe that the two descriptions of $\Theta(\tilde{\pi}_{3/2}^{\pm})$ in (b) and (c) are consistent due to Prop. 10.9(iii). The theorem is proved completely. ∎

(10.11) Remark (i) For the proof of (b), one may also use the see-saw argument as in the proof of Thm. 9.1(c). This will give:

$$\Theta(\omega_{\psi}^{+}) \oplus \Theta(\omega_{\psi}^{-}) = \text{restriction of minimal representation of split } Spin_8(\mathbb{R}) \text{ to } G_2(\mathbb{R})$$

and

$$\Theta(\omega_{\overline{\psi}}^{+}) \oplus \Theta(\omega_{\overline{\psi}}^{-}) = \text{restriction of minimal representation of quasi-split } Spin_8(\mathbb{R}) \text{ to } G_2(\mathbb{R}).$$

This is certainly consistent with Theorem 10.10, but falls slightly short of proving it. To complete the proof, one would still need to resort to the examination of K-types.

(ii) In [LS, §3.6, Remarks (1)], it was claimed that $\Theta(\tilde{\pi}_{1/2}^{+})$ is irreducible. As we saw above, this is not the case.

(10.12) Complex case. Finally, we come to the case $F = \mathbb{C}$. In this case, the covering $\tilde{SL}(W)$ is split, so that $\omega_{\psi}^{(7)}$ is a representation of $SL_2(\mathbb{C}) \times G_2(\mathbb{C})$. The irreducible infinite-dimensional unitary representations of $SL_2(\mathbb{C})$ are given by:

- the tempered principal series $\tilde{\pi}(\mu)$ where μ is a unitary character;
- the complementary series $\tilde{\pi}(\mu)$ with $\mu = |-|^s$ and $0 < s < 1$.

Note that the absolute value on $\mathbb{C}$ is defined by $|z| = z \cdot \overline{z}$.

To identify the even and odd Weil representations among these, let us define a unitary character μ_0 of $\mathbb{C}^{\times}$ by:

$$\mu_0(z) = |z|^{1/2} \cdot \overline{z}^{-1}$$

so that $\mu_0(re^{i\theta}) = e^{i\theta}$. Then we have:

$$\begin{cases} \omega_{\psi}^{+} = \tilde{\pi}(|-|^{1/2}); \\ \omega_{\psi}^{-} = \tilde{\pi}(\mu_0). \end{cases}$$

As in the real case, there is a Waldspurger map Wd_{ψ}, which is simply defined by:

$$Wd_{\psi}(\tilde{\pi}(\mu)) = \pi(\mu, \mu^{-1}),$$

where $\mu \neq |-|^{1/2}$. Thus, for an infinite-dimensional representation τ of $PGL_2(\mathbb{C})$, the packet $\tilde{A}_\tau = Wd_\psi^{-1}(\tau)$ is a singleton set.

Finally, $G_2(\mathbb{C})$ has 3 special unipotent representations which were studied by Barbasch-Vogan [BV]. These can be obtained from the minimal representation Π of $Spin_8(\mathbb{C})$ as in the other cases:

$$\Pi = J_{P_1}(\pi(1,1),1) \oplus 2 \cdot J_{P_2}(St, 1/2) \oplus I_{P_2}(\tau)$$

where τ is the standard representation of $GL_2(\mathbb{C})$. These 3 representations can be distinguished from each other by their minimal K-types. Indeed, the minimal K-types of the 3 representations are the trivial character $\mathbf{1}$, the 7-dimensional standard representation V_7 and the adjoint representation Ad respectively. The K-types $\mathbf{1}$ and V_7 occur in exactly one of the unipotent representations, but Ad occur in both $J_{P_1}(\pi(1,1),1)$ and $I_{P_2}(\tau)$.

Here is the main theorem for $F = \mathbb{C}$:

(10.13) Theorem *Let $\sigma = \tilde{\pi}(\mu)$ be an irreducible infinite-dimensional unitary representation of $SL_2(\mathbb{C})$, and let $\Theta(\sigma)$ be the restriction to $G_2(\mathbb{C})$ of the theta lift of σ to $SO(V_7)$. Then we have:*

$$\Theta(\tilde{\pi}(\mu)) = I_{P_1}(\mu).$$

Moreover, $\Theta(\sigma)$ is irreducible unless σ is the even or odd Weil representation, in which case it is the sum of unipotent representations:

$$\begin{cases} \Theta(\omega_\psi^+) = J_{P_1}(\pi(1,1),1) \oplus J_{P_2}(St, 1/2) \\ \Theta(\omega_\psi^-) = J_{P_2}(St, 1/2) \oplus J_{P_2}(\tau). \end{cases}$$

Proof. The theta correspondence for complex groups has been completely determined by Adams-Barbasch [AB]. From their results, one sees that the theta lift of $\tilde{\pi}(\mu)$ to $SO(V_7)$ is given by the analogous degenerate principal series as in the proof of Theorem 10.10(a). Hence, it follows that $\Theta(\tilde{\pi}(\mu))$ is equal to $I_{P_1}(\mu)$. It remains to check that $I_{P_1}(\mu)$ is irreducible except when $\mu = |-|^{1/2}$ or μ_0.

In [CBD, Cor. 2.10], Conze-Berline and Duflo gave a sufficient condition for a degenerate principal series of a complex group to be irreducible. This condition is satisfied by the representation $I_{P_1}(\mu)$ under our consideration, provided that μ is not equal to $|-|^{1/2}$ or μ_0. For the two exceptional cases, $I_{P_1}(\mu)$ has the same infinitesimal character as the 3 unipotent representations and thus must be a linear combination of these. To determine the precise combination, one shows that $I_{P_1}(|-|^{1/2})$ contains the K-types $\mathbf{1}$, V_7 and Ad with multiplicity one, whereas $I_{P_1}(\mu_0)$ contains V_7 and Ad with multiplicity one. We leave this easy verification to the reader. ∎

(10.14) Remark The reducibility of $I_{P_1}(|-|^{1/2})$ was also shown by Duflo in [D, Prop. 6 (4), Pg. 84]. However, [D, Thm. 3 (3), Pg. 91] seemed to suggest that $I_{P_1}(\mu_0)$ is irreducible. As we saw above, this is not the case.

§11. Definition of local A-packets

In this section, we shall explain how our local results (Theorems 9.1, 10.10 and 10.13) allow us to give a definition for the A-packets associated to a short root parameter of G_2.

(11.1) Local A-parameters. Let F be a local field and let

$$L_F = \begin{cases} \text{the Weil group } W_F, \text{ if } F \text{ is archimedean;} \\ \text{the Weil-Deligne group } W_F \times SL_2(\mathbb{C}), \text{ if } F \text{ is } p\text{-adic.} \end{cases}$$

If τ is an irreducible unitary representation of $PGL_2(F)$, then τ corresponds to a map

$$\phi_\tau : L_F \longrightarrow SL_2(\mathbb{C}).$$

Now consider the (local) A-parameter

$$\psi_\tau : L_F \times SL_2(\mathbb{C}) \xrightarrow{\phi_\tau \times id} SL_{2,l}(\mathbb{C}) \times SL_{2,s}(\mathbb{C}) \xrightarrow{\quad i \quad} G_2(\mathbb{C}).$$

If S_{ψ_τ} is the component group of the centralizer of ψ_τ, then

$$S_{\psi_\tau} = \begin{cases} 1, & \text{if } \phi_\tau \text{ is reducible} \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } \phi_\tau \text{ is irreducible.} \end{cases}$$

Note the the condition $\phi_{\tau,v}$ is irreducible is equivalent to τ_v being a discrete series representation of $PGL_2(F)$.

(11.2) Local A-packets. Now Arthur's conjecture (cf. [A1,2]) predicts that there is a finite set A_τ of unitary representations of $G_2(F)$ associated to ψ_τ. The representations should be indexed by the irreducible characters of S_{ψ_τ}. Hence, in our case, A_τ should have the form:

$$A_\tau = \begin{cases} \{\pi_\tau^+\}, & \text{if } \tau \text{ is not discrete series} \\ \{\pi_\tau^+, \pi_\tau^-\}, & \text{if } \tau \text{ is discrete series.} \end{cases}$$

Here, π_τ^+ is indexed by the trivial character of S_τ.

The set A_τ is called a local A-packet, and should satisfy

- if τ is unramified with Satake parameter $t_\tau \in SL_2(\mathbb{C})$, then π_τ^+ is irreducible and unramified with Satake parameter

$$s_\tau = i\left(t_\tau \times \begin{pmatrix} q^{1/2} & \\ & q^{-1/2} \end{pmatrix}\right) \in G_2(\mathbb{C})$$

  where q is the order of the residue field of F.

- the distribution $\pi_\tau^+ - \pi_\tau^-$ is stable.
- certain identities involving transfer of character distributions to endoscopic groups of $G(F)$ should hold.

(11.3) Definition of local A-packets. Now we can use Theorems 9.1, 10.10 and 10.13 to give a natural candidate for the packet A_τ. Recall that τ determines a set $\tilde{A}_\tau$ of representations of $\widetilde{SL}_2(F)$. This has 2 or 1 elements $\sigma_\tau^\pm$, depending on whether τ is discrete series or not. We set

$$\pi_\tau^\pm = \Theta(\sigma_\tau^\pm).$$

This defines the packet A_τ.

Why is this a reasonable definition? For one thing, when τ is unramified, then $\Theta(\sigma_\tau^+)$ is indeed irreducible and unramified with the required Satake parameter s_τ. Secondly, when F is archimedean, this definition agrees with that given by Adams-Johnson in [AJ]; in particular, our local packets are stable in the archimedean case. Yet another justification is our global theorem, which is the subject matter of the following sections.

(11.4) A special case. We would like to highlight the case when τ is the Steinberg representation St. When F is p-adic, the parameter ψ_τ is quite special. It is given by

$$\psi_\tau : W_F \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \longrightarrow SL_{2,l}(\mathbb{C}) \times SL_{2,s}(\mathbb{C}) \longrightarrow G_2(\mathbb{C})$$

which is trivial on the Weil group W_F. According to our definition,

$$\begin{cases} \pi_\tau^+ = \Theta(\omega_{\bar{\psi}}^-) = J_{P_2}(St, 1/2) + \pi_\epsilon \\ \pi_\tau^- = \Theta(sp_1) = J_{P_1}(St, 1/2). \end{cases}$$

In particular, π_τ^+ is reducible. For the case of split p-adic groups, this is the first instance we know in which the representation in a packet can be reducible, and is quite surprising at first sight. The initial guess would be to take $\pi_{\tau_v}^+$ simply as $J_{P_2}(St, 1/2)$. We set

$$\pi_\tau^{++} = \begin{cases} J_{P_2}(St, 1/2), \text{ if } \tau = St; \\ \pi_\tau^+ \text{ otherwise.} \end{cases}$$

Thus, π_τ^{++} is simply a distinguished constituent of π_τ^+.

We note:

(11.5) Proposition *Assume that the packet of unipotent representations in Prop. 7.10(i) is indeed an Arthur packet, so that $J_{P_1}(\pi(1,1),1) + 2J_{P_2}(St, 1/2) + \pi_\epsilon$ is stable (which holds when $F = \mathbb{R}$). Then $(J_{P_2}(St, 1/2) + \pi_\epsilon) - J_{P_1}(St, 1/2)$ is stable.*

PROOF. By Lemma 7.3(ii), we see that in the p-adic case,

$$I_{P_1}(|-|^{1/2}) = J_{P_1}(St, 1/2) \oplus J_{P_1}(\pi(1,1),1) \oplus J_{P_2}(St, 1/2)$$

after semisimplification. The same equation also holds when $F = \mathbb{R}$,; this can be easily checked because $I_{P_1}(|-|^{1/2})$ is a sum of unipotent representations. Now the character distribution of $I_{P_1}(|-|^{1/2})$ is stable because any admissible representation of GL_2 is stable and parabolic induction preserves stability. Thus under the assumption of the proposition, the result follows by taking the difference of the two stable distributions in sight. ∎

The proposition provides some justification for our definition of $\pi_{\tau_v}^+$. A more powerful justification is given by our global result Thm. 13.1; see also (14.1).

§12. Automorphic Forms and Global Lifting

Henceforth, we let F be a number field with adele ring $\mathbb{A}$ and fix a non-trivial additive character ψ of $F\backslash\mathbb{A}$. If v is an archimedean place, we assume that ψ_v is as given in the last section. In this section, we provide some preparation for the statement of our main global result.

(12.1) Cusp forms of $\widetilde{SL}_2(\mathbb{A})$. Let $\mathcal{A}^2(\tilde{SL}_2)$ denote the space of square-integrable genuine automorphic forms on $\widetilde{SL}_2(\mathbb{A})$. Then there is an orthogonal decomposition

$$\mathcal{A}^2(\tilde{SL}_2) = \mathcal{A}_{00} \bigoplus \left(\bigoplus_{\chi} \mathcal{A}_{\chi} \right).$$

Here, χ runs over all quadratic characters of $F^{\times}\backslash\mathbb{A}^{\times}$.

Let us describe the space $\mathcal{A}_{\chi}$ more concretely. If $\omega_{\chi} = \otimes_v \omega_{\chi_v}$ is the global Weil representation of $\tilde{SL}_2(\mathbb{A})$ attached to χ, then the formation of theta series gives a map

$$\theta_{\chi} : \omega_{\chi} \to \mathcal{A}^2(\tilde{SL}_2),$$

whose image is the space $\mathcal{A}_{\chi}$. To describe the decomposition of $\mathcal{A}_{\chi}$, for a finite set S of places of F, let us set

$$\omega_{\chi,S} = (\otimes_{v\in S}\omega^{-}_{\chi_v}) \otimes (\otimes_{v\notin S}\omega^{+}_{\chi_v})$$

so that

$$\omega_{\chi} = \bigoplus_{S} \omega_{\chi,S}.$$

Then we have

$$\mathcal{A}_{\chi} \cong \bigoplus_{\#S \text{ even}} \omega_{\chi,S}.$$

Moreover, $\omega_{\chi,S}$ is cuspidal if and only if S is non-empty.

(12.2) Near equivalence classes. In the beautiful paper [W2], Waldspurger has described the near equivalence clases of representations in $\mathcal{A}_{00}$. Earlier, in [W1], he has shown that $\mathcal{A}_{00}$ satisfies multiplicity one. Let us describe his results.

Given a cuspidal automorphic representation $\tau = \otimes_v \tau_v$ of $PGL_2(\mathbb{A})$, we define a set of irreducible unitary representations of $\tilde{SL}_2(\mathbb{A})$ as follows. Recall that for each place v, we have a local packet

$$\tilde{A}_{\tau_v} = \{\sigma^{+}_{\tau_v}, \sigma^{-}_{\tau_v}\}$$

where $\sigma^{-}_{\tau_v} = 0$ if τ_v is not discrete series. Now set

$$\tilde{A}_{\tau} = \{\sigma = \otimes_v \sigma^{\epsilon_v}_{\tau_v} : \epsilon_v = \pm \text{ and } \epsilon_v = + \text{ for almost all } v\}.$$

This is called the global packet associated to τ.

For $\sigma = \otimes_v \sigma_{\tau_v}^{\epsilon_v} \in \tilde{A}_\tau$, let us set

$$\epsilon_\sigma = \prod_v \epsilon_v.$$

Then we have:

$$\mathcal{A}_{00} = \bigoplus_{\text{cuspidal } \tau} \mathcal{A}(\tau)$$

where each $\mathcal{A}(\tau)$ is a near equivalence class of cuspidal representations and

$$\mathcal{A}(\tau) = \bigoplus_{\sigma \in \tilde{A}_\tau : \epsilon_\sigma = \epsilon(\tau, 1/2)} \sigma.$$

(12.3) Fourier coefficients on $\tilde{SL}_2$. For a character χ of $N(F)\backslash N(\mathbb{A})$, the χ-Fourier coefficient of an automorphic form f of $\widetilde{SL}(\mathbb{A})$ is the function defined by

$$f_\chi(h) = \int_{N(F)\backslash N(\mathbb{A})} \overline{\chi(n)} \cdot f(nh)\, dn.$$

Since we have fixed the additive character ψ, any non-trivial character χ of $N(\mathbb{A})$ trivial on $N(F)$ is of the form $\chi(n(x)) = \psi(ax)$ for some $a \in F^\times$; we denote this character by ψ_a. In particular, using ψ as the base point, the $\tilde{T}(F)$-orbits of non-trivial characters χ are parametrized naturally by $F^\times/F^{\times 2}$, or equivalently by étale quadratic algebras.

Say that σ has missing χ-coefficient if $f_\chi = 0$ for all $f \in \sigma$. Otherwise, say that σ has χ-coefficient. Also, set

$$\widehat{F}(\sigma) = \{a \in F^\times : \ \sigma \text{ has } \psi_a\text{-coefficient}\}.$$

Clearly, $\widehat{F}(\sigma)$ is a union of square classes.

If $\sigma \subset \mathcal{A}_{00}$, then it is known that $\widehat{F}(\sigma)$ contains at least 2 square classes, whereas $\widehat{F}(\omega_{\chi,S})$ contains only the square class determined by the quadratic character χ.

(12.4) Fourier coefficient on G_2. On G_2, we shall consider Fourier expansion along U_2 (the unipotent radical of the Heisenberg parabolic P_2). As explained in (5.3)-(5.5), the characters of $U_2(A)$ trivial on $U_2(F)$ can be identified with the vector space $U_2(F)/Z_2(F)$ (using the fixed character ψ and the alternating form $\langle -, - \rangle$ on U_2/Z_2), and the $L_2(F)$-orbits of Fourier coefficients are naturally parametrized by cubic F-algebras.

Suppose $a \in F^\times$ corresponds to an étale quadratic algebra K. For the purpose of later computation, it is necessary to specify exactly a character Ψ_a of $U_2(\mathbb{A})$ which lies in the orbit corresponding to $F \times K$. We shall take Ψ_a to be the character associated to the element

$$\frac{1}{3} X_{\alpha+\beta} - a X_{3\alpha+\beta} \in U_2(F)/Z_2(F).$$

In particular, using the épinglage of U_2 defined in (5.3), we have:

$$\Psi_a(x_\beta(u_1)x_{\alpha+\beta}(u_2)x_{2\alpha+\beta}(u_3)x_{3\alpha+\beta}(u_4)x_{3\alpha+2\beta}(u_5)) = \psi(-au_1 + u_3).$$

If f is an automorphic form on G_2, its Ψ_a-Fourier coefficient f_{Ψ_a} is defined by:

$$f_{\Psi_a}(h) = \int_{U_2(F)\backslash U_2(\mathbb{A})} \overline{\Psi_a(u)} \cdot f(uh)\, du.$$

If $\pi \subset \mathcal{A}(G_2)$ is an automorphic subrepresentation (not necessarily irreducible), we set

$$\widehat{F}(\pi) = \{a \in F^\times : \text{ some } f \in \pi \text{ has non-zero } \Psi_a\text{-coefficient}\}.$$

(12.5) Global theta lift. Let $\omega_\psi^{(7)} = \otimes_v \omega_{\psi_v}^{(7)}$ be the global Weil representation of $\widetilde{Sp}_{14}(\mathbb{A})$ associated to ψ. By the formation of theta series, we have a map

$$\theta : \omega_\psi^{(7)} \longrightarrow \mathcal{A}(\tilde{S}p_{14}).$$

Now if $\sigma \subset \mathcal{A}^2(\tilde{S}L_2)$ is a cuspidal representation of $\tilde{S}L_2(\mathbb{A})$, then we let $V(\sigma)$ denote the linear span of all functions on $G_2(\mathbb{A})$ of the form

$$\theta(\varphi, f)(g) = \int_{\widetilde{SL}_2(F)\backslash\widetilde{SL}_2(\mathbb{A})} \theta(\varphi)(gh) \cdot \overline{f(h)} dh, \qquad \text{for } \varphi \in \omega_\psi^{(7)} \text{ and } f \in \sigma.$$

The complex conjugate over $f(h)$ is introduced to ensure the compatibility of global and local theta lifts.

From the results of [RS], one has:

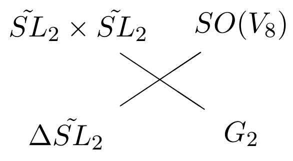
(12.6) Theorem *(i) The space $V(\sigma)$ is non-zero and is contained in the space of square-integrable automorphic forms on G_2, so that $V(\sigma)$ is semisimple and is a non-zero summand of $\Theta(\sigma) = \otimes_v \Theta(\sigma_v)$.*

(ii) $V(\sigma)$ is contained in the space of cusp forms if and only if σ has missing ψ-coefficient.

(iii) Let $\pi \subset \mathcal{A}_{cusp}(G_2)$ be a cuspidal representation of G_2. Then π is not orthogonal to some $V(\sigma)$ iff π has non-zero SU_3^K period for some étale quadratic algebra K. Moreover, if π is not orthogonal to $V(\sigma)$ and the square class determined by K is contained in $\widehat{F}(\sigma)$, then π is distinguished with respect to SU_3^K.

(12.7) Regularized theta lift. It is desirable to extend the definition of the theta lift to all summands of $\mathcal{A}^2(\tilde{S}L_2)$, i.e. for the non-cuspidal representations $\omega_{\chi,S}$ (S empty). Let us explain how this is done.

For simplicity, let us take the case when $\chi = 1$ is trivial, so that $\omega_1 = \omega_\psi^{(1)}$. With $V_8 := V_7 \oplus (-V_1) \cong \mathbb{H}^4$, we have the following seesaw diagram:



As a representation of $G_2(\mathbb{A}) \times \Delta \tilde{SL}_2(\mathbb{A})$, $\omega_\psi^{(7)} \otimes \overline{\omega_\psi^{(1)}}$ is (a dense subspace of) the restriction to $SO(V_8)(\mathbb{A}) \times \Delta \tilde{SL}_2(\mathbb{A})$ of the Weil representation W_ψ of $\widetilde{Sp}_{16}(\mathbb{A})$. Let $\Theta : W_\psi \to \mathcal{A}(\widetilde{Sp}_{16})$ be the usual theta map. In particular, for $\varphi \in \omega_\psi^{(7)}$ and $f \in \omega_\psi^{(1)}$, the function

$$(g, h) \mapsto \theta(\varphi)(gh) \cdot \overline{f(h)}$$

is the restriction to $G_2(\mathbb{A}) \times \Delta \tilde{SL}_2(\mathbb{A})$ of the element $\Theta(\varphi \otimes f)$. It follows that the absolute convergence of $\theta(\varphi, f)$ is equivalent to the absolute convergence of

$$\int_{\Delta \tilde{SL}_2(F) \backslash \Delta \tilde{SL}_2(\mathbb{A})} \Theta(\varphi \otimes f)(gh) dh$$

for all $g \in G_2(\mathbb{A})$.

Now the convergence of this latter integral is a well-studied problem in the theory of regularized Siegel-Weil formula. In our case, if we realize W_ψ in the Schrodinger model $\mathcal{S}(V_8(\mathbb{A}))$, it is easy to see that the integral of $\Theta(\Phi)$ (for $\Phi \in W_\psi$) over $\Delta \tilde{SL}_2$ converges absolutely iff $W_\psi(h)\Phi(0) = 0$ for all $h \in \Delta \tilde{SL}_2(\mathbb{A})$. So if $\varphi \otimes f$ has this property, then the integral $\theta(\varphi, f)$ converges. This is the case, for example, if f lies in $\omega_{\chi,S}$ with S non-empty.

In general, the map sending Φ to the function $h \mapsto W_\psi(h)\Phi(0)$ gives a $(SO(V_8) \times \Delta \tilde{SL}_2)$-equivariant map

$$T : W_\psi \longrightarrow Ind_B^{SL_2} \delta_B^2 \qquad \text{(unnormalized induction).}$$

Now fix an archimedean place v_0. It is clear that one can find an element Z in the center of the universal enveloping algebra of $\Delta \tilde{SL}_2(F_{v_0})$ such that

$$Z = \begin{cases} 1 \text{ on the trivial representation;} \\ 0 \text{ on the above principal series.} \end{cases}$$

Then $W_\psi(Z)\Phi$ lies in $ker(T)$ and this allows us to define the regularized theta lift:

$$\theta^{reg}(\varphi, f)(g) := \int_{\Delta \tilde{SL}_2(F) \backslash \Delta \tilde{SL}_2(\mathbb{A})} \Theta(W_\psi(Z)(\varphi \otimes f))(gh) dh.$$

Note that there is a unique equivariant extension of the theta integral from $Ker(T)$ to W_ψ. Hence the regularized theta lift defined here is in fact canonical. Moreover, when $f \in \omega_{\chi,S}$ with S non-empty, we have:

$$\theta^{reg}(\varphi, f) = \theta(\varphi, f).$$

We shall set:

$$V_\chi = \text{regularized theta lift of } \mathcal{A}_\chi.$$

It is a subspace of $\mathcal{A}(G_2)$.

§13. The Main Global Theorem

We are now ready to state the main global theorem:

(13.1) Theorem *(i) For each quadratic character χ, the space V_χ is equal to the space of automorphic forms obtained by restricting the automorphic minimal representation of the quasi-split $Spin_8^\chi$. The latter space consists of square-integrable automorphic forms and was determined as an abstract representation in [GGJ].*

(ii) If $\sigma \subset \mathcal{A}_{00}$, then

$$V(\sigma) \cong \Theta(\sigma).$$

Moreover,

$$\widehat{F}(V(\sigma)) = \widehat{F}(\sigma).$$

(13.2) Beginning of proof. The rest of the section is devoted to the proof of this theorem. In fact, a consideration of the see-saw diagram and [GRS, Theorem 6.9] (which says that the regularized theta lift of the trivial representation of $\Delta \tilde{SL}_2$ is a minimal representation of SO_8) gives part (i) of the theorem. Moreover, if $\sigma \subset \mathcal{A}_{00}$ is such that $\Theta(\sigma)$ is irreducible, then (ii) holds without any work. Hence the content of (ii) is the assertion for those σ for which $\Theta(\sigma)$ is reducible. Our earlier local results show that this holds precisely when σ_v is the odd Weil representation $\omega_{\psi_v}^-$ for some place v.

(13.3) The space $\mathcal{W}(\sigma, \Psi_a)$. Suppose $a \in F^\times$ corresponds to the quadratic algebra K. Recall that we have defined a character Ψ_a of $U_2(\mathbb{A})$ in (12.4). For $\theta(\varphi, f) \in V(\sigma)$, we have its Ψ_a-coefficient $\theta(\varphi, f)_{\Psi_a}$ which is a function on $G_2(\mathbb{A})$. Let $\mathcal{W}(\sigma, \Psi_a)$ denote the span of all the functions $\theta(\varphi, f)_{\Psi_a}$ with varying φ and f. Then we have a $G_2(\mathbb{A})$-equivariant surjective map

$$V(\sigma) \longrightarrow \mathcal{W}(\sigma, \Psi_a),$$

so that $\mathcal{W}(\sigma, \Psi_a)$ is a semisimple representation of $G_2(\mathbb{A})$ and is a summand of $\Theta(\sigma)$.

For $\sigma \subset \mathcal{A}_{00}$, if we can show that

$$\mathcal{W}(\sigma, \Psi_a) \cong \Theta(\sigma),$$

then the theorem will be proven. Hence, we shall analyze the space $\mathcal{W}(\sigma, \Psi_a)$.

(13.4) The space $\mathcal{W}(\sigma_v, \Psi_{a,v})$. We first introduce a local analog of $\mathcal{W}(\sigma, \Psi_a)$. Fix a place v of F, and realize the local Weil representation $\omega_v^{(7)}$ using the mixed model associated to the decomposition

$$V_7 = X' \oplus V_3 \oplus X'^*,$$

where we recall from (cf. 4.4) that:

$$\begin{cases} X' = \langle e_3, e_1^* \rangle \\ V_3 = \langle e_2, e_0, e_2^* \rangle. \end{cases}$$

To simplify the notation we shall write (x_1, x_2, x_3, x_4) for the element

$$x_1 f_1 \otimes e_3 + x_2 f_2 \otimes e_3 + x_3 f_1 \otimes e_1^* + x_4 f_2 \otimes e_1^* \in W \otimes X'.$$

Similarly we write (x_5, x_6, x_7) for the element

$$x_5 f_2 \otimes e_2 + x_6 f_2 \otimes e_0 + x_7 f_2 \otimes e_2^* \in f_2 \otimes V_3.$$

In this way, we identify $U = (W \otimes X') \oplus (f_2 \otimes V_3)$ with $\mathbb{G}_a^7$.

In this mixed model, the space of $\omega_v^{(7)}$ is the space of Schwarz functions on $U(F_v)$. If $P_7(X'^*)$ is the parabolic subgroup of $SO(V_7)$ stabilizing X'^*, then the action of $\tilde{SL}(W) \times P_7(X'^*)$ in this model are given by the formulas of (3.4). Recall also that the Heisenberg parabolic P_2 of G_2 is contained in $P_7(X'^*)$.

Now suppose σ_v is an irreducible genuine representation of $\tilde{SL}(F_v)$, and $a \in F_v^\times$ is such that $a \in \widehat{F}(\sigma)$. Because of the uniqueness of local Whittaker functionals, one has an embedding

$$L_v : \sigma_v \hookrightarrow Ind_{\tilde{N}}^{\tilde{SL}_2}(sgn \otimes \psi_a)$$

which is unique up to scaling. The image of L_v is the Whittaker model of σ_v.

For $\varphi_v \in S(U(F_v))$ and $f_v \in \sigma_v$, let us define a function on $G_2(F_v)$ by:

$$\begin{aligned} \mathcal{W}_{\Psi_{a,v}}(\varphi_v, f_v, g_v) &= \int_{N(F_v)\backslash SL_2(F_v)} (\omega_v^{(7)}(g_v h_v)\varphi_v)(f_1 \otimes e_1^*)(ae_2 + e_2^*) \cdot \overline{L_v(f_v)(h_v)}\, dh_v. \\ &= \int_{N(F_v)\backslash SL_2(F_v)} (\omega_v^{(7)}(g_v h_v)\varphi_v)(0,0,1,0)(a,0,1) \cdot \overline{L_v(f_v)(h_v)}\, dh_v. \end{aligned}$$

Here dh_v is the invariant measure on $N(F_v)\backslash SL_2(F_v)$ which gives the image of a maximal compact subgroup of $SL_2(F_v)$ volume 1. It is not difficult to see that this integral converges absolutely, using the growth properties of Whittaker functions. Moreover, for $u_v \in U_2(F_v)$, one has:

$$\mathcal{W}_{\Psi_{a,v}}(\varphi_v, f_v, u_v g_v) = \Psi_a(u_v) \cdot \mathcal{W}_{\Psi_{a,v}}(\varphi_v, f_v, g_v),$$

which explains the inclusion of $\Psi_{a,v}$ in the notations.

Let

$$\mathcal{W}(\sigma_v, \Psi_{a,v}) = \{\mathcal{W}_{\Psi_{a,v}}(\varphi_v, f_v, -) : \ \varphi_v \in S(U(F_v)) \text{ and } f_v \in \sigma_v\}.$$

Then $\mathcal{W}(\sigma_v, \Psi_{a,v})$ is a smooth representation of $G_2(F_v)$. Moreover, we have:

(13.5) Lemma *(i) Suppose that $a \in \widehat{F}(\sigma_v)$, then the space $\mathcal{W}(\sigma_v, \Psi_{a,v})$ is non-zero.*

(ii) Let F_v be a p-adic field with $p \neq 2$. Suppose that the character ψ_v of F_v has conductor equal to the ring of integers A_v of F_v and that $a \in A_v^\times$. Suppose also that σ_v is unramified with spherical vector f_v^0, and L_v is normalized so that $L_v(f_v^0)(1) = 1$. Finally, suppose that φ_v^0 is the characteristic function of A_v^7. Then

$$\mathcal{W}_{\Psi_{a,v}}(\varphi_v^0, f_v^0, 1) = 1.$$

In the following, we shall assume that L_v is normalized as in the lemma. The proof of the lemma will be deferred to (13.11). Now we have the following key formula for $\theta(\varphi, f)_{\Psi_a}$:

(13.6) Proposition *Let σ be any irreducible cuspidal representation of $\tilde{SL}_2(\mathbb{A})$. The space $\mathcal{W}(\sigma, \Psi_a)$ is non-zero iff σ has non-zero ψ_a-Fourier coefficient. In this case, if $a \notin F^{\times 2}$, $f = \otimes_v f_v$ and $\varphi = \otimes_v \varphi_v$, then one has an expression:*

$$\theta(\varphi, f)_{\Psi_a}(g) = \prod_v \mathcal{W}_{\Psi_{a,v}}(\varphi_v, f_v, g_v).$$

As a representation of $G_2(\mathbb{A})$,

$$\mathcal{W}(\sigma, \Psi_a) \cong \otimes_v \mathcal{W}(\sigma_v, \Psi_{a,v}) \qquad \textit{(restricted tensor product)}$$

and $\mathcal{W}(\sigma_v, \Psi_{a,v})$ is a non-zero summand of $\Theta(\sigma_v)$.

(13.7) Remark (i) The product on the right hand side is a finite one, because of Lemma 13.5(ii).

(ii) It is crucial that the space $\mathcal{W}(\sigma_v, \Psi_{a,v})$ depends only on the local representation σ_v (and the character $\Psi_{a,v}$) and not on the cuspidal representation of which σ_v is a local component.

(13.8) The proof of Theorem 13.1. We shall postpone the proof of Prop. 13.6 to (13.12), and use it to complete the proof of Thm. 13.1. Hence suppose that $\sigma \subset \mathcal{A}_{00}$. Choose $a \notin F^{\times 2}$ so that σ has a ψ_a-Fourier coefficient. Then to prove the theorem for σ, it suffices to show that:

$$\mathcal{W}(\sigma_v, \Psi_{a,v}) \cong \Theta(\sigma_v) \qquad \text{for all } v.$$

Again this is clear if $\Theta(\sigma_v)$ is irreducible.

From our local results, we are reduced to showing this for the representation $\omega^-_{\psi_v}$. Note that in this case, $a \in F_v^{\times 2}$. More precisely, we need to show that

$$\mathcal{W}(\omega^-_{\psi_v}, \Psi_{a,v}) \cong \Theta(\omega^-_{\psi_v}) = J_{P_2}(St_v, 1/2) \oplus \pi_{\epsilon_v}.$$

One can give a purely local proof of this statement. However, we shall present a local-global argument which we find rather amusing.

(13.9) A local-global argument. Suppose we want to prove the local statement above for a place v_0. Choose a quadratic field K split at v_0, and let χ_K be the quadratic character corresponding to K. Let v_1 be another place where K is split and set $S_0 = \{v_0, v_1\}$.

Consider the two representations π_1 and π_2 of $G_2(\mathbb{A})$ defined as follows:

$$(\pi_1)_v = \begin{cases} \pi_{\epsilon_v} \text{ if } v \in S_0; \\ J_{P_1}(\pi(1, \chi_v), 1) \text{ if } v \notin S_0 \end{cases} \qquad \text{and} \qquad (\pi_2)_v = \begin{cases} \pi_{\epsilon_v} \text{ if } v = v_1; \\ J_{P_2}(St_v, 1/2) \text{ if } v = v_0; \\ J_{P_1}(\pi(1, \chi_v), 1) \text{ if } v \notin S_0. \end{cases}$$

By the results of [GGJ], we know that π_1 and π_2 occur with multiplicity one in the restriction of the minimal representation of $Spin_8^K$. Hence, by Theorem 13.1(i), π_1 and π_2 occur with multiplicity one in V_{χ_K}. So they must occur in $V(\omega_{\chi_K,S})$ for some S of even cardinality. By our local results, one sees that the only possibility for S is S_0. Hence we have:

$$\pi_1 \oplus \pi_2 \subset V(\omega_{\chi_K,S_0}).$$

Suppose that K corresponds to $a \in F^\times$. Since ω_{χ_K,S_0} has non-zero ψ_a-coefficient, we have a surjective map from $V(\omega_{\chi_K,S_0})$ onto the non-zero space $\mathcal{W}(\omega_{\chi_K,S_0}, \Psi_a)$. In fact, by results of [GGJ], this map must be non-zero when restricted to any irreducible summand of $V(\omega_{\chi_K,S_0})$, so that it is injective. Hence, we have:

$$V(\omega_{\chi_K,S_0}) \cong \mathcal{W}(\omega_{\chi_K,S_0}, \Psi_a)$$

so that

$$\pi_1 \oplus \pi_2 \subset \mathcal{W}(\omega_{\chi_K,S_0}, \Psi_a).$$

Thus we conclude that

$$(\pi_1)_{v_0} \oplus (\pi_2)_{v_0} \subset \mathcal{W}(\omega^-_{\psi_{v_0}}, \Psi_{a,v_0}),$$

which is what we desire to prove. Theorem 13.1 is proven. ∎

(13.10) Remark In a similar way, one can show that

$$\mathcal{W}(\omega^+_{\psi_v}, \Psi_{1,v}) \cong \Theta(\omega^+_{\psi_v}).$$

(13.11) Proof of Lemma 13.5. We first give the proof of (ii). Indeed, using Iwasawa decomposition, we obtain

$$\mathcal{W}_{\Psi_{a,v}}(\varphi_v^0, f_v^0, 1) = \int_{T(F_v)} \omega^{(7)}(t)\varphi_v^0(0,0,1,0)(a,0,1) \cdot \overline{L_v(f_v^0)(t)} \cdot \delta_B^{-1}(t)\, dt.$$

Here dt is the measure giving $T(A_v)$ volume 1. Let $t = t(r)$ with $r \in F^\times$. Observe that

$$\omega^{(7)}(t(r))\varphi_v^0(0,0,1,0)(a,0,1) = |r|^{3/2} \cdot \gamma_\psi(r) \cdot \varphi_v^0(0,0,r^{-1},0)(ra,0,r).$$

This vanishes unless $|r| = 1$, in which case it is equal to 1. Thus the integral can be taken over $T(A_v)$ and (ii) follows easily.

To prove (i), we need to find some φ_v and f_v such that $\mathcal{W}_{\Psi_{a,v}}(\varphi_v, f_v, 1) \neq 0$. To do this, we shall use the fact that the representation $\omega^{(7)}$ is actually a representation of $\tilde{Sp}_{14}(F_v) = \tilde{Sp}(W \otimes V_7)$. In fact, we will only need to know the action of a certain unipotent subgroup, which we now describe. Let Q be the maximal parabolic subgroup of $Sp(W \otimes V_7)$ stabilizing the 4-dimensional isotropic space $W \otimes X'^*$. The unipotent radical of Q is two-step nilpotent and its center is isomorphic to the group $S^2(W \otimes X')$ of symmetric bilinear forms on $W \otimes X'$. This contains the subgroup $Z = S^2(W \otimes e_1^*)$, consisting of those symmetric bilinear forms which contain $W \otimes e_3$ in their kernels. For $S \in S^2(W \otimes e_1^*)$, we write $z(S)$ for the corresponding element in Z. Then the action of $z(S)$ on $\omega^{(7)}$ is given by:

$$\omega^{(7)}(z(S))\varphi(w_1 \otimes e_1^* + w_2 \otimes e_3) = \psi(S(w_1, w_1)) \cdot \varphi(w_1 \otimes e_1^* + w_2 \otimes e_3)$$

with $w_1, w_2 \in W$. Moreover, under adjoint action, $SL(W)$ normalizes Z; indeed the action of $SL(W)$ on Z is the natural one on $S^2(W)$ (the symmetric square action).

Now let us return to the proof of (i). Choose φ_v and f_v such that

$$\varphi_v(0,0,1,0)(a,0,1) \cdot \overline{L_v(f_v)(1)} \neq 0.$$

Let Φ be a Schwarz function on $S^2(W \otimes e_1^*)$ and let us replace φ_v by

$$\Phi * \varphi_v = \int_{S^2(W \otimes e_1^*)} \Phi(S) \cdot \omega^{(7)}(z(S))\varphi_v \, dS.$$

Then a short computation gives:

$$\mathcal{W}_{\Psi_{a,v}}(\Phi * \varphi_v, f_v, 1) = \int_{N(F_v)\backslash SL_2(F_v)} \omega^{(7)}(h)\varphi_v(0,0,1,0)(a,0,1) \cdot \overline{L_v(f_v)(h)} \cdot \widehat{\Phi}(h) \, dh$$

where

$$\widehat{\Phi}(h) = \int_{S^2(W)} \Phi(S) \cdot \psi(S(h^{-1}(f_1), h^{-1}(f_1))) \, dS$$

Note that the map

$$l_0 : S \mapsto S(f_1, f_1)$$

is an element of the dual space $S^2(W)^*$, and

$$\widehat{\Phi}(h) = \int_{S^2(W)} \Phi(S) \cdot \psi(\langle h \cdot l_0, S \rangle) \, dS$$

is simply the restriction of the Fourier transform of Φ to the $SL(W)$-orbit $\mathcal{O}$ of l_0 in $S^2(W)^*$. Since $\mathcal{O}$ is a locally closed subset of $S^2(W)^*$, we see that

$$S(\mathcal{O}) \subset \{\text{Restriction of } \widehat{\Phi} \text{ to } \mathcal{O}, \text{ for all Schwarz functions } \Phi\}.$$

It is then clear that the integral for $\mathcal{W}_{\Psi_{a,v}}(\Phi * \varphi_v, f_v, 1)$ is non-zero for some choice of Φ. This proves (i) and hence the lemma. ∎

(13.12) Proof of Prop. 13.6. We realize the global Weil representation $\omega^{(7)}$ using the mixed model as in (13.4). Hence $\omega^{(7)}$ is realized on the space of Schwarz functions on $U_{\mathbb{A}} = (W_{\mathbb{A}} \otimes X'_{\mathbb{A}}) \oplus (f_2 \otimes V_{3,\mathbb{A}})$, and we shall use coordinates on this adelic vector space as in (13.4).

The automorphic realization of the minimal representation $\omega^{(7)}$:

$$\theta : \mathcal{S}(U_{\mathbb{A}}) \longrightarrow \mathcal{A}(\tilde{Sp}(W \otimes V_7))$$

is given by:

$$\theta(\phi)(g) = \sum_{x \in (W\otimes X)(F)} \sum_{u \in f_2 \otimes V_3(F)} \omega^{(7)}(g)\phi(x)(u).$$

Hence, for $\varphi \in S(U_{\mathbb{A}})$ and $f \in \sigma$, $\theta(\varphi, f)_{\Psi_a}(g)$ is equal to

$$\text{(6)} \qquad \int_{U_2(F)\backslash U_2(\mathbb{A})} \int_{SL_2(F)\backslash SL_2(\mathbb{A})} \left(\sum_{x_i \in F} \omega^{(7)}(uhg)\varphi(x_1,x_2,x_3,x_4)(x_5,x_6,x_7) \right) \cdot \overline{f(h)} \cdot \overline{\Psi_a(u)}\, dh\, du.$$

This is in turn equal to the sum of the following two terms:

$$\text{(7)} \qquad \int_{U_2(F)\backslash U_2(\mathbb{A})} \int_{N(F)\backslash SL_2(\mathbb{A})} \left(\sum_{x_3,x_4} \sum_{x_5,x_6,x_7} \omega^{(7)}(uhg)\varphi(1,0,x_3,x_4)(x_5,x_6,x_7) \right) \cdot \overline{f(h)} \cdot \overline{\Psi_a(u)}\, dh\, du$$

and

$$\text{(8)} \qquad \int_{U_2(F)\backslash U_2(\mathbb{A})} \int_{SL_2(F)\backslash SL_2(\mathbb{A})} \sum_{x_3,x_4} \sum_{x_5,x_6,x_7} \omega^{(7)}(uhg)\varphi(0,0,x_3,x_4)(x_5,x_6,x_7) \cdot \overline{f(h)} \cdot \overline{\Psi_a(u)}\, dh\, du.$$

We examine each of these two terms in turn.

First we have:

(13.13) Lemma *(i) If $a \notin F^{\times 2}$, then (7) vanishes.*

(ii) If $a = 1$, then (7) equals to the sum of the following two terms:

$$\text{(9)} \qquad \int_{N(\mathbb{A})\backslash SL_2(\mathbb{A})} \omega^{(7)}(gh)\varphi(1,0,1,0)(1,-\frac{1}{2},0) \cdot \overline{f_{\psi_{\frac{1}{4}}}(h)}\, dh$$

and

$$\text{(10)} \qquad \int_{N(\mathbb{A})\backslash SL_2(\mathbb{A})} \omega^{(7)}(gh)\varphi(1,0,-1,0)(-1,-\frac{1}{2},0) \cdot \overline{f_{\psi_{\frac{1}{4}}}(h)}\, dh$$

Proof. We shall perform the integral over $U_2(F)\backslash U_2(\mathbb{A})$ first. For $u = x_{3\alpha+\beta}(r_1)x_{3\alpha+2\beta}(r_2)$, one has

$$\omega^{(7)}(u)\varphi(1,0,x_3,x_4)(x_5,x_6,x_7) = \psi(r_2x_4 - r_1x_7)\varphi(1,0,x_3,x_4)(x_5,x_6,x_7).$$

Since Ψ_a is trivial when restricted to the root subgroups $U_{3\alpha+\beta}$ and $U_{3\alpha+2\beta}$, when we integrate out these two root subgroups, we are left with a sum over those $x \in U(F)$ with $x_4 = x_7 = 0$.

For $u = x_{2\alpha+\beta}(r)$, one has

$$\omega^{(7)}(u)\varphi(1,0,x_3,0)(x_5,x_6,0) = \psi(-2rx_6)\cdot\varphi(1,0,x_3,0)(x_5,x_6,0).$$

Since

$$\Psi_a(x_{2\alpha+\beta}(r)) = \psi(r),$$

we see that only those $x \in U(F)$ with $x_6 = -1/2$ contributes to the integral.

Now for $u = x_\beta(r)$, we have:

$$\omega^{(7)}(x_\beta(r))\varphi(1,0,x_3,0)(x_5,-1/2,0) = \psi(-rx_3x_5)\cdot\varphi(1,0,x_3,0)(x_5,-1/2,0).$$

Since

$$\Psi_a(x_\beta(r)) = \psi(-ar),$$

we see that only those points $x \in U(F)$ with $x_3x_5 = a$ contribute to the integral.

Finally, when $u = x_{\alpha+\beta}(r)$, we have:

$$\omega^{(7)}(x_{\alpha+\beta}(r))\varphi(1,0,x_3,0)(ax_3^{-1},-1/2,0) = \psi(r(ax_3^{-1}-x_3))\cdot\varphi(1,0,x_3,0)(ax_3^{-1},-1/2,0).$$

Since Ψ_a is trivial on $U_{\alpha+\beta}$, one sees that the integral over $U_{\alpha+\beta}(F)\backslash U_{\alpha+\beta}(\mathbb{A})$ vanishes unless $a = x_3^2 \in F^{\times 2}$. This proves (i).

On the other hand, if $a = 1$, then only those x with $x_3 = \pm 1$ can contribute. In this case, for $n = n(r) \in N(\mathbb{A})$, we have:

$$\omega^{(7)}(n(r))\varphi(1,0,\pm1,0)(\pm1,-1/2,0) = \psi(r/4)\cdot\varphi(1,0,\pm1,0)(\pm1,-1/2,0).$$

From this, the desired expression follows. This proves the lemma. ∎

Let us proceed with (8). We see that it is equal to the sum of the following two terms:

$$(11)\qquad \int\limits_{U_2(F)\backslash U_2(\mathbb{A})}\int\limits_{N(F)\backslash SL_2(\mathbb{A})}\sum_{x_i\in F}\omega^{(7)}(uhg)\varphi(0,0,1,0)(x_5,x_6,x_7)\cdot\overline{f(h)}\cdot\overline{\Psi_a(u)}\,dh\,du$$

and

$$(12)\qquad \int\limits_{U_2(F)\backslash U_2(\mathbb{A})}\int\limits_{SL_2(F)\backslash SL_2(\mathbb{A})}\sum_{x_i\in F}\omega^{(7)}(uhg)\varphi(0,0,0,0)(x_5,x_6,x_7)\cdot\overline{f(h)}\cdot\overline{\Psi_a(u)}\,dh\,du.$$

The summand (12) vanishes because

$$\omega^{(7)}(u)\varphi(0,0,0,0)(x_5,x_6,x_7) = \varphi(0,0,0,0)(x_5,x_6,x_7)\quad\text{ for all } u\in U_2(\mathbb{A})$$

whereas Ψ_a is a non-trivial character. Let us proceed with (11). For

$$u = x_\beta(r_1)x_{\alpha+\beta}(r_2)x_{2\alpha+\beta}(r_3)x_{3\alpha+\beta}(r_4)x_{3\alpha+2\beta}(r_5)$$

one finds that:

$$\omega^{(7)}(u)\varphi(0,0,1,0)(x_5,x_6,x_7) = \psi(-r_1x_5 - 2r_2x_6 + r_3x_7)\cdot\varphi(0,0,1,0)(x_5,x_6,x_7).$$

Hence, in considering the integral over $U_2(F)\backslash U_2(\mathbb{A})$, only the point $x = (0,0,1,0;a,0,1)$ contributes.

Finally, for $u = n(r) \in N(\mathbb{A})$, we have:

$$\omega^{(7)}(n(r))\varphi(0,0,1,0)(a,0,1) = \psi(ra)\cdot\varphi(0,0,1,0)(a,0,1).$$

In particular, if $a \notin F^{\times 2}$, then $\theta(\varphi,f)_{\Psi_a}$ is given by:

$$\theta(\varphi,f)_{\Psi_a}(g) = \int_{N(\mathbb{A})\backslash SL_2(\mathbb{A})} \omega^{(7)}(hg)\phi(0,0,1,0)(a,0,1)\cdot\overline{f_{\psi_a}(h)}\,dh. \tag{13}$$

On the other hand, if $a = 1$, then it is equal to the sum of this term and the two expressions (9) and (10).

This final formula proves that if $a \notin \widehat{F}(\sigma)$, then $a \notin \widehat{F}(V(\sigma))$. On the other hand, if a is in $\widehat{F}(\sigma)$ and $f = \otimes_v f_v$ is a decomposable vector, then the Fourier coefficient f_{ψ_a} can be expressed as the product:

$$f_{\psi_a}(h) = \prod_v L_v(f_v)(h_v)$$

where L_v is normalized as in Lemma 13.5(ii) for almost all v. Hence, if $\varphi = \otimes_v\varphi_v$ is also decomposable and $a \notin F^{\times 2}$, then we have the desired identity:

$$\theta(\varphi,f)_{\Psi_a}(g) = \prod_v \mathcal{W}_{\Psi_{a,v}}(\varphi_v,f_v,g_v).$$

Together with Lemma 13.5, this shows that $a \in \widehat{F}(V(\sigma))$.

To complete the proof of the proposition, it remains to show that if $1 \in \widehat{F}(\sigma)$, then $1 \in \widehat{F}(V(\sigma))$. Recall that in this case, $\theta(\varphi,f)_{\Psi_a}$ is the sum of the 3 terms (9), (10) and (13). By Lemma 13.5, we know that (13) is non-zero for some choices of φ and f. Hence, we need to show that this non-zero contribution is not cancelled by the other two terms.

Denote the functions in (9),(10) and (13), by $F_1(\varphi,f,g)$, $F_2(\varphi,f,g)$ and $F_3(\varphi,f,g)$ respectively. Let us define a functional L_i on $\omega^{(7)}\otimes\overline{\sigma}$ by:

$$L_i(\varphi\otimes f) = F_i(\varphi,f,1).$$

We would like to show that $L_1 + L_2 + L_3$ is non-zero. Now let $N(X'^*)$ be the unipotent radical of the parabolic $P_7(X'^*)$ stabilizing X'^*. Then it is easy to check that for $n \in N(X'^*)(\mathbb{A})$, one has

$$L_i(\omega^{(7)}(n)\varphi\otimes f) = \chi_i(n)\cdot L_i(\varphi\otimes f),$$

where χ_i are three distinct characters of $N(X'^*)(\mathbb{A})$. Thus we deduce that $L_1+L_2+L_3 = 0$ if and only if each L_i is zero. Since L_3 is not zero, neither is the sum $L_1+L_2+L_3$.

The proposition is proved completely. ∎

§14. Global A-Packets and Periods

In these section, we collect some consequences of our global theorem.

(14.1) Global A-packets. With the local packets $A_{\tau,v}$ defined in (11.3), the global A-packet is simply given by:

$$A_\tau = \{\pi = \otimes_v \pi^{\epsilon_v} : \pi^{\epsilon_v} \in A_{\tau,v}, \epsilon_v = \pm \text{ and } \epsilon_v = + \text{ for almost all } v\}.$$

It is a set of nearly equivalent representations of $G_2(\mathbb{A})$. If S is the set of places v where τ_v is discrete series, then $\# A_\tau = 2^{\# S}$.

To each $\pi \in A_\tau$, Arthur attached a multiplicity $m(\pi)$ as follows. Firstly, one can attach to ψ_τ a quadratic character ϵ_{ψ_τ} of the component group S_{ψ_τ}. In the case at hand, ϵ_{ψ_τ} is the non-trivial character of $S_{\psi_\tau} \cong \mathbb{Z}/2\mathbb{Z}$ if and only if $\epsilon(\tau, 1/2) = -1$. Now if $\pi = \otimes_v \pi^{\epsilon_v} \in A_\tau$, set

$$m(\pi) = \begin{cases} 1, & \text{if } \epsilon_\pi := \prod_v \epsilon_v = \epsilon(\tau, 1/2); \\ 0, & \text{if } \epsilon_\pi = -\epsilon(\tau, 1/2). \end{cases}$$

If we let

$$V_\tau = \bigoplus_{\pi \in A_\tau : \epsilon_\pi = \epsilon(\pi, 1/2)} \pi,$$

then Arthur conjectures that there is a $G_2(\mathbb{A})$-equivariant embedding

$$\iota_\tau : V_\tau \hookrightarrow L^2_{disc}(G_2(F) \backslash G_2(\mathbb{A})).$$

Now our global theorem 13.1 says that for the given τ, the global theta correspondence constructs a subspace of $L^2_{disc}(G_2(F)\backslash G_2(\mathbb{A}))$ isomorphic to

$$\bigoplus_{\sigma \in \tilde{A}_\tau : \epsilon_\sigma = \epsilon(\tau, 1/2)} \Theta(\sigma).$$

This is isomorphic to V_τ with our definition of the local packets $A_{\tau,v}$, and thus we have constructed the embedding ι_τ as required by Arthur's conjecture. In particular, we deduce the following corollary.

(14.2) Corollary *If π_0 is an irreducible constituent of $\pi \in \mathcal{A}_\tau$, then*

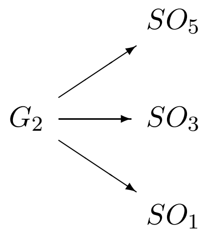
$$m_{disc}(\pi_0) \geq m(\pi).$$

This provides compelling global justification for our definition, especially for taking $\pi^+_{\tau_v}$ to be reducible when τ_v is the Steinberg representation. However, when $m(\pi) = 0$, the inequality of the corollary does not exclude the occurence of π_0 in the discrete spectrum.

In the next two sections, we shall show how one can strengthen the inequality to an equality.

(14.3) SL_3 **period and tower of correspondences.** We conclude this section with another consequence of Theorem 13.1. If $\sigma \subset \mathcal{A}_{00}$ is such that $1 \in \widehat{F}(\sigma)$, then we know that $V(\sigma)$ is not totally contained in $\mathcal{A}_{cusp}(G_2)$. However, our global theorem shows that it is possible that $V(\sigma) \cap \mathcal{A}_{cusp}(G_2)$ is non-zero for such σ's. This can occur, for example, when σ_v is the odd Weil representation $\omega^-_{\psi_v}$ for some finite place v. This is because $V(\sigma) \cong \Theta(\sigma)$ and $\Theta(\sigma_v)$ contains the supercuspidal representation π_{ϵ_v} for such a place v. This answers the question raised in the remark in [RS, Pg. 823].

Further, if $\pi \subset V(\sigma) \cap \mathcal{A}_{cusp}(G_2)$, then it follows from Theorem 12.6(iii) that π is distinguished with respect to SL_3. Further, π is orthogonal to $\oplus_\chi V_\chi$. Such π's can be neatly viewed in the framework of a tower of theta correspondences studied in [GRS2]:



It was shown in [GRS2] that $\pi \subset \mathcal{A}_{cusp}(G_2)$ has a non-zero lift to SO_3 iff π is SL_3-distinguished. Examples of such π's are those which are obtained by the restriction of the minimal representation of split SO_8, since such π's lift non-trivially to SO_3 by the tower principle. However, their lifts to SO_3 are non-cuspidal. The question arises whether there exists cuspidal π whose theta lift to SO_3 is non-zero and cuspidal. Our discussion above shows that such π's do exist.

§15. A Rankin-Selberg Integral

In the remainder of the paper, we shall establish the multiplicity formula for most of the representations in our A-packets. Recall that at the moment, we only have a lower bound on multiplicities given by Cor. 14.2. Indeed, we shall address a more general question: showing that the image V_τ of ι_τ is a full near equivalence class. As we explain in (1.17), showing this will provide conclusive evidence for the authenticity of our A-packets. Our approach exploits a Rankin-Selberg integral representation. The purpose of this section is to treat this Rankin-Selberg integral in detail.

(15.1) Eisenstein series on SU_3^K. Let K be an étale quadratic algebra, corresponding to $a \in F^\times$. It determines a quasi-split special unitary group SU_3^K. Let $B_K = T_K \cdot N_K$ be a Borel subgroup of SU_3^K with modulus character δ_K. The maximal torus T_K is isomorphic to $Res_{K/F}\mathbb{G}_m$, so that $T_K(F) \cong K^\times$ and $\delta_K(t) = |Nm_{K/F}(t)|^2$.

One may consider the family of induced representations

$$I_K(s) = Ind_{B_K(\mathbb{A})}^{G_2(\mathbb{A})} \delta_K^s.$$

For a standard section f_s, one has the Eisenstein series $E(f,s,g)$ which is meromorphic in s, and is given by the sum

$$E_K(f,s,g) = \sum_{\gamma \in B_K(F)\backslash SU_3^K(F)} f_s(\gamma g)$$

when $Re(s)$ is sufficiently large.

The behaviour of $E(f,s,g)$ at $s=1$ is given by the following proposition.

(15.2) Proposition *(i) Let $K = F \times F$. For any standard section f_s, $E_K(f,s,g)$ has at most a double pole at $s=1$. This double pole is attained by the spherical section.*

(ii) Let K be a field. For any standard section f_s, $E_K(f,s,g)$ has at most a simple pole at $s=1$. This simple pole is attained by some standard section.

iii) The residue representation in each case is the trivial representation.

This can be easily checked by examining the constant term of the Eisenstein series, as usual.

(15.3) A Rankin-Selberg integral. The group SU_3^K has a natural embedding into G_2, well-determined up to conjugacy. In (5.6), we have defined a specific embedding $SU_3^K \hookrightarrow G_2$. In the sequel, we shall work with this specific embedding.

Let $\pi \subset \mathcal{A}_{cusp}(G_2)$ and let $\varphi \in \pi$. In this section, we consider the Rankin-Selberg integral:

$$J_K(f,\varphi,s) = \int_{SU_3^K(F)\backslash SU_3^K(\mathbb{A})} E(f,s,g)\cdot\varphi(g)dg.$$

We first note:

(15.4) Lemma *$J_K(f,\varphi,s)$ is a meromorphic function on $\mathbb{C}$.*

PROOF. It suffices to show that the cusp form φ is rapidly decreasing on a Siegel domain for SU_3^K. A priori, we only know that φ is rapidly decreasing on a Siegel domain for G_2. However, the property of rapid decrease can be extended to a Siegel domain for SU_3^K using the action of the Weyl group of G_2 and the automorphy of φ. We omit the details. ∎

(15.5) Unfolding. On unfolding the Rankin-Selberg integral, assuming that $Re(s)$ is sufficiently large, we obtain:

$$J_K(f,\varphi,s) = \int_{T_K(F)N_K(\mathbb{A})\backslash SU_3^K(\mathbb{A})} f_s(g)\cdot\left(\sum_{\gamma\in T_K(F)}\varphi_{\Psi_a}(\gamma g)\right)dg,$$

where Ψ_a is the unitary character on $U_2(\mathbb{A})$ given by:

$$\Psi_a(x_\beta(u_1)x_{\alpha+\beta}(u_2)x_{2\alpha+\beta}(u_3)x_{3\alpha+\beta}(u_4)x_{3\alpha+2\beta}(u_5)) = \psi(-au_1+u_3).$$

Hence, $J_K(f,\varphi,s)$ is zero unless π has non-vanishing $(F\times K)$-coefficient. In the sequel, we assume this whenever we consider J_K.

Proceeding formally at the moment, we collapse the integral and the sum to obtain:

$$J_K(f,\varphi,s) = \int_{N_K(\mathbb{A})\backslash SU_3^K(\mathbb{A})} f_s(g)\cdot\varphi_{\Psi_a}(g)\,dg \tag{14}$$

when $Re(s)$ is large. This will be justified if we show that the integral on the right hand side is absolutely convergent.

Convergence issues aside, this is as far as we can go, since the Fourier coefficient φ_{Ψ_a} may not be Eulerian in general. However, we shall consider examples of π for which φ_{Ψ_a} is almost Eulerian. Examples are those π's which are nearly equivalent to the representations in V_χ or V_τ. For such π's, we shall address the convergence issues above, as well as obtain an expression for $J(f,\varphi,s)$ in terms of the (partial) L-function of χ or τ.

(15.6) Convergence issues. Suppose that the Fourier coefficient φ_{Ψ_a} is almost Eulerian. In other words, there is a large finite set S of places of F, including the archimedean ones, so that:

$$\varphi_{\Psi_a}(g) = L_S(g_S\cdot\varphi_S)\cdot\prod_{v\notin S} l_v^0(g_v\cdot\varphi_v)$$

where $L_S\in Hom_{U_2(\mathbb{A}_S)}(\pi_S,\mathbb{C}_{\Psi_a})$, and for $v\notin S$, l_v^0 is a distinguished element of $Hom_{U_2(F_v)}(\pi_v,\mathbb{C}_{\Psi_a})$. In this case,

$$J_K(f,\varphi,s) = d_S(f_S,\varphi_S,L_S,s)\cdot\left(\prod_{v\notin S} J_{K_v}(f_v,\varphi_v,s)\right)$$

where

$$d_S(f_S,\varphi_S,L_S,s) = \int_{N_K(\mathbb{A}_S)\backslash SU_3^K(\mathbb{A}_S)} f_s(g)\cdot L_S(g_S)\,dg_S$$

and $J_{K,v}$ is the analogously defined local integral over F_v.

To show the convergence of the right-hand-side of (14), we need to show the absolute convergence of the integrals d_S (and $J_{K,v}$), as well as the absolute convergence of the Euler product over $v \notin S$. We first address the convergence of d_S in general. Using the Iwasawa decomposition for $SU_3^K(\mathbb{A}_S)$, we have:

$$d_S(f_S, \varphi_S, L_S, s) = \int_{K_S} f_S(k) \cdot G_S(k\varphi_S, L_S, s)\, dk$$

where $K_S \subset SU_3^K(\mathbb{A}_S)$ is an appropriate maximal compact subgroup and

$$G_S(\varphi_S, L_S, s) = \int_{T_K(\mathbb{A}_S)} \delta_K(t)^{s-1} \cdot L_S(t\varphi_S)\, dt.$$

Thus it suffices to prove that $G_S(\varphi_S, L_S, s)$ converges for all φ when $Re(s) >> 0$.

(15.7) Lemma *(i) There exists a constant c so that for all φ_S (not necessarily K-finite), $G_S(\varphi_S, L_S, s)$ converges absolutely whenever $Re(s) > c$. Moreover, for fixed s, $\varphi_S \mapsto G_S(\varphi_S, L_S, s)$ is a non-zero continuous linear functional on π_S. For fixed φ_S, $s \mapsto G_S(\varphi_S, L_S, s)$ is a holomorphic function on $Re(s) > c$.*

(ii) For any φ_S and f_S, $d_S(f_S, \varphi_S, L_S, s)$ converges absolutely when $Re(s) > c$ and defines a holomorphic function of s in the half-plane $Re(s) > c$. Moreover, given s_0 with $Re(s_0) > c$, there exists K-finite φ_S and f_S such that $d_S(f_S, \varphi_S, L_S, s_0) \neq 0$.

Proof. Clearly, (ii) is an immediate consequence of (i). To prove (i), we need some analytic properties of $L_S(t\varphi_S)$ as a function of $T_K(\mathbb{A}_S)$. The first thing to note is that since φ is of moderate growth, there is a constant k (independent of the set S and the vector φ_S) such that

$$|L_S(t\varphi_S)| \leq C_{\varphi_S} \cdot ||t||^k.$$

We need to refine this bound further.

Note that Ψ_a gives a character on $U_2(\mathbb{A}_S)/N_K(\mathbb{A}_S) \cong \prod_{v\in S} K_v$ and $T_K(\mathbb{A}_S) \cong \prod_{v\in S} K_v^\times$. We fix these isomorphisms so that the adjoint action of $T_K(\mathbb{A}_S)$ on $U_2(\mathbb{A}_S)/N_K(\mathbb{A}_S)$ is isomorphic to the action of $\prod_{v\in S} K_v^\times$ on $\prod_{v\in S} K_v$ by multiplication. Thus, we regard $L(t\varphi_S)$ as a function on $\prod_{v\in S} K_v^\times$.

By a famous result of Dixmier-Malliavin [DM, Thm. 3.3], one can express the smooth vector φ_S as:

$$\varphi_S = \sum_{i=1}^{n} \phi_i * \varphi_i$$

where $\phi_i \in C_c^\infty(U_2(\mathbb{A}_S))$, $\varphi_i \in \pi_S$ is a smooth vector and

$$\phi_i * \varphi_i = \int_{U_2(\mathbb{A}_S)} \phi_i(u) \cdot u\varphi_i\, du.$$

Let $\phi_i' \in C_c^\infty(N_K(\mathbb{A}_S)\backslash U_2(\mathbb{A}_S)) = C_c^\infty(\prod_{v\in S} K_v)$ be defined by:

$$\phi_i'(x) = \int_{N_K(\mathbb{A}_S)} \phi_i(nx)\, dn.$$

Then an easy calculation shows that

$$L_S(t \cdot (\phi_i * \varphi_i)) = \widehat{\phi_i'}(t) \cdot L_S(t\varphi_i),$$

where

$$\widehat{\phi_i'}(t) = \int_{N_K(\mathbb{A}_S)\backslash U_2(\mathbb{A}_S)} \phi_i'(x) \cdot \Psi_a(tx)\, dx$$

is the Fourier tansform of ϕ_i'. In particular, $\widehat{\phi_i'}$ is a Schwarz function on $\prod_{v\in S} K_v$. In conclusion, we see that there exists a non-negative Schwarz function Φ_{φ_S} on $\prod_{v\in S} K_v$ and a positive integer k so that

$$|L(t\varphi_S)| \leq \Phi_{\varphi_S}(t) \cdot ||t||^k.$$

Note that k does not depend on φ_S. This proves the existence of c such that whenever $Re(s) > c$, $G_S(\varphi_S, L_S, s)$ converges absolutely and locally uniformly in s for any φ_S, and shows that $G_S(\varphi_S, L_S, s)$ is holomorphic in s.

We now show that $\varphi_S \mapsto G_S(\varphi_S, L_S, s)$ defines a continuous linear functional. In fact, the proof that we describe below gives an alternatve proof of the absolute convergence of G_S when $Re(s)$ is large. Fix $\varphi_{fin} \in \pi_{S,fin}$ and let l_s be the linear functional on π_∞ defined by

$$l_s : \varphi_\infty \mapsto \int_{T_K(\mathbb{A}_{S,fin})} \delta(t)^{s-1} \cdot L_S(\varphi_\infty \otimes (t\varphi_{fin}))\, dt.$$

Note that l_s is continuous (for $Re(s) >> c$). Indeed, since φ_{fin} is fixed by an open compact subgroup of $U_2(\mathbb{A}_{S,fin})$, the integral over $T_K(\mathbb{A}_{S,fin}) = \prod_{v\in S_{fin}} K_v^\times$ can be replaced by one over the intersection of $T_K(\mathbb{A}_{S,fin})$ with a compact subset of $\prod_{v\in S_{fin}} K_v$. On this set, we have:

$$|L_S(\varphi_\infty \otimes (t\varphi_{fin}))| \leq \beta_{\varphi_{fin}}(\varphi_\infty) \cdot ||t||^k$$

where $\beta_{\varphi_{fin}}$ is a continuous seminorm on π_∞ and $k > 0$ is independent of $\varphi_\infty \otimes \varphi_{fin}$. This shows that the integral defining l_s converges for large s and defines a continuous functional on π_∞.

Now we need to show that

$$\varphi_\infty \mapsto G_{S_\infty}(\varphi_\infty, l_s, s)$$

is continuous. Note that for each archimedean v, $K_v^\times = \mathbb{R}^\times \times \mathbb{R}^\times$, $\mathbb{C}^\times$ or $\mathbb{C}^\times \times \mathbb{C}^\times$. Suppose that $t \in K_v^\times$ lies in $\mathbb{R}^\times$ or $\mathbb{C}^\times$. Then one can find an element in the complexified Lie algebra of $U_2(F_v)$ such that

$$d\Psi_a(X) = 1 \quad \text{and} \quad Ad(t)(X) = tX.$$

This implies that for any $n \geq 1$,

$$l_s(t\varphi_\infty) = l_s(X^n t\varphi_\infty) = t^{-n} \cdot l_s(tX^n\varphi_\infty).$$

Together with the continuity of l_s, this shows that there is a continuous seminorm β_1 on π_∞ such that when $|t| >> 1$,

$$|l_s(t\varphi_\infty)| \leq |t|^{k-n} \cdot \beta_1(\varphi_\infty),$$

where k is independent of φ_∞ and $n >> k$. On the other hand, when $|t| << 1$, there is a continuous seminorm β_2 such that

$$l_s(t\varphi_\infty) \leq |t|^{-k} \cdot \beta_2(\varphi_\infty),$$

where k is independent of φ_∞. Iterating this argument over all the copies of $\mathbb{R}^\times$ or $\mathbb{C}^\times$, we see that there is a continuous seminorm β such that for all φ_∞,

$$|G_{S_\infty}(\varphi_\infty, l_s, s)| \leq \beta(\varphi_\infty)$$

which proves the desired continuity.

It remains to show that for a given s, there exists φ such that $G_S(\varphi_S, L_S, , s) \neq 0$. Pick φ_S so that $L_S(\varphi_S) \neq 0$, and for a Schwarz function ϕ on $U_2(\mathbb{A}_S)$, consider

$$G_S(\phi * \varphi_S, L_S, s) = \int_{T_K(\mathbb{A}_S)} \widehat{\phi}'(t) \cdot \delta_K(t)^{s-1} \cdot L_S(t\varphi_S)\, dt. \tag{15}$$

Since the image of $T(\mathbb{A}_S) \cong \prod_{v\in S} K_v^\times$ in $\prod_{v\in S} K_v$ is locally closed, we may choose ϕ so that $\widehat{\phi}$ is supported on a relatively compact neighbourhood of $1 \in T_K(\mathbb{A}_S)$. By making the support of $\widehat{\phi}$ sufficiently small, we can make the right-hand-side of (15) non-zero. ∎

(15.8) A special case. Let us examine the special case $\pi = V(\sigma)$, where σ is a cuspidal representation of $\tilde{SL}_2(\mathbb{A})$. Assume that K is a field, so that $a \notin F^{\times 2}$.

In this case, if $\varphi = \theta(\Phi, \phi)$, then we have seen in Prop. 13.6 that for $a \notin F^{\times 2}$,

$$\varphi_{\Psi_a}(g) = \prod_v \mathcal{W}_{\Psi_{a,v}}(\Phi_v, \phi_v, g_v)$$

with

$$\mathcal{W}_{\Psi_{a,v}}(\Phi_v, \phi_v, g_v) = \int_{N(F_v)\backslash SL_2(F_v)} \omega(g,h)\Phi_v(0,0,1,0)(a,0,1) \cdot \overline{W_{\psi_a}(\phi_v, h_v)}\, dh.$$

Thus, the Fourier coefficient φ_{Ψ_a} is Eulerian, despite the fact that when σ is not an elementary Weil representation, the multiplicity-one result for local functionals fails at those places v which split in K. Indeed, at such places v, it follows from Prop. 8.3(ii) that

$$\dim \mathrm{Hom}_{U_2(F_v)}(\pi_v, \mathbb{C}_{\Psi_a}) = 3.$$

Regardless, we have:

$$J_K(\varphi, f, s) = \prod_v J_{K,v}(\Phi_v, \phi_v, f_v, s)$$

where the local factor is

$$\int\limits_{N_K(F_v)\backslash SU_K^3(F_v)} \left(\int\limits_{N(F_v)\backslash SL_2(F_v)} \omega(g,h)\Phi(0,0,1,0)(a,0,1) \cdot \overline{W_{\psi_a}(\phi_v, h)}\, dh \right) \cdot f_s(g)\, dg$$

It is not difficult to see that this double integral is absolutely convergent for $Re(s)$ large. Thus in this special case, $J(f, \varphi, s)$ is Eulerian, as long as we show that the Euler product converges absolutely for $Re(s)$ large. We shall show this next by computing the local factors explicitly for almost all v.

(15.9) Unramified computation. We compute the integral $J_{K,v}(\Phi_v, \phi_v, f_v)$ for the places where ψ_v has conductor A_v, ϕ_v and f_v are spherical and Φ_v is a characteristic function of A_v^7. Since the computation is local, we will omit v everywhere. The result is:

(15.10) Proposition *With all the data unramifed as above,*

$$J_K(\Phi, \phi, f) = \begin{cases} \frac{\zeta_K(2s-1)}{L(\chi_K, 4s-1)} \cdot L(Wd_\psi(\sigma), 4s - 3/2) \ \textit{if } \sigma \neq \omega^+_{\chi_K}; \\ \\ \zeta_K(2s-1) \cdot L(\chi_K, 4s-2) \ \textit{if } \sigma = \omega^+_{\chi_K}. \end{cases}$$

Proof. Using the Iwasawa decompositions for $SU_{3,K}$ and SL_2, we see that:

$$J_K = \int\limits_{K^\times}\int\limits_{F^\times} \omega(t(\alpha), t(r))\Phi(0,0,1,0)(a,0,1) \cdot \overline{W_{\psi_a}(\phi, t(r))} \cdot |r|^{-2} \cdot |Nm_{K/F}\alpha|^{s-1}\, dr\, d\alpha$$

(15.11) Lemma *One has*

$$\omega_{\psi,q}((t(\alpha), t(r))\Phi(0,0,1,0)(a,0,1) = \begin{cases} |Nm_{K/F}\alpha| \cdot |r|^{3/2} \cdot \gamma_\psi(r)^{-1}, \ \textit{if } |Nm_{K/F}\alpha| \leq |r|^2 \leq 1; \\ 0, \ \textit{otherwise.} \end{cases}$$

The next fact is known from [W]:

(15.12) Lemma *For* $Re\, s >> 0$ *one has*

$$\int\limits_{F^\times} |r|^{s-1} \cdot \gamma_\psi(r)^{-1} \cdot \overline{W_{\psi_a}(\phi, t(r))}\, dr = \begin{cases} \frac{L(Wd_\psi(\sigma), s)}{L(\chi_K, s+1/2)}, \ \textit{if } \sigma \neq \omega^+_{\chi_K}; \\ \\ L(\chi_K, s-1/2), \ \textit{if } \sigma = \omega^+_{\chi_K}. \end{cases}.$$

In particular,

$$J_K = \int\limits_{|r|\le 1} \left(\int\limits_{|Nm\alpha)|\le |r|^2} |Nm\alpha||Nm\alpha|^{2s-2} d\alpha \right) \cdot \gamma_\psi(r)^{-1} \cdot |r|^{-1/2} \cdot \overline{W_{\psi_a}(\phi, t(r))} dr$$

which is in turn equal to

$$\zeta_K(2s-1) \times \int\limits_{|r|\le 1} \gamma_\psi^{-1}(r) \cdot |r|^{4s-5/2} \cdot \overline{W_{\psi_a}(\phi, t(r))}\, dr.$$

This gives the proposition, by virtue of Lemma 15.12. ∎

(15.13) Absolute convergence. Finally, we address the convergence of the product (outside a finite set S) of the local factors:

$$J^A_{K_v}(\varphi_v, f_v, s) := \int\limits_{K_v^\times} \int\limits_{F_v^\times} |\omega(t(\alpha), t(r))\Phi_v(0,0,1,0)(a,0,1)| \cdot |W_{\psi_{a,v}}(\phi, t(r))| \cdot |r|^{-2} \cdot |Nm_{K/F}\alpha|^{s-1}\, dr\, d\alpha$$

with s a sufficiently large real number.

If we put in absolute values in the local computation above, we see that

$$J^A_{K_v}(\varphi_v, f_v, s) = |\zeta_{K_v}(2s-1)| \times \int\limits_{|r|\le 1} |r|^{4s-5/2} \cdot |W_{\psi_{a,v}}(\phi_v, t(r))|\, dr := |\zeta_{K_v}(2s-1)| \cdot I^A(\varphi_v, s).$$

It was shown in [W1, Lemma 49, Pg. 88-89] that $\prod_{v\notin S} I^A(f_v, s)$ converges for s large, and thus the Euler product expression for $J_K(\varphi, f, s)$ is absolutely convergent. This justifies our earlier formal manuveres.

(15.14) More general π. We now consider the more general situation where $\pi \subset L^2_{cusp}$ is nearly equivalent to the representations in V_χ or V_τ.

Let's first consider the easier case when π is nearly equivalent to the representations in V_χ (where χ is possibly trivial). The $(F \times K)$-coefficient of π is zero unless $\chi = \chi_K$ is the quadratic character associated to K. Assuming this, we see that $\pi_v \cong \pi_v^K := J_{P_1}(\pi(1, \chi_K), 1)$ (cf. Prop. 7.10) for almost all v. On the other hand, we know from [HMS] that

$$\dim \mathrm{Hom}_{U_2(F_v)}(\pi_v^K, \mathbb{C}_{\Psi_a}) = 1.$$

A non-zero element in this 1-dimensional space can be described as follows. We know by our main local theorem that

$$\pi_v^K \hookrightarrow \theta(\omega^+_{\chi K_v}) \cong (\omega_v^{(7)} \otimes \omega^\vee_{\chi K_v})_{\tilde{SL_2}}.$$

Moreover, the linear form on $\omega_v^{(7)} \otimes \omega_{\chi_{K_v}}^\vee$ given by (15.9):

$$l^0_{K_v} : \Phi_v \otimes \overline{\phi_v} \mapsto \mathcal{W}_{\Psi_{a,v}}(\Phi_v, \phi_v, 1)$$

factors to the quotient $(\omega_v^{(7)} \otimes \omega_{\chi_{K_v}}^\vee)_{\tilde{SL_2}}$, and restricts to a non-zero element of the one-dimensional space $\mathrm{Hom}_{U_2(F_v)}(\pi_v^K, \mathbb{C}_{\Psi_a})$. From this, we deduce that:

- the Fourier coefficient φ_{Ψ_a} is almost Eulerian: for S a large finite set of places, including the archimedean ones,

$$\varphi_{\Psi_a}(1) = \left(\prod_{v \notin S} l^0_{K_v}(\varphi_v) \right) \cdot L_S(\varphi_S)$$

  for some non-zero $L_S \in Hom_{U_2(\mathbb{A}_S)}(\pi_S, \mathbb{C}_{\Psi_a})$.

- For $v \notin S$, the local factor is equal to $J_{K,v}(\Phi_v^0, \phi_v^0, f_v^0, s)$, which was computed in Prop. 15.10.

Hence, we conclude:

(15.15) Proposition *Supppose that π is nearly equivalent to the representations in V_{χ_K}. Then for a sufficiently large finite set S of places, including the archimedean ones,*

$$J_K(\varphi, f, s) = \zeta_K^S(2s-1) \cdot L^S(\chi_K, 4s-2) \cdot d_S(f_S, \varphi_S, L_S, s) \qquad Re(s) >> 0.$$

where

$$d_S(f_S, \varphi_S, L_S, s) = \int_{N_K(\mathbb{A}_S)\backslash SU_3^K(\mathbb{A}_S)} f_{S,s}(g) \cdot L_S(g \cdot \varphi_S)\, dg.$$

(15.16) The harder case. We now assume that π is nearly equivalent to the representations in V_τ. In this case, we shall argue that for $\varphi \in \pi$, the Fourier coefficient φ_{Ψ_a} is almost Eulerian if $a \notin F^{\times 2}$, even though the local multiplicity-one result fails for half of the places.

We first note the following lemma.

(15.17) Lemma *If π is nearly equivalent to the representations in V_τ, then π has a non-zero $(F \times K)$-coefficient for some K (possibly split).*

Proof. Indeed, since π is non-generic cuspidal, π must support some E-coefficient, with E an étale cubic algebra [G2, Prop. 15.1]. On the other hand, $\pi_v \cong I_{P_1}(\mu_v)$ for almost all v, and these degenerate principal series does not support (U_2, ψ_{E_v})-functionals when E_v is a field [HMS]. This proves the lemma. ∎

(15.18) Fourier-Jacobi coefficient. For $f \in \pi$, consider

$$V_{Z,\psi}(\pi) = \{f_{Z,\psi}|_{P_{1,ss}} : f \in \pi\}.$$

We have a $P_{1,ss}(\mathbb{A})$-equivariant map

$$P_{Z,\psi} : \pi \longrightarrow V_{Z,\psi}(\pi).$$

Define $FJ_\psi(\pi)$ to be the span of all functions on $L_{1,ss}(F)\backslash\tilde{L}_{1,ss}(\mathbb{A})$ of the form:

$$F_{f,\varphi}(h) = \int_{U_1(F)\backslash U_1(\mathbb{A})} f_{Z,\psi}(uh) \cdot \overline{\theta(\varphi)(uh)}\, du.$$

Then by a result of Ikeda [I],

$$V_{Z,\psi}(\pi) \cong FJ_\psi(\pi)\hat{\otimes}\omega_\psi.$$

(15.19) Lemma *Let π be nearly equivalent to $V(\sigma) \subset V_\tau$. Then*

(i) $FJ_\psi(\pi)$ is non-zero and cuspidal, and

$$FJ_\psi(\pi) \subset \overline{\mathcal{A}_{\chi_0}} \oplus \overline{\mathcal{A}(\tau)},$$

where χ_0 is the trivial character.

(ii) Under the assumption that π has non-zero $(F \times K)$-coefficient for some quadratic field K, the projection

$$P_\tau : FJ_\psi(\pi) \longrightarrow \overline{\mathcal{A}(\tau)}$$

is non-zero. Conversely, if the projection P_τ is non-zero on $FJ_\psi(\pi)$, then π has some non-zero $(F \times K)$-coefficient with K a field.

Proof. (i) By the previous lemma, π supports an $(F \times K)$-coefficient for some K (possibly split). This implies that $FJ_\psi(\pi)$ supports a ψ_K-coefficient, and thus it is non-zero. On the other hand, since π non-generic cuspidal, all its degenerate coefficients vanish, so that $FJ_\psi(\pi)$ is cuspidal.

For almost all v, we know that:

$$FJ_{\psi_v}(\pi_v) = FJ_{\psi_v}(I_{P_1}(\mu_v)) \cong \omega_{\psi_v}^\vee \oplus \tilde{\pi}(\mu_v)^\vee.$$

So if σ' is any irreducible summand of $FJ_\psi(\pi)$, then for almost all v, $\sigma'_v \cong (\omega^+_{\psi_v})^\vee$ or $\tilde{\pi}(\mu_v)^\vee$. If $\sigma'_v = (\omega^+_{\psi_v})^\vee$ for at least one place v, then $\sigma' \subset \overline{\mathcal{A}_{\chi_0}}$. If not, then σ' is contained in $\overline{\mathcal{A}(\tau)}$. This proves (i).

(ii) If π has non-zero $(F \times K)$-coefficient with K a field, we cannot have $FJ_\psi(\pi) \subset \overline{\mathcal{A}_{\chi_0}}$, since otherwise the only non-zero coefficient of π is the one corresponding to $F \times F \times F$. The converse follows from the fact that any $\sigma \subset \overline{\mathcal{A}(\tau)}$ has some non-zero Fourier coefficient corresponding to a non-square class. ∎

(15.20) Fourier coefficient. Let us use π in the Rankin-Selberg integral J_K considered above, assuming that π has non-zero $(F \times K)$ coefficient, with K a field. Consider the linear form $L_K : \pi \to \mathbb{C}$ defined by

$$L_K(\varphi) = \varphi_{\Psi_a}(1).$$

Clearly, L_K factors through the projection $P_{Z,\psi} : \pi \to V_{Z,\psi}(\pi)$. Since K is a field, it further factors through the projection

$$P_\tau : V_{Z,\psi}(\pi) \longrightarrow \overline{\mathcal{A}(\tau)}\hat{\otimes}\omega_\psi.$$

In other words,

$$L_K : \pi \longrightarrow V_{Z,\psi}(\pi) \longrightarrow \overline{\mathcal{A}(\tau)}\hat{\otimes}\omega_\psi \longrightarrow \mathbb{C}.$$

Outside a finite set of places, there is a 1-dimensional subspace of $Hom_{U_2}(\pi_v, \mathbb{C}_{\Psi_K})$ which factors as:

$$l_{K_v} : \pi_v \longrightarrow \pi_{Z,\psi_v} \longrightarrow \tilde{\pi}(\mu_v)^\vee \otimes \omega_{\psi_v} \longrightarrow \mathbb{C}.$$

Indeed, l_{K_v} is precisely the linear form

$$\Phi_v \otimes \phi_v \mapsto \mathcal{W}_{\Psi_{a,v}}(\Phi_v, \phi_v, 1)$$

studied in (15.9).

As before, this implies:

(15.21) Proposition *Suppose that π is nearly equivalent to the representations in $\mathcal{A}_\tau$, and π possesses non-zero $(F \times K)$-coefficient with K a quadratic field. Then for $Re(s) >> 0$ and for a sufficiently large finite set S of places, including the archimedean ones,*

$$J_K(\varphi, f, s) = \frac{\zeta_K^S(2s-1)}{L^S(\chi_K, 4s-1)} \cdot L^S(Wd_\psi(\sigma), 4s-3/2) \cdot d_S(f_S, \varphi_S, L_S, s)$$

where

$$d_S(f_S, \varphi_S, L_S, s) = \int_{N_K(\mathbb{A}_S)\backslash SU_3^K(\mathbb{A}_S)} f_{S,s}(g) \cdot L_S(g \cdot \varphi_S)\, dg.$$

(15.22) The bad factor d_S. In Lemma 15.7, we showed that $d_S(f_S, \varphi_S, L_S, s)$ defines a holomorphic function when $Re(s)$ is sufficiently large. However, the equalities in Prop. 15.15 and Prop. 15.21 provide a meromorphic continuation of d_S to the whole complex plane. In particular, the identities in these propositions hold for all $s \in \mathbb{C}$.

Note that our application of the Rankin-Selberg integral here is quite different from the usual one. Usually, one is hoping to use the Rankin-Selberg integral to deduce the meromorphic continuation of the (partial) L-function involved, and thus one needs to establish the meromorphic continuation of the bad factor d_S independently. In our case, the meromorphic continuation of the L-function is already known, and this gives the continuation of d_S as a consequence.

(15.23) The bad factor G_S. Recall that by the Iwasawa decomposition, we have:

$$d_S(f_S, \varphi_S, L_S, s) = \int_{K_S} f_S(k) \cdot G_S(k\varphi_S, L_S, s)\, dk \tag{16}$$

with

$$G_S(k\varphi_S, L_S, s) = \int_{T_K(\mathbb{A}_S)} \delta_K(t)^{s-1} \cdot L_S(tk\varphi_S)\, dt.$$

Now if φ_S is K_S-finite, then $G_S(\varphi_S, L_S, s)$ is a finite linear combination of $d_S(f_i, \varphi_S, L_S, s)$ and thus is also a meromorphic function. Hence, the equation (16) holds for all s when f_S and φ_S are K-finite. We have:

(15.24) Proposition *Given $s_0 \in \mathbb{C}$, there exist K-finite f_S and φ_S such that $d_S(f_S, \varphi_S, L_S, s)$ is non-zero at $s = s_0$.*

Proof. It suffices to show the existence of a K-finite φ_S such that $G_S(\varphi_S, L_S, s_0) \neq 0$. In Lemma 15.7(i), we have shown this for s_0 in some half-plane $Re(s_0) > c$. We shall show that one can replace c by $c - 1$.

Let us start with $\varphi_S = \varphi_\infty \otimes \varphi_{fin}$, and fix φ_{fin} so that the linear form

$$l_{\varphi_{fin}}(\varphi_\infty) = L_S(\varphi_\infty \otimes \varphi_{fin}).$$

on π_∞ is non-zero. Analogous to the proof of Lemma 15.7(i), one can find a Schwarz function ϕ on $U_2(\mathbb{A}_{S,fin})$ so that

$$G_S(\varphi_\infty \otimes (\phi * \varphi_{fin}), L_S, s) = G_{S_\infty}(\varphi_\infty, l_{\varphi_{fin}}, s).$$

Now, because the action of $T_K(F_v)$ on $U_2(F_v)/N_K(F_v)$ is isomorphic to the action of $K_v^\times$ on K_v by multiplication, one can find an element X in the universal enveloping algebra of the complexified Lie algebra of $U_2(\mathbb{A}_\infty)$, such that

$$G_{S_\infty}(X\varphi_\infty, l_{\varphi_{fin}}, s) = G_{S_\infty}(\varphi_\infty, l_{\varphi_{fin}}, s+1).$$

Thus, if $Re(s_0) > c - 1$, then one can find φ_∞^0 such that

$$G_S(X\varphi_\infty^0, (\phi * \varphi_{fin}), L_S, s_0) \neq 0.$$

Iterating this procedure, we obtain the desired result for all $s_0 \in \mathbb{C}$. ∎

§16. Near Equivalence Classes and Multiplicity Formula

In this final section, we shall exploit the Rankin-Selberg integral studied in the last section to prove the following theorem.

(16.1) Theorem *Let $\pi \subset \mathcal{A}_{cusp}(G_2)$ be a cuspidal representation of G_2.*

(i) If π is nearly equivalent to the representations in V_χ, for some quadratic character χ, then $\pi \subset V_\chi$. In other words, V_χ is a full near equivalence class.

(ii) Suppose that π is nearly equivalent to the representations in V_τ, for some cuspidal representation τ of $PGL_2(\mathbb{A})$. Then π is not orthogonal to V_τ.

Proof. Let us first consider the proof of (ii). Assume first that π has a non-zero Fourier coefficient corresponding to $F \times K$, for some quadratic field K. Then we may use π in the Rankin-Selberg integral J_K of the last section and Props. 15.2, 15.21 and 15.24 imply that π has non-zero SU_3^K-period. Thus, by Theorem 12.6(iii), we conclude that π is not orthogonal to some $V(\sigma)$. To finish the proof in this case, we need to convince ourselves that $\sigma \subset \mathcal{A}(\tau)$.

Now $\mathcal{A}(\tau)$ is a full near equivalence class of $\widetilde{SL}_2$. If $\tau_v = \pi(\mu_v, \mu_v^{-1})$ for almost all v, then the representations in $\mathcal{A}(\tau)$ have local components $\widetilde{\pi}(\mu_v)$ for almost all v. In the case at hand, we know that $\sigma_v \otimes \pi_v$ is a quotient of $\omega_v^{(7)}$, and for almost all v, $\pi_v = I_{P_1}(\mu_v) = I_{P_1}(\mu_v^{-1})$ Now our local Thm. 9.1 implies that the only σ_v for which $I_{P_1}(\mu_v) \subset \Theta(\sigma_v)$ is the representation $\widetilde{\pi}(\mu_v)$. Hence, $\sigma \subset \mathcal{A}(\tau)$ and (ii) is proved, assuming that π has some non-zero $(F \times K)$-coefficient with K a field.

If all the $(F \times K)$-coefficients of π vanish when K is a field, then π only supports $(F \times F \times F)$-coefficient. This implies, in the language of Lemma 15.19, that $FJ_\psi(\pi) \subset \overline{\mathcal{A}_{\chi_0}}$, with χ_0 the trivial character, and thus

$$V_{Z,\psi}(\pi) \subset \overline{\mathcal{A}_{\chi_0}} \hat{\otimes} \omega_\psi.$$

In this case, we shall argue that π has non-zero SL_3-period. It is worth remarking that one cannot resort to the Rankin-Selberg analysis of the last section here. This is because the analysis there (in particular in (15.20)) rests on the assumption that $FJ_\psi(\pi)$ is not totally contained in $\overline{\mathcal{A}_{\chi_0}}$. The argument we present below is decidedly roundabout.

Consider the following linear functional β on the space $V_{Z,\psi}(\pi) \subset C^\infty(P_{1,ss}(F)\backslash P_{1,ss}(\mathbb{A}))$:

$$\beta(\phi) = \int_{SL_2(F)\backslash SL_2(\mathbb{A})} \phi(g)dg$$

where $SL_2 = L_{1,ss}$. Because $\mathcal{A}_{\chi_0} \subset \omega_\psi$, we see that

$$V_{Z,\psi}(\pi) \cong \overline{V} \hat{\otimes} \omega_\psi$$

with $V \subset \omega_\psi$. It is thus clear that β is non-zero on $V_{Z,\psi}(\pi)$.

Now we consider the dual pair $G_2 \times PGL_3 \subset E_6$. The theta correspondence of this dual pair (given by the minimal representation of E_6) has been studied in a number of papers. In particular, it was shown in [GJ] that the non-vanishing of β implies that the global theta lift of π to PGL_3 has non-zero Whittaker coefficient; in particular, it is non-zero. To be honest, [GJ] works only with generic representations of G_2, but the above implication does not require genericity. Now we claim that this theta lift has zero intersection with the cuspidal space of PGL_3. Indeed, if it were not, there would be a cuspidal representation π' of $PGL_3(\mathbb{A})$ such that $\pi \otimes \pi'$ is a quotient of the minimal representation of E_6. This would mean that the local theta lift of π_v contains a generic representation of $PGL_3(F_v)$. However, one knows that a generic spherical representation of $PGL_3(F_v)$ cannot be paired with a non-generic representation of $G_2(F_v)$ (cf. [GS] for example).

Since the global theta lift of π to PGL_3 is non-cuspidal, it follows from the tower property of [GRS, Theorem 3.1] that π has non-zero theta lift to PGL_2 (which is the lower step of the tower). Then [GRS, Theorem 4.1 (5)] implies that π has non-zero SL_3-period, as desired.

The same line of reasoning shows that if π is as in (i), then π is not orthogonal to V_χ. The proof that π is actually contained in V_χ depends on the results of [G]. Since the details are explicated in [G], we shall not give the proof of (i) here.

∎

(16.2) Multiplicity formula. We note some consequences of Theorem 16.1:

(16.3) Theorem *(i) Let $P_{cusp}(V_\tau)$ denote the projection of V_τ onto $\mathcal{A}_{cusp}(G_2)$. Then $P_{cusp}(V_\tau)$ is a full near equivalence class in $\mathcal{A}_{cusp}(G_2)$. In other words, if $\pi \subset \mathcal{A}_{cusp}(G_2)$ is nearly equivalent to the representations in V_τ, then $\pi \subset P_{cusp}(V_\tau)$.*

(ii) Suppose π_0 is an irreducible constituent of $\pi \in \mathcal{A}_\tau$. Then

$$m_{disc}(\pi_0) = m(\pi).$$

unless possibly in the following exceptional case: $\pi_0 = \pi_\tau^{++} := \otimes_v \pi_{\tau,v}^{++}$ and $L(\tau, 1/2) \neq 0$. *In this exceptional case, one has $m(\pi) = 1$, but we only know that $m_{disc}(\pi_0) = 1$ or 2.*

Proof. (i) This is clear from Theorem 16.1.

(ii) Suppose that π_0 is not in the exceptional case. Then we know from (i) that $m_{cusp}(\pi_0) = m(\pi)$, since the multiplicity of π_0 in V_τ is equal to $m(\pi)$. On the other hand, by the results of Kim [K] and Zampera [Z] on the residual spectrum of G_2, we see that π_0 does not occur in L^2_{res}, so that $m_{disc}(\pi_0) = m_{cusp}(\pi_0)$. In the exceptional case, π_0 does occur in the residual spectrum. So $m_{disc}(\pi_0) = m_{cusp}(\pi_0) + 1$, but we only know from (i) that $m_{cusp}(\pi_0) \leq 1$. This is because while we know that the space affording π_τ^{++} in V_τ is not cuspidal, we do not know that it is orthogonal to the space of cusp forms. ∎

This theorem shows that the local A-packets defined in this paper are the correct ones, since they capture all possible candidates which could appear as local components of representations in the discrete spectrum which are nearly equivalent to those in the short root A-packets. Of course, it would also be important to check the local requirements of Arthur's conjecture, namely stability and endoscopic transfer. For these, we have nothing to contribute.

(16.4) Distinguished representations. In the proof of Theorem 16.1, we were led to consider two cases, depending on whether π has non-zero $(F \times K)$-coefficient or not. In fact, a further analysis shows that the second case never occurs, so that if π is as in Theorem 16.1, then π always supports some $(F \times K)$-coefficient. Indeed, one can prove:

(16.5) Theorem *Suppose that π is a cuspidal representation of G_2 which is not nearly equivalent to a generic representation. If the only non-zero Fourier coefficients of π are those corresponding to $F \times F \times F$, then $\pi \subset V_{\chi_0}$ with χ_0 the trivial character. In other words, π is an element of the cubic unipotent A-packet associated to $F \times F \times F$.*

Recall that the generic Fourier coefficients along the Heisenberg parabolic P of G_2 are parametrized by étale cubic F-algebras. One says that a cuspidal representation is *E-distinguished* if the only non-zero generic Fourier coefficients supported by π are those corresponding to the étale cubic algebra E. It is natural to conjecture that the E-distinguished representations are precisely those contained in the cubic unipotent packets attached to E. The above theorem is a preliminary step towards a proof of this. We postpone its proof to a future paper, where hopefully we have a more complete result.

References

[A1] J. Arthur, *On some problems suggested by the trace formula*, in Lie Groups Representations II, Lecture Notes in Math. 1041 (1983), Springer-Verlag, 1-49.

[A2] J. Arthur, *Unipotent automorphic representations: conjectures*, in Orbites Unipotentes et Representations, Asterique Vol. 171-172 (1989), Lecture Notes in Math. 1041 (1983),

[AB] J. Adams and D. Barbasch, *Reductive dual pair correspondence for complex groups*, Journal of Functional Analysis 132 (1995), 1-42.

[AJ] J. Adams and J. Johnson, *Endoscopic groups and packets of nontempered representations*, Compositio Math. 64 (1987), 271-309.

[B] C. Bushnell, *Representations of reductive p-adic groups: localizations of Hecke algebras and applications*, J. of London Math. Soc. 63, No. 2 (2001), 364-386.

[BV] D. Barbasch and D. Vogan, *Unipotent representations of complex semisimple groups*, Annals of Math. 121 (1985), 41-110.

[CBD] N. Conze-Berline and M. Duflo, *Sur les representations induites des groupes semi-simples complexes*, Compositio Math. 34 (1977), 307-336.

[DM] J. Dixmier and P. Malliavin, *Factorisations de fonctions et de vecteurs indefiniment differentiables*, Bull. Sciences. Math. 102 (1978), No. 4, 305-330.

[D] M. Duflo, *Representation unitaires irreductibles des groupes simples complexes de rang deux*, Bull. Math. Soc. France 107 (1979), 55-96.

[G] W. T. Gan, *Multiplicity formula for cubic unipotent Arthur packets*, preprint (2003).

[G2] W. T. Gan, *A Siegel-Weil formula for exceptional groups*, J. Reine Angew. Math. 528 (2000), 149-181.

[G3] W. T. Gan, *Saito-Kurokawa space of $PGSp_4$ and its transfer to inner forms*, preprint (2004), available at **http://www.math.ucsd.edu/˜ wgan**.

[GGJ] W. T. Gan, N. Gurevich and D.-H. Jiang, *Cubic unipotent Arthur parameters and multiplicities of square-integrable automorphic forms*, Invent. Math. 149 (2002), 225-265.

[GGS] W. T. Gan, B. H. Gross and G. Savin, *Fourier coefficients of modular forms on G_2*, Duke Math. J. 115, No. 1 (2002), 105-169.

[GJ] D. Ginzburg and D.-H. Jiang, *Periods and liftings from G_2 to C_3*, Israel Math. J. 123 (2001), 29-59.

[GS] . T. Gan and G. Savin, *Endoscopic lifts from PGL_3 to G_2*, Compositio Math. 140 (2004), 793-808.

[GRS] D. Ginzburg, S. Rallis and D. Soudry, *On the automorphic theta representation for simply-laced groups*, Israel J. of Math. 100 (1997), 61-116.

[GRS2] D. Ginzburg, S. Rallis and D. Soudry, *A tower of theta correspondences for G_2*, Duke Math. J. 88 (1997), 537-624.

[HMS] J.-S. Huang, K. Magaard and G. Savin, *Unipotent representations of G_2 arising from the minimal representations of D_4^E*, J. Reine Angew Math. 500 (1998), 65-81.

[I] T. Ikeda, *On the theory of Jacobi forms and the Fourier-Jacobi coefficients of Eisenstein series*, J. Math. Kyoto Univ. 34 (1994), 615-636.

[J] C. Jantzen, *Some remarks on degenerate principal series*, Pacific Math. J. 186, No. 1 (1998), 67-87.

[K1] S. Kudla, *On the local theta correspondence*, Invent. Math. 83 (1986), 229-255.

[K2] S. Kudla, *Notes on the local theta correspondence*, personal notes (1996), available at **http://www.math.umd.edu/˜ssk, castle.dvi**.

[KV] A. Knapp and D. Vogan, *Cohomological induction and unitary representations*, Princeton Math. Series 45, Princeton Univ. Press (1995).

[LS] J.-S. Li and J. Schwermer, *Construction of automorphic forms and related cohomology classes for arithmetic subgroups of G_2*, Compositio Math. 87 (1993), 45-78.

[LTZ] J.-S. Li, E.-C. Tan and C.-B. Zhu, *Tensor product of degenerate principal series and local theta correspondence*, Journal of Functional Analysis 186 (2001), 381-431.

[M] G. Muic, *Unitary dual of p-adic G_2*, Duke Math. J. 90 (1997), 465-493.

[MVW] C. Moeglin, M.-F. Vigneras and J.-L. Waldspurger, *Correspondances de Howe sur un corps p-adique*, Lecture Notes in Math. 1291 (1987), Springer Verlag.

[RS] S. Rallis and G. Schiffmann, *Theta correspondence associated to G_2*, American J. of Math. 111 (1989), 801-849.

[RS2] S. Rallis and G. Schiffmann, *Discrete spectrum of the Weil representation*, Bull. Amer. Math. Soc. 83 (1977), 267-270.

[V] D. Vogan, *The unitary dual of G_2*, Invent Math. 116 (1994), 677-791.

[VZ] D. Vogan and G. Zuckerman, *Unitary representations with non-zero cohomology*, Compositio Math. 53 (1984), 51-90.

[W1] J.-L. Waldspurger, *Correspondance de Shimura*, J. Math. Pures et Appl. 59 (1980), 1-133.

[W2] J.-L. Waldspurger, *Correspondance de Shimura et quaternions*, Forum Math. 3 (1991), 219-307.

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