Multivariable Matrix-valued moment problems

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The classical moment problem

Full moment problem:

(i) Given \( \{s_m\}_{m \in \mathbb{N}_0} \) we wish to determine whether or not there exists \( \sigma \in m^+ (\mathbb{R}) \) such that

\[
s_m = \int_{\mathbb{R}} x^m d\sigma(x), \quad m \in \mathbb{N}_0 := \{0, 1, \ldots\}. \tag{1}
\]

(ii) Describe all \( \sigma \) which obey (1).

The cases when \( \text{supp} \sigma \subseteq [0, \infty), \mathbb{R}, \) and \( [c, d] \) are due to Stieltjes, Hamburger, and Hausdorff, respectively.

Observation: If \( \sigma \) satisfies (1) then

\[
\sum_{a,b=0}^n z_a \bar{z}_b s_{a+b} = \sum_{a,b=0}^n z_a \bar{z}_b \int_{\mathbb{R}} x^{a+b} d\sigma(x)
\]

\[
= \int_{\mathbb{R}} \left| \sum_{a=0}^n z_a x^a \right|^2 d\sigma(x) \geq 0.
\]
**Hamburger’s theorem**

Theorem (Hamburger’s theorem, 1921): \( \{s_m\}_{m \in \mathbb{N}_0} \) has a representing measure if and only if

\[
\sum_{a,b=0}^{n} z_a \bar{z}_b s_{a+b} \geq 0 \iff (s_{a+b})_{a,b=0}^n := \begin{pmatrix}
    s_0 & \cdots & s_n \\
    \vdots & \ddots & \vdots \\
    s_n & \cdots & s_{2n}
\end{pmatrix} \succeq 0,
\]

for all finite subsets \( \{z_1, \ldots, z_n\} \subset \mathbb{C} \).

Remarks on proofs of Hamburger’s theorem:

- Hamburger’s original proof is around 150 pages.

- The operator theory proof is much shorter. [Krein, 1949] generalized this result to the case when one is a given \( \{S_m\}_{m \in \mathbb{N}_0} \).
The full multidimensional $K$-moment problem

**Full $K$-moment problem on $\mathbb{R}^d$:** Given $\{s_m\}_{m \in \mathbb{N}_0^d}$ and $K \subseteq \mathbb{R}^d$, we wish to determine whether or not there exists $\sigma$ such that

$$s_m = \int_{\mathbb{R}^d} x^m d\sigma(x) := \int \cdots \int_{\mathbb{R}^d} x_1^{m_1} \cdots x_d^{m_d} d\sigma(x_1, \ldots, x_d), \quad m \in \mathbb{N}_0^d.$$  \hfill (2)

and

$$\text{supp } \sigma \subseteq K.$$  \hfill (3)

**Observation:** If $\{s_m\}_{m \in \mathbb{N}_0^d}$ has a representing measure then

$$\sum_{0 \leq |m|, |\tilde{m}| \leq n} z_m \tilde{z}_m s_{m+\tilde{m}} \geq 0, \quad n \in \mathbb{N}_0.$$  \hfill (4)

**Caveat:** There do exist sequences $\{s_m\}_{m \in \mathbb{N}_0^d}$ which satisfy (4) yet do not satisfy (2).
Solutions to the full multidimensional $K$-moment problem

- A solution to the full $K$-moment problem on $\mathbb{R}$ in [M. Riesz, 1923].

- A $d$-variable ($d > 1$) generalization was achieved in [Haviland, 1936].

- When $K$ is compact and semi-algebraic, [Schmüdgen, 1991] has a solution to the full $K$-moment problem based on the approach in [Haviland, 1936].

- Subsequently, [Putinar & Vasilescu, 1999] improved upon this approach.
**Truncated Hamburger Moment Problem**

**Problem:** Given a real-valued sequence \( \{ s_k \}_{0 \leq k \leq n} \), we wish to determine necessary and sufficient conditions on \( \{ s_k \}_{0 \leq k \leq n} \) so that there exists \( \sigma \) so that

\[
s_k = \int_{\mathbb{R}} x^k d\sigma(x), \quad 0 \leq k \leq n.
\]

**Remarks:** When \( n = 2m \) then a solution exists if and only if we can choose \( s_{2m+1} \) and \( s_{2m+2} \) such that

\[
\begin{pmatrix}
S_0 & \cdots & S_{m-1} & S_m & S_{m+1} \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
S_{m-1} & \cdots & S_{2m-2} & S_{2m-1} & S_{2m} \\
S_m & \cdots & S_{2m-1} & S_{2m} & S_{2m+1} \\
S_{m+1} & \cdots & S_{2m} & S_{2m+1} & S_{2m+2}
\end{pmatrix} \succeq 0.
\]

In this case, one can find \( \sigma = \sum_{j=1}^r \rho_j \delta_{x_j} \), where \( r = \text{rank } (s_{i+j})_{i,j=0}^m \).
**Multivariable THMPs**

Truncated Hamburger moment problem (even total degree): Given $\{s_m\}_{0 \leq |m| \leq 2n}$ we wish to determine whether or not there exists $\sigma$ such that

$$s_m = \int_{\mathbb{R}^d} x^m d\sigma(x) := \int \cdots \int_{\mathbb{R}^d} x_1^{m_1} \cdots x_d^{m_d} d\sigma(x_1, \ldots, x_d).$$

Given $\{s_m\}_{0 \leq |m| \leq 2n}$ we can construct the following moment matrix:

$$\mathcal{M}(n) := (s_m + \tilde{m})_{0 \leq |m|, |\tilde{m}| \leq n}.$$

When $d = 2$ and $n = 1$,

$$\mathcal{M}(1) := \begin{pmatrix} s_{00} & s_{01} & s_{10} \\ s_{01} & s_{02} & s_{11} \\ s_{10} & s_{11} & s_{20} \end{pmatrix}$$
Flat extension theory of Curto and Fialkow

Given \( \{s_m\}_{0 \leq |m| \leq 2n} \), we call \( \mathcal{M}(n + 1) \) a flat extension of \( \mathcal{M}(n) \) when there exist new data \( \{s_{\tilde{m}}\}_{2n+1 \leq |\tilde{m}| \leq 2n+2} \) such that

1. \( \mathcal{M}(n + 1) = (s_{m+\tilde{m}})_{0 \leq |m|, |\tilde{m}| \leq n+1} \geq 0 \)
2. rank \( \mathcal{M}(n + 1) = \text{rank} \mathcal{M}(n) \).

For instance, when we are given \( \{s_{00}, s_{01}, s_{10}, s_{02}, s_{11}, s_{20}\} \) so that \( \mathcal{M}(1) \geq 0 \), we wish to find \( \{s_{03}, s_{12}, s_{21}, s_{30}, s_{04}, s_{13}, s_{22}, s_{31}, s_{40}\} \) so that

\[
\mathcal{M}(2) := \begin{pmatrix}
    s_{00} & s_{01} & s_{10} & s_{02} & s_{11} & s_{20} \\
    s_{01} & s_{02} & s_{11} & s_{03} & s_{12} & s_{21} \\
    s_{10} & s_{11} & s_{20} & s_{12} & s_{21} & s_{30} \\
    s_{02} & s_{03} & s_{12} & s_{04} & s_{13} & s_{22} \\
    s_{11} & s_{12} & s_{21} & s_{13} & s_{22} & s_{31} \\
    s_{20} & s_{21} & s_{30} & s_{22} & s_{31} & s_{40}
\end{pmatrix} \geq 0 \text{ and rank } \mathcal{M}(1) = \text{rank} \mathcal{M}(2). \]

Theorem (Curto and Fialkow, 1996): The even total degree HMP has a solution if and only if \( \mathcal{M}(n) \) eventually admits a flat extension.
Problem: Given \( \{S_m\}_{0 \leq m \leq n} \), we wish to find \( \Sigma := (\sigma_{jk})_{j,k=1}^p \) such that

\[
S_m = \int_{\mathbb{R}} x^m d\Sigma(x) := \left( \int_{\mathbb{R}} x^m d\sigma_{jk}(x) \right)_{j,k=1}^p , \quad 0 \leq m \leq n.
\]

This problem has been studied by

- [Dym, 1989];
- [Bolotnikov, 1996];
- [Bakonyi & Woerdeman, 2011].

Under analogous conditions, one can find \( \sigma = \sum_{j=1}^r T_j \delta_{x_j} \), where \( \sum_{j=1}^r \text{rank } T_j = \text{rank } (S_{i+j})_{i,j=0}^n \).
An illustrative example

Given the sequence \( \{s_{(k,\ell)}\}_{0 \leq k+\ell \leq 3} \), suppose

\[
\Phi = \begin{pmatrix}
  s_{00} & s_{01} & s_{10} \\
  s_{01} & s_{02} & s_{11} \\
  s_{10} & s_{11} & s_{20}
\end{pmatrix} = \begin{pmatrix}
  3 & 1 & 2 \\
  1 & 1 & 1 \\
  2 & 1 & 2
\end{pmatrix} > 0,
\]

\[
\Phi_1 = \begin{pmatrix}
  s_{10} & s_{11} & s_{20} \\
  s_{11} & s_{12} & s_{21} \\
  s_{20} & s_{21} & s_{30}
\end{pmatrix} = \begin{pmatrix}
  2 & 1 & 2 \\
  1 & 1 & 1 \\
  2 & 1 & 2
\end{pmatrix},
\]

and

\[
\Phi_2 = \begin{pmatrix}
  s_{01} & s_{02} & s_{11} \\
  s_{02} & s_{03} & s_{12} \\
  s_{11} & s_{12} & s_{21}
\end{pmatrix} = \begin{pmatrix}
  1 & 1 & 1 \\
  1 & 1 & 1 \\
  1 & 1 & 1
\end{pmatrix}.
\]
An illustrative example continued

Realize that

\[ \Theta_1 = \Phi^{-1}\phi_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \]

and

\[ \Theta_2 = \Phi^{-1}\phi_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \]

commute.

Also the eigenvalues of \( \Theta_1 \) and \( \Theta_2 \) are \( \{0, 1, 1\} \) and \( \{0, 0, 1\} \), respectively.

Put \((x_1, y_1) = (0, 0), (x_2, y_2) = (1, 0) \) and \((x_3, y_3) = (1, 1)\).

Then \( \sigma = \sum_{j=1}^{3} \delta_{(x_j,y_j)} \) is a solution!
Indexing sets

- We say a finite set \( \Gamma \subset \mathbb{N}_0^d \) is a lattice set when for all \( \gamma \in \Gamma \) there exist \( \gamma_1 = 0_d, \gamma_2, \ldots, \gamma_k \in \Gamma \) and \( j_1, \ldots, j_k \in \{1, \ldots, d\} \) so that \( \gamma_2 = \gamma + e_{j_1}, \ldots, \gamma = \gamma_k + e_{j_k} \), where \( k = |\gamma| \).

- We say a finite set \( \Gamma \subset \mathbb{N}_0^d \) is lower inclusive when for any \( m = (m_1, \ldots, m_d) \in \mathbb{N}_0^d \) and \( \gamma = (g_1, \ldots, g_d) \in \Gamma \) with \( m_j \leq g_j, 1 \leq j \leq d \), we have that \( m \in \Gamma \).

- Note that \( \{(0, 0), (0, 1), (1, 1)\} \subset \mathbb{N}_0^2 \) is a lattice set but not a lower inclusive set. Also \( \{(0, 0), (1, 1)\} \subset \mathbb{N}_0^2 \) is not a lattice set.
Problem: Let $K \subseteq \mathbb{R}^d$, $\Gamma \subset \mathbb{N}_0^d$ be a lattice set and $\{S_\gamma\}_{\gamma \in \Gamma}$ be given. We wish to determine whether or not there exists $\Sigma$ so that

(i) $S_\gamma = \int_{\mathbb{R}^d} x^\gamma d\Sigma(x)$, for all $\gamma \in \Gamma$;

(ii) supp $\Sigma \subseteq K$.

- When $\Gamma = \{m \in \mathbb{N}_0^d : 0 \leq |m| \leq 2n\}$ and $\{S_\gamma\}_{\gamma \in \Gamma}$ is scalar-valued then [Curto and Fialkow, 2000 & 2005] analyzed the $K$-moment problem on $\mathbb{R}^d$ and its multidimensional complex analogue.

- [Stochel, 2001] showed that the full $K$-moment problem on $\mathbb{R}^d$ or $\mathbb{C}^d$ has a solution if and only if the truncated $K$-moment problem, where the moments given have indices which are of total degree at most $2n$, has a solution for every $n \in \mathbb{N}$. 


CONSTRUCTION OF MOMENT MATRICES FROM GIVEN DATA

\begin{itemize}
\item Given a lattice set $\Lambda \subset \mathbb{N}_0^d$, we define
\[ \Lambda + \Lambda = \{ \lambda + \tilde{\lambda} : \lambda, \tilde{\lambda} \in \Lambda \} \text{ and } \Lambda + \Lambda + e_j = \{ \lambda + \tilde{\lambda} + e_j : \lambda, \tilde{\lambda} \in \Lambda \}, \]
\[1 \leq j \leq d.\]

\item Put \( \Gamma = (\Lambda + \Lambda) \cup (\Lambda + \Lambda + e_1) \cup \cdots \cup (\Lambda + \Lambda + e_d) \), which will serve as the indexing set for \( \{ S_\gamma \}_{\gamma \in \Gamma} \).

\item Introduce \( \Phi, \Phi_1, \ldots, \Phi_d \) as follows. Index the rows and columns of \( \Phi \) by \( \Lambda \). Let the entry in the row indexed by \( \lambda \) and the column indexed by \( \tilde{\lambda} \) be given by \( S_{\lambda + \tilde{\lambda}} \). That is,
\[ \Phi = (S_{\lambda + \tilde{\lambda}})_{\lambda, \tilde{\lambda} \in \Lambda} \geq 0. \]

\item Similarly, index the rows and columns of \( \Phi_j \) by \( \Lambda \). That is,
\[ \Phi_j = (S_{\lambda + \tilde{\lambda} + e_j})_{\lambda, \tilde{\lambda} \in \Lambda}, \quad 1 \leq j \leq d. \]
\end{itemize}
An example of the construction

Let \( \{S_\gamma\}_{\gamma \in \Gamma} \), where \( \Gamma = \{\gamma \in \mathbb{N}_0^d : 0 \leq |\gamma| \leq 3\} \). Put \( \Lambda = \{(0,0), (0,1), (1,0)\} \) and so then we get the following matrices:

\[
\Phi = (S_{\lambda + \tilde{\lambda}})_{\lambda, \tilde{\lambda} \in \Lambda} = \begin{pmatrix}
S_{00} & S_{01} & S_{10} \\
S_{01} & S_{02} & S_{11} \\
S_{10} & S_{11} & S_{20}
\end{pmatrix} \geq 0,
\]

\[
\Phi_1 = (S_{\lambda + \tilde{\lambda} + e_1})_{\lambda, \tilde{\lambda} \in \Lambda} = \begin{pmatrix}
S_{10} & S_{11} & S_{20} \\
S_{11} & S_{12} & S_{21} \\
S_{20} & S_{21} & S_{30}
\end{pmatrix},
\]

and

\[
\Phi_2 = (S_{\lambda + \tilde{\lambda} + e_2})_{\lambda, \tilde{\lambda} \in \Lambda} = \begin{pmatrix}
S_{01} & S_{02} & S_{11} \\
S_{02} & S_{03} & S_{12} \\
S_{11} & S_{12} & S_{21}
\end{pmatrix}.
\]
Given \( \{S_\gamma\}_{\gamma \in \Gamma} \), where \( \Gamma = \Lambda + \Lambda \), build \( \Phi = (S_{\lambda+\mu})_{\lambda,\mu \in \Lambda} \). Suppose \( \{S_\gamma\}_{\gamma \in \Gamma} \) has a representing measure \( \Sigma = \sum_{q=1}^{k} T_q \delta_{w_q} \), where \( T_1, \ldots, T_k \geq 0 \) and \( w_1, \ldots, w_k \) are distinct points in \( \mathbb{R}^d \).

We say that \( \Sigma \) is minimal when \( \text{rank} \Phi = \sum_{q=1}^{k} \text{rank} T_q \).

Why? For \( m \in \mathbb{N}_0^d \), realize \( S_m := \int \xi^m d\sigma(\xi) = \sum_{q=1}^{k} T_q w_q^m \). One can check that \( \Phi = (V \otimes I_p)^T (T_1 \oplus \cdots \oplus T_k) (V \otimes I_p) \), where

\[
V := \begin{pmatrix}
    w_1^{\lambda_1} & \cdots & w_1^{\lambda_k} \\
    \vdots & \ddots & \vdots \\
    w_k^{\lambda_1} & \cdots & w_k^{\lambda_k}
\end{pmatrix}.
\]
**K-INCLUSIVE EIGENVALUE PROPERTY**

Let $K \subseteq \mathbb{R}^d$ and $W_1, \ldots, W_n \in \mathbb{C}^{n \times n}$. We say $W_1, \ldots, W_d$ has the $K$-inclusive eigenvalue property with respect to the subspace $M \subseteq \mathbb{C}^n$, with $\dim M = k$, if the following conditions are satisfied:

1. $W_j^*$ is $M$-invariant, i.e.

   $$W_j = \begin{pmatrix} \tilde{W}_j & 0 \\ * & * \end{pmatrix} : \bigoplus_{M} M \rightarrow \bigoplus_{M} M, \quad 1 \leq j \leq d;$$

2. There exists an invertible matrix $S$ so that

   $$S^{-1} \tilde{W}_1 S = \text{diag} \left( x_1^{(1)}, \ldots, x_1^{(k)} \right);$$

   $$\vdots$$

   $$S^{-1} \tilde{W}_d S = \text{diag} \left( x_d^{(1)}, \ldots, x_d^{(k)} \right).$$

3. $\left( x_1^{(q)}, \ldots, x_d^{(q)} \right) \in K, \quad 1 \leq q \leq k$. 
Minimal truncated matrix-valued $K$-moment problem on $\mathbb{R}^d$ solution

Theorem (Kimsey & Woerdeman, 2010): Let $K \subseteq \mathbb{R}^d$ and 
$\{S_\gamma\}_{\gamma \in \Gamma}$, where $\Gamma = (\Lambda + \Lambda) \cup (\Lambda + \Lambda + e_1) \cup \ldots \cup (\Lambda + \Lambda + e_d)$, 
be given. There exists a minimal $K$-representing measure $\Sigma$ if:

(i) $\Phi \geq 0$;  
(ii) There exist $\Theta_1, \ldots, \Theta_d$ so that 
    (a) $\Phi \Theta_j = \Phi_j$, $1 \leq j \leq d$;  
    (b) $\Theta_1, \ldots, \Theta_d$ have the $K$-inclusive eigenvalue property with respect to $\mathcal{M} = \text{Ran} \Phi$.

Conversely, if $\Sigma = \sum_{q=1}^k T_q \delta_{w_q}$ is given then there exists a lower inclusive set $\Lambda$ which gives rise to $\Phi, \Phi_1, \ldots, \Phi_d$ which satisfy conditions (i) and (ii).
Remarks on the proof

To prove the sufficiency of conditions (i) and (ii), we use the fact that $\tilde{W}_1, \ldots, \tilde{W}_d$ commute to write

$$\Phi = CC^* \text{ and } \Phi_j = CD_j C^*, \ 1 \leq j \leq d,$$

where $C$ is an injective matrix and $D_1, \ldots, D_d$ are real diagonal matrices.

Next, write $C = \text{col}(C_{\lambda})_{\lambda \in \Lambda}$ and it turns out that

$$S_\gamma = C_0 D_1^{g_1} \cdots D_d^{g_d} C_0^*,$$

for all $\gamma = (g_1, \ldots, g_d) \in \Gamma$. 
To prove the necessity of conditions (i) and (ii), use a construction in [Sauer, 1997] to produce a lower inclusive set \( \tilde{\Lambda} = \{\lambda_1, \ldots, \lambda_k\} \subset \mathbb{N}_0^d \) so that

\[
V := \begin{pmatrix}
    w_1^{\lambda_1} & \cdots & w_1^{\lambda_k} \\
    \vdots & & \vdots \\
    w_k^{\lambda_1} & \cdots & w_k^{\lambda_k}
\end{pmatrix}
\]

is invertible.

Next put \( S_m = \int x^m d\Sigma(x) \) and note that \( S_m = \sum_{q=1}^k T_q w_q^m \). Write \( w_q = (w_1^{(q)}, \ldots, w_d^{(q)}) \in K, 1 \leq q \leq k \). So then we get

\[
\Phi = (V \otimes I_p)^T R (V \otimes I_p) \geq 0 \quad \text{and} \quad \Phi_j = (V \otimes I_p)^T RX_j (V \otimes I_p),
\]

where \( R = T_1 \oplus \cdots \oplus T_k \) and \( X_j = x_j^{(1)} I_p \oplus \cdots \oplus x_j^{(k)} I_p, 1 \leq j \leq d \). Put

\[
\Theta_j = (V \otimes I_p)^{-1} X_j (V \otimes I_p)
\]

so that condition (ii) is satisfied.
Truncated matrix-valued $K$-moment problem on $\mathbb{C}^d$

The power moments of $\Sigma$ are defined by the formula

$$\hat{\Sigma}(m, \tilde{m}) = \int_{\mathbb{C}^d} \bar{z}^m z^{\tilde{m}} d\Sigma(z) := \int \cdots \int_{\mathbb{C}^d} \bar{z}_1^{m_1} \cdots \bar{z}_d^{m_d} z_1^{\tilde{m}_1} \cdots z_d^{\tilde{m}_d},$$

where $m, \tilde{m} \in \mathbb{N}_0^d$.

Note that $\hat{\Sigma}(m, \tilde{m})^* = \hat{\Sigma}(\tilde{m}, m)$.

Problem: Given a lattice set $\Gamma \subset \mathbb{N}_0^d \times \mathbb{N}_0^d$ and $K \subseteq \mathbb{C}^d$, and

$\{S_{(\gamma, \tilde{\gamma})}\}_{(\gamma, \tilde{\gamma}) \in \Gamma}$, we look for $\Sigma$ so that

(i) $\hat{\Sigma}(\gamma, \tilde{\gamma}) = S_{(\gamma, \tilde{\gamma})}$, for all $(\gamma, \tilde{\gamma}) \in \Gamma$;

(ii) $\text{supp } \Sigma \subseteq K$. 

Indexing sets and moment matrices

- Given a lattice set \( \Lambda \subset \mathbb{N}_0^d \times \mathbb{N}_0^d \), we define the set

\[
\Lambda^T = \{ (\mu, \lambda) : (\lambda, \mu) \in \Lambda \}.
\]

For example, if \( \Lambda = \{(0,0), (0,1), (1,0), (2,0)\} \) then \( \Lambda^T = \{(0,0), (0,1), (0,2)(1,0)\} \).

- Put

\[
\Gamma = (\Lambda + \Lambda^T) \cup \bigcup_{j=1}^{d}(\Lambda + \Lambda^T + (0_d, e_j)) \cup (\Lambda + \Lambda^T + (e_j, 0_d)),
\]

which will serve as an indexing set for \( \{S_{(\gamma, \tilde{\gamma})}\}_{(\gamma, \tilde{\gamma})\in \Gamma} \).

- Index the rows of \( \Phi \) by \( \Lambda \) and the columns by \( \Lambda^T \). Let the entry in the row indexed by \( (\alpha, \beta) \in \Lambda \) and the column indexed by \( (\mu, \lambda) \in \Lambda^T \) be given by \( S_{(\alpha, \beta) + (\mu, \lambda)} \). That is,

\[
\Phi = \left( S_{(\alpha+\mu, \beta+\lambda)} \right)_{(\alpha, \beta)\in \Lambda, (\mu, \lambda)\in \Lambda^T}.
\]
Index the rows of $\Phi_{zj}$ by $\Lambda$ and the columns by $\Lambda^T$. Let the entry in the row indexed by $(\alpha, \beta) \in \Lambda$ and the column indexed by $(\mu, \lambda) \in \Lambda^T$ be given by $S(\alpha, \beta) + (\mu, \lambda) + (0_d, e_j)$. That is,

$$\Phi_{zj} = (S(\alpha, \beta) + (\mu, \lambda) + (0_d, e_j))(\alpha, \beta) \in \Lambda, (\mu, \lambda) \in \Lambda^T, \ 1 \leq j \leq d.$$ 

Index the rows of $\Phi_{\bar{z}j}$ by $\Lambda$ and the columns by $\Lambda^T$. Let the entry in the row indexed by $(\alpha, \beta) \in \Lambda$ and the column indexed by $(\mu, \lambda) \in \Lambda^T$ be given by $S(\alpha, \beta) + (\mu, \lambda) + (e_j, 0_d)$. That is,

$$\Phi_{\bar{z}j} = (S(\alpha, \beta) + (\mu, \lambda) + (e_j, 0_d))(\alpha, \beta) \in \Lambda, (\mu, \lambda) \in \Lambda^T, \ 1 \leq j \leq d.$$ 

Note that we necessarily have $\Phi = \Phi^*$ because if $\{S(\gamma, \tilde{\gamma})\}(\gamma, \tilde{\gamma}) \in \Gamma$ has a representing measure then $S(\gamma, \tilde{\gamma}) = S^*(\tilde{\gamma}, \gamma)$, for all $(\gamma, \tilde{\gamma}) \in \Gamma$. Similarly, we have $\Phi_{zj}^* = \Phi_{\bar{z}j}$. 

**Moment matrix construction continued**
An example of the construction

Let $\Lambda = \{(0, 0), (0, 1), (1, 0)\}$. Then $\Lambda^T = \{(0, 0), (1, 0), (0, 1)\}$ and

$$\Gamma = (\Lambda + \Lambda^T) \cup (\Lambda + \Lambda^T + (1, 0)) \cup (\Lambda + \Lambda^T + (0, 1))$$

$$= \{(m, \tilde{m}) \in \mathbb{N}_0^2 : 0 \leq m + \tilde{m} \leq 3\}.$$ 

We get the following matrices:

$$\Phi = \begin{pmatrix}
S_{00} & S_{10} & S_{01} \\
S_{01} & S_{11} & S_{02} \\
S_{10} & S_{20} & S_{11}
\end{pmatrix} \succeq 0,$$

$$\Phi_{\tilde{z}} = \begin{pmatrix}
S_{10} & S_{20} & S_{11} \\
S_{11} & S_{21} & S_{12} \\
S_{20} & S_{30} & S_{21}
\end{pmatrix},$$

and

$$\Phi_z = \begin{pmatrix}
S_{01} & S_{11} & S_{02} \\
S_{02} & S_{12} & S_{12} \\
S_{11} & S_{21} & S_{12}
\end{pmatrix}.$$
Matrix-valued truncated $K$-moment problem on $\mathbb{C}^d$

**Theorem (Kimsey & Woerdeman, 2010):** Let $K \subseteq \mathbb{C}^d$, $\Lambda \subseteq \mathbb{N}_0^d \times \mathbb{N}_0^d$ be a lattice set, and suppose the $\mathbb{C}^{p \times p}$-valued sequence $\{S_{(\gamma, \tilde{\gamma})}\}_{(\gamma, \tilde{\gamma}) \in \Gamma \times \tilde{\Gamma}}$ is given.

If

1. $\Phi \geq 0$;

2. There exist matrices $\Theta_{z_1}, \ldots, \Theta_{z_d}, \Theta_{\tilde{z}_1}, \ldots, \Theta_{\tilde{z}_d}$ which commute with respect to $M = \text{Ran} \Phi$ so that $\Phi \Theta_{z_j} = \Phi_{z_j}$ and $\Phi \Theta_{\tilde{z}_j} = \Phi_{\tilde{z}_j}$, $1 \leq j \leq d$. In addition, $\Theta_{z_1}, \ldots, \Theta_{z_d}$ must satisfy the $K$-inclusive eigenvalue property with respect to $M = \text{Ran} \Phi$.

then there exists a minimal $\Sigma$ such that $S_{(\gamma, \tilde{\gamma})} = \int_{\mathbb{C}^d} \tilde{z}^\gamma z^{\tilde{\gamma}} d\Sigma(z)$ and $\text{supp} \Sigma \subseteq K$. 


Remarks on a converse result

We only have a converse when \( d = 1 \). The following question would lead to a \( d > 1 \) generalization.

**Question:** Given distinct \( u_1, \ldots, u_k \in \mathbb{C}^d \). Can we choose a lattice set \( \Lambda = \{ (\lambda_1, \mu_1), \ldots, (\lambda_k, \mu_k) \} \in \mathbb{N}_0^d \times \mathbb{N}_0^d \) (lower inclusive set) so that

\[
V = \begin{pmatrix}
\bar{u}_1^{\lambda_1} u_1^{\mu_1} & \cdots & \bar{u}_1^{\lambda_k} u_1^{\mu_k} \\
\vdots & \ddots & \vdots \\
\bar{u}_k^{\lambda_1} u_k^{\mu_1} & \cdots & \bar{u}_k^{\lambda_k} u_k^{\mu_k}
\end{pmatrix}
\]
is invertible?
Cubic complex moment problem

Problem: Given \( s := \{ s_{(m_1,m_2)} \}_{0 \leq m_1 + m_2 \leq 3} \) we wish to find \( \sigma \) so that

\[
s(m,\tilde{m}) = \int_{\mathbb{C}} \bar{z}^m z^{\tilde{m}} d\sigma(z).
\]

Let

\[
\Phi = \begin{pmatrix} s_{00} & s_{10} & s_{01} \\ s_{01} & s_{11} & s_{02} \\ s_{10} & s_{20} & s_{11} \end{pmatrix}, \quad \Phi_{\bar{z}} = \begin{pmatrix} s_{10} & s_{20} & s_{11} \\ s_{11} & s_{21} & s_{12} \\ s_{20} & s_{30} & s_{21} \end{pmatrix}, \quad \text{and} \quad \Phi_z = \begin{pmatrix} s_{01} & s_{11} & s_{02} \\ s_{02} & s_{12} & s_{12} \\ s_{11} & s_{21} & s_{12} \end{pmatrix}.
\]

Theorem (Kimsey, 2011): Suppose \( r := \text{rank} \ \Phi = 1, 2 \). Then the following are equivalent:

(i) \( s \) has a representing measure;

(ii) \( s \) has a \( r \)-atomic representing measure;

(iii) \( \Phi \succeq 0 \) and there exist commuting \( \Theta_z, \Theta_{\bar{z}} \) such that

\[
\Phi \Theta_z = \Phi_z \quad \text{and} \quad \Phi \Theta_{\bar{z}} = \Phi_{\bar{z}}.
\]

Suppose \( r = 3 \). Then \( s \) has a 3-atomic representing measure if and only if \( \Phi^{-1} \Phi_z \) and \( \Phi^{-1} \Phi_{\bar{z}} \) commute.
Data that does not admit a minimal representing measure

Example: There does exist a sequence \( s \) with a representing measure, with \( \text{rank } \Phi = 3 \), yet \( s \) does not have a 3-atomic representing measure. Let \( s \) be given by

\[
\Phi = \begin{pmatrix}
4 & 0 & 0 \\
0 & 54 & -50 + 4i \\
0 & -50 - 4i & 54 \\
\end{pmatrix}, \quad \Phi_z = \begin{pmatrix}
0 & -50 - 4i & 54 \\
54 & 0 & 0 \\
-50 - 4i & 0 & 0 \\
\end{pmatrix},
\]

and

\[
\Phi_{\bar{z}} = \begin{pmatrix}
0 & 54 - 4i & -50 + 4i \\
-50 + 4i & 0 & 0 \\
54 & 0 & 0 \\
\end{pmatrix}.
\]

Hence \( \Theta_z = \Phi^{-1} \Phi_z = \begin{pmatrix}
0 & -\frac{25}{2} - i & \frac{27}{2} \\
1 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix} \) and

\[
\Theta_{\bar{z}} = \Phi^{-1} \Phi_{\bar{z}} = \begin{pmatrix}
0 & 0 & 0 \\
0 & \frac{27}{2} & -\frac{25}{2} + i \\
0 & \frac{25}{2} + i & -\frac{27}{2} \\
\end{pmatrix} \text{ which do not commute.}
\]

Note that a 4-atomic representing measure for \( s \) is given by

\[
\sigma = \delta_{5i} + \delta_{1-i} + \delta_{-5i} + \delta_{-1+i}.
\]
Example continued

Suppose there does exist a 3-atomic representing measure for $s$. Then we must have the existence of $s_{22} > 0$, $s_{23}, s_{32}, s_{13}, s_{31} \in \mathbb{C}$ so that

$$
\tilde{\Phi} = \begin{pmatrix}
    s_{00} & s_{01} & s_{02} \\
    s_{10} & s_{11} & s_{12} \\
    s_{20} & s_{21} & s_{22}
\end{pmatrix} = \begin{pmatrix}
    4 & 0 & -50 - 4i \\
    0 & 54 & 0 \\
    -50 + 4i & 0 & s_{22}
\end{pmatrix} > 0,
$$

$$
\tilde{\Phi}_z = \begin{pmatrix}
    s_{01} & s_{02} & s_{03} \\
    s_{11} & s_{12} & s_{13} \\
    s_{21} & s_{22} & s_{23}
\end{pmatrix} = \begin{pmatrix}
    0 & -50 - 4i & 0 \\
    54 & 0 & s_{13} \\
    0 & s_{22} & s_{23}
\end{pmatrix},
$$

and

$$
\tilde{\Phi}_{\bar{z}} = \begin{pmatrix}
    s_{10} & s_{11} & s_{12} \\
    s_{20} & s_{21} & s_{22} \\
    s_{30} & s_{31} & s_{32}
\end{pmatrix} = \begin{pmatrix}
    0 & 54 & 0 \\
    -50 + 4i & 0 & s_{22} \\
    0 & s_{31} & s_{32}
\end{pmatrix}
$$

admit commuting $\tilde{\Theta}_z$, $\tilde{\Theta}_{\bar{z}}$. 
**Tchakaloff’s Theorem**

**Theorem (Kimsey, 2011):** Suppose $\{S_m\}_{0 \leq |m| \leq n}$ is a truncated sequence with representing measure $\Sigma \in M^+(\mathbb{R}^d)$. There exists a finitely atomic $\tilde{\Sigma} \in M^+(\mathbb{R}^d)$, such that $\tilde{\Sigma}$ is a representing measure for $\{S_\lambda\}_{0 \leq |m| \leq n}$ and $\text{supp } \tilde{\Sigma} \subseteq \text{supp } \Sigma$.

Remarks:

- **[Tchakaloff, 1957]** proved a scalar-valued result when the representing measure is absolutely continuous with respect to the Lebesgue measure.

- **[Putinar, 1997]** proved a scalar-valued result when the representing measure has compact support.

- **[Bayer & Teichmann, 2006]** proved a scalar-valued result with no assumptions on the representing measure.

- **[Laurent, 2006]** provided an alternative proof of Bayer & Teichmann’s result which is more algebraic.
Remarks on the proof on Tchakaloff’s theorem

The basic idea is to build a convex cone \( C \) from matrices of the form

\[
\text{col}(x^m)_{0 \leq |m| \leq n} \otimes vv^*
\]

where \( x \in \text{supp } \Sigma \) and \( v \in \mathbb{C}^p \).

Next we have to show that

\[
(S_m)_{0 \leq |m| \leq n} \in \text{ri } C.
\]

Finally use Carathéodory’s theorem to produce the finitely atomic representing measure.