משוואות לינאריות (המשך)

משוואת אוילר מסדר 2 $x^2 y''+a_1 x\,y'+a_0\,y=0 \quad \left(a_1,\,a_0\in R\right)$.

דרך 1 אם אם $x>0$ ההצבה $y=x^\lambda$ (אם $x<0$ אז ההצבה $y=(-x)^\lambda$) במשוואת אוילר נותנת

$\lambda^2+(a_1-1)\lambda+a_0=0$.

λ_1 , λ_2 פתרונות של משואה זאת. אזי

1) $\lambda_1,\lambda_2\in R \quad ,\lambda_1\neq\lambda_2\Rightarrow y=C_1|x|^{\lambda_1}+C_2|x|^{\lambda_2}$

2) $\lambda_1,\lambda_2\in R \quad ,\lambda_1=\lambda_2=\lambda\Rightarrow y=|x|^\lambda\,(C_1+C_2\ln x)$

3) $\alpha,\beta\in R, \quad \lambda_{1,2}=\alpha\pm\beta i\Rightarrow y=|x|^\alpha\,(C_1\cos(\beta\ln|x|)+C_2\sin(\beta\ln|x|))$

$$\left(a_1,\,a_0,a,b\in R\right)\qquad \begin{cases}(ax+b)^2 y"+a_1(ax+b)\,y'+a_0\,y=0\\ y=(ax+b)^\lambda\end{cases}\Rightarrow \lambda^2+(a_1-1)\lambda+a_0=0$$

דרך 2 ההצבה $x=e^t$

$$x=e^t\Rightarrow\ t=\ln x\Rightarrow\ y=y(t(x))\Rightarrow\ y'_x=y'_t\cdot t'_x\Rightarrow\ y'_x=y'_t\cdot\frac{1}{x},\quad y''_{xx}=y''_{tt}\cdot\frac{1}{x^2}-y'_t\cdot\frac{1}{x^2}$$

$$\{x^2y"+a_1xy'+a_0\,y=0,\ \ y'_x=y'_t/x,\quad y''_{xx}=(y''_{tt}-y'_t)/x^2\}\quad\Rightarrow\ y''_{tt}+(a_1-1)\,y'_t+a_0\,y=0$$

משוואת אוילר מסדר n

$$(ax+b)^n y^{(n)}+a_{n-1}(ax+b)^{n-1}y^{(n-1)}+\cdots+a_1(ax+b)\,y'+a_0y=0$$

$$a_{n-1},a_{n-2},\ldots,a_0\in R,$$

ההצבה $y=(ax+b)^\lambda$

דוגמאות

1. $2(x+5)^2y"+3(x+5)\,y'-y=0,\quad \left.\begin{matrix}2(x+5)^2y"+3(x+5)\,y'-y=0\\ y=(x+5)^\lambda\end{matrix}\right\}\Rightarrow$

$$2\lambda(\lambda-1)+3\lambda-1=0\Rightarrow 2\lambda^2+\lambda-1=0\Rightarrow\lambda_1=\frac{1}{2},\lambda_2=-1\Rightarrow y=C_1|x+5|^{0.5}+C_2(x+5)^{-1}$$

2. $x^3y'''+xy'-y=0$

$$\left.\begin{matrix}x^3y'''+xy'-y=0\\ y=x^\lambda\end{matrix}\right\}\Rightarrow\lambda(\lambda-1)(\lambda-2)+\lambda-1=0\Rightarrow(\lambda-1)(\lambda^2-2\lambda+1)=0\Rightarrow$$

$$\lambda_1=\lambda_2=\lambda_3=1\ \Rightarrow\qquad y=C_1x+C_2x\ln|x|+C_3x\ln^2|x|$$

3. $x^2y"+3xy'+5y=0$

$$\left.\begin{matrix}x^2y"+3xy'+5y=0\\ y=x^\lambda\end{matrix}\right\}\Rightarrow\lambda^2+2\lambda+5=0\Rightarrow\lambda_{1,2}=-1\pm2i\Rightarrow y=\frac{C_1\cos(2\ln|x|)+C_2\sin(2\ln|x|)}{x}$$

4. $x^3y'''-3x^2y''+6xy'-6y=x^{3.5}$ $\quad y=y_h+y_p$

$a)$ $\left.\begin{matrix}x^3y'''-3x^2y''+6xy'-6y=0\\ y=x^\lambda\end{matrix}\right\}\Rightarrow \lambda(\lambda-1)(\lambda-2)-3\lambda(\lambda-1)+6\lambda-6=0\Rightarrow$

$(\lambda-1)[\lambda(\lambda-2)-3\lambda+6]=0\Rightarrow(\lambda-1)(\lambda^2-5\lambda+6)=0\Rightarrow \lambda_1=1,\lambda_2=2,\lambda_3=3\Rightarrow$

$y_1=x, y_2=x^2, y_3=x^2$

$b)$ $y_p=M(x)x+N(x)x^2+K(x)x^3$

$$\begin{cases}M'x+N'x^2+K'x^3=0\\ M'+2N'x+3K'x^2=0\\ \qquad 2N'+6K'x=x^{0.5}\end{cases}\Rightarrow M'=0.5x^{1.5},\ N'=-x^{0.5},\ K'=0.5x^{-0.5}\Rightarrow$$

$$M=0.2x^{2.5},\quad N=-\frac{2}{3}x^{1.5},\quad K=x^{0.5}\Rightarrow y_p=\frac{1}{5}x^{2.5}x-\frac{2}{3}x^{1.5}x^2+x^{0.5}x^3=\frac{8}{15}x^{3.5}$$

$$\underline{y=C_1x+C_2x^2+C_3x^3+\frac{8}{15}x^{3.5}}$$

תרגילים

I . פתור את המשוואות

1) $x^2y''-4xy'+6y=0,$

2) $x^2y'''=2y',$

3) $x^2y''-3xy'+5y=0,$

4) $x^3y''''-6xy''=0,$

5) $x^4y''''+6x^3y'''+6x^2y''-2y=0,$

6) $(x-2)^2y''-3(x-2)y'+4y=0,$

7) $(2x+3)^2y''+2(2x+3)y'-4y=0,$

8) $x^2y''+xy'+4y=10x,$

9) $x^3y''-2xy=6\ln x$

10) $x^2y''-6y=5x^3+8x^2,$

11) $x^2y''-xy'+y=2x,$

12) $x^2y''-2xy'+2y+x-2x^3=0,$

13) $x^2y''-xy'-3y=0$

14) $x^3y'''+xy'-y=0$

15) $x^2y''-xy'+y=\frac{\ln x}{x}+\frac{x}{\ln x}$

16) $(x-2)^2y''-3(x-2)y'+4y=x,$

17) $x^2y''-2y=\sin(\ln x)$

18) $x^3y'''-3x^2y''+6xy'-6y=\frac{x^4}{\sqrt{x}+1}$

II חזרה

פתור משוואות :

1. $xy' - y = xe^{y/x}$
2. $t\,s' = s + t^2 \sin^2 t$
3. $(1-x^2)t' - xt = xt^2,\ t(0) = 0.5$
4. $2x\cos^2 y\,dx + (8\sqrt[3]{y} - x^2 \sin 2y)\,dy = 0,\ y(1) = 0$
5. $\cos y\,dx + (\sin y + e^x)\,dy = 0$
6. $2xy\,dx - (3y + x^2)\,dy = 0$
7. $(x + y + 1)\,dx = (2x + 2y - 1)\,dy$
8. $(x + ye^{-1/y})\,y' = y^2$
9. $r' - r\tan\varphi + r^2\cos\varphi = 0$
10. $y''' = x^{-3}$
11. $xy'' = y'(\ln y' - \ln x)$
12. $y''\tan y = 2(y')^2$
13. $2y(y')^3 + y'' = 0,\ y(0) = 0,\ y'(0) = -3$
14. $x^2 y'' - xy' + y = 0$
15. $y'' - 16y = 0$
16. $y'' + 16y = 0$
17. $y''' + 9y' = 0$
18. $y^V + 16y' = 0$
19. $y^{IV} - 4y''' + 13y'' = 0$
20. $y'' + 4y' + 4y = 3xe^{-2x}$
21. $y'' - 4y' = 4x - 5 + 10e^x \cos x$
22. $x'' + x = \cos t + \cos 2t$
24. $y'' + y = \cos^{-3} x$
23. $y'' - 2y' + 10y = 74\sin 3x,\ y(0) = 6,\ y'(0) = 3$
25. $x^2 y'' - 3x\,y' + 4y = \ln x$
26. $x^2 y'' + x\,y' + 4y = \sin(\ln x)$
27. $y'' + \dfrac{\mu}{x^2} y = 0 \qquad a)\,\mu > 0.25 \quad b)\,\mu = 0.25 \quad c)\,\mu < 0.25$
28. $y''' + y' = \dfrac{\sin x}{\cos^2 x}$
29. $x^3 y''' - x\,y' - 3y = x^2$

תשובות :

I

1) $y = C_1 x^2 + C_2 x^3$
2) $y = C_1 + C_2 \ln|x| + C_3 x^3$
3) $y = x^2(C_1 \cos(\ln|x|) + C_2 \sin(\ln|x|))$
4) $y = C_1 + C_2 \ln|x| + C_3 x + C_4 x^5$
5) $y = C_1 |x|^{\sqrt{2}} + C_2 |x|^{-\sqrt{2}} + C_3 \cos(\ln|x|) + C_4 \sin(\ln|x|)$
6) $y = C_1 (x-2)^2 + C_2 (x-2)^2 \ln|x-2|$
7) $y = C_1(2x+3) + C_2(2x+3)^{-1}$
8) $y = C_1 \cos(2\ln|x|) + C_2 \sin(2\ln|x|) + 2x$
9) $y = C_1 x^2 + \dfrac{1}{x}\left(C_2 - \dfrac{2}{3}\ln x - \ln^2 x\right)$
10) $y = C_1 x^{-2} + C_2 x^3 + x^3 \ln|x| - 2x^2$
11) $y = C_1 x + C_2 x \ln|x| + x\ln^2|x|$
12) $y = C_1 x + C_2 x^2 + x\ln|x| + x^3$
13) $y = C_1 x^{-1} + C_2 x^3$
14) $y = x(C_1 + C_2 \ln|x| + C_3 \ln^2|x|)$
15) $y = x[C_1 + (C_2 + \ln|\ln x|)\ln x] + \dfrac{1 + \ln x}{4x}$
16) $y = (x-2)^2(C_1 + C_2 \ln|x-2|) + x + 1.5$
17) $y = C_1 x^{-1} + C_2 x^2 + 0.1\cos(\ln x) - 0.3\sin(\ln x)$
18) $y = C_1 x + C_2 x^2 + C_3 x^3 + \dfrac{8}{15}x^{3.5} - \dfrac{5}{3}x^{2.5} + x^{1.5} - (x - 2x^2 + x^3)\ln(\sqrt{x} + 1)$

II תשובות (חזרה)

1. $1+e^{y/x}\ln|Cx|=0$ 2. $s=\frac{t^2}{2}-\frac{t}{4}\sin 2t+Ct$ 3. $t=\frac{1}{3\sqrt{|1-x^2|}-1}$

4. $x^2\cos^2 y+6y\sqrt[3]{y}-1=0$ 5. $y=C+e^{-x}\cos y$ 6. $x^2=y(C+3\ln|y|),\ y=0$

7. $C+2y=x+\ln|x+y|\ ,\ y=-x$ 8. $C+\ln|y|=xe^{1/y}$

9. $r=\frac{1}{(\varphi+C)\cos\varphi}\ ,\ r=0$ 10. $y=0.5\ln|x|+C_1x^2+C_2x+C_3$

11. $y=\frac{1}{C_1}e^{C_1x+1}\left(x-\frac{1}{C_1}\right)+C_2,\ \ y=\frac{e}{2}x^2+C$

12. $C_1x+C_2=-\cot y,\begin{cases}y=C\\ C\neq 0.5\pi+k\pi,\ k\in Z\end{cases}$ 13. $3x=y^3-y$

14. $y=x(C_1+C_2\ln|x|)$

15. $y=C_1e^{4x}+C_2e^{-4x}$ 16. $y=C_1\cos 4x+C_2\sin 4x$

17. $y=C_1+C_2\cos 3x+C_3\sin 3x$

18. $y=C_1+e^{\sqrt{2}x}\left(C_2\cos(x\sqrt{2})+C_3\sin(x\sqrt{2})\right)+e^{-\sqrt{2}x}\left(C_4\cos(x\sqrt{2})+C_5\sin(x\sqrt{2})\right)$

19. $y=C_1+C_2\,x+e^{2x}\left(C_3\cos 3x+C_4\sin 3x\right)$

20. $y=e^{-2x}(C_1+C_2\,x+0.5x^3)$

21. $y=C_1+C_2e^{4x}-0.5x^2+x-e^x(2\cos x+\sin x)$

22. $x=C_1\cos t+C_2\sin t+\frac{1}{2}t\sin t-\frac{1}{3}\cos 2t$

23. $y=e^x(\sin 3x-6\cos 3x)+12\cos 3x+2\sin 3x$

24. $y=C_1\cos x+C_2\sin x+\frac{1}{2\cos x}$

25. $y=x^2(C_1+C_2\ln x)+0.25\ln x+0.25$

26. $y=C_1\cos(2\ln x)+C_2\sin(2\ln x)+\frac{1}{3}\sin(\ln x)$

27. *a*) $r_1=0.5+\sqrt{0.25-\mu}\quad r_2=0.5-\sqrt{0.25-\mu},\quad y=C_1|x|^{r_1}+C_2|x|^{r_2}$

b) $r_{1,2}=0.5,\qquad y=|x|^{0.5}(C_1+C_2\ln|x|)$

c) $r_{1,2}=0.5\pm\sqrt{0.25-\mu}=0.5\pm\tau\,i,\quad y=|x|^{0.5}(C_1\cos(\tau\ln|x|)+C_2\sin(\tau\ln|x|))$

28. $y_p=\frac{1}{\cos x}+\cos x\ln|\cos x|+\sin x(x-\tan x)=\cos x+\cos x\ln|\cos x|+x\sin x$

$y=\cos x(C_1+\ln|\cos x|)+(C_2+x)\sin x+C_3$

29. $y=C_1x^3+C_2\cos(\ln|x|)+C_3\sin(\ln|x|)-0.2x^2$