RESEARCH REPORT FOR "HILBERT MODULES" WORKSHOP APRIL 2014

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This is a short report on some topics in *complex geometry and operator theory* that I worked on this year. These topics are (1) spectral sets and distinguished varieties in the symmetrized bidisc, (2) the isomorphism problem for complete Pick algebras, and (3) Arveson's conjecture on essential normality.

1. Spectral sets and distinguished varieties in the symmetrized bidisc

This is joint work with Sourav Pal, who is a postdoc at BGU. We were inspired by the works of Agler–Young and Agler–McCarthy, e.g. [1, 2].

The *closed symmetrized bidisc* is defined to be the set

$$\Gamma = \{(z_1 + z_2, z_1 z_2) : \ |z_1| \leq 1, |z_2| \leq 1\} \subseteq \mathbb{C}^2.$$

Agler and Young proved that every Γ-contraction (that is, a pair of commuting operators for which Γ is a spectral set) has Γ as a complete spectral set, in other words it has a normal $\partial\Gamma$-dilation. Our main result is a refinement of this fact in the flavour of [1].

In the paper [14], we show that for every pair of commuting matrices (S, P), having the closed symmetrized bidisc Γ as a spectral set, there is a one dimensional complex algebraic variety Λ in Γ, such that for every matrix valued polynomial $f(z_1, z_2)$,

$$\|f(S,P)\| \leq \max_{(z_1,z_2)\in\Lambda} \|f(z_1, z_2)\|.$$

The variety Λ is shown to have the determinantal representation

$$\Lambda = \{(s,p) \in \Gamma : \det(F + pF^* - sI) = 0\}, \tag{1.1}$$

where F is the unique matrix of numerical radius not greater than 1 that satisfies

$$S - S^*P = (I - P^*P)^{\frac{1}{2}} F (I - P^*P)^{\frac{1}{2}}.$$

When (S, P) is a strict Γ-contraction, then Λ is a *distinguished variety* in the symmetrized bidisc, i.e. a one dimensional algebraic variety that exits the symmetrized bidisc through its distinguished boundary. We also showed that the distinguished varieties of the symmetrized bidisc can be characterized by a determinantal representation as (1.1) above.

2. The isomorphism problem for complete Pick algebras

The following problem, regarding quotients of the multiplier algebra of Drury-Arveson space, has interested me for the last five years; see [8, 9, 13].

Let H^2_d denote the Drury-Arveson space in d variables, and let $\mathcal{M}_d$ denote its multiplier algebra. If V is a subvariety of the unit ball we denote by $\mathcal{M}_V$ the restriction algebra $\mathcal{M}_d\big|_V$. $\mathcal{M}_V$ can also be identified with the multiplier algebra of the RKHS $\overline{\text{span}}\{k_\lambda : \lambda \in V\}$, where k_λ denotes the kernel function of H^2_d at a point $\lambda \in \mathbb{B}_d$. The questions we are interested in are variants of the following: *What is the relationship between the geometry of V and the structure of the operator algebra $\mathcal{M}_V$?*

In a recent work [7], joint with Ken Davidson and Michael Hartz, we prove the following results (this work might be presented in the workshop by Michael Hartz).

2.1. **An "automatic transversality" result.** The prototypical result in the isomorphism problem is the following theorem due to Alpay, Putinar and Vinnikov [3], which says, roughly, that if V is a disc nicely embedded in $\mathbb{B}_d$ (where $d < \infty$) then $H^\infty(V) = \mathcal{M}_V \cong H^\infty(\mathbb{D})$.

Theorem 2.1 (Alpay-Putinar-Vinnikov). *Suppose that f is an injective holomorphic function of $\mathbb{D}$ onto $V \subset \mathbb{B}_d$ such that*

(1) *f extends to an injective C^2 function on $\overline{\mathbb{D}}$,*
(2) *$f'(z) \neq 0$ on $\overline{\mathbb{D}}$,*
(3) *$\|f(z)\| = 1$ if and only if $|z| = 1$,*
(4) *$\langle f(z), f'(z)\rangle \neq 0$ when $|z| = 1$.*

Then $\mathcal{M}_V$ is equal to $H^\infty(V)$ and isomorphic to $H^\infty(\mathbb{D})$.

Generalizations of this result (with similar conditions) were obtained for the case where the disc is replaced by a planar domain or a finite Riemann surface [4, 13]. We showed that condition (4) is satisfied automatically in this setting. In fact, we proved that if $f : \mathbb{D} \to \mathbb{B}_d$ is a proper analytic map which extends to be C^1 on $\overline{\mathbb{D}}$, then $f(\mathbb{D})$ meets the boundary transversally, that is, $\langle f(z), f'(z)z\rangle > 0$ for all $z \in \partial\mathbb{D}$.

2.2. **An example of an embedded disc V with $\mathcal{M}_V \neq H^\infty(V)$.** We construct an example (actually a class of examples) of a mapping f satisfying all conditions in Theorem 2.1 except that it is not injective on $\partial\mathbb{D}$, for which the conclusion does not hold. This is the first example in finite dimensions of an algebra $\mathcal{M}_V$ for which V is biholomorphic to $\mathbb{D}$, but $\mathcal{M}_V$ is not isomorphic to $H^\infty(\mathbb{D})$.

2.3. **Isomorphisms induce bi-Lipschitz maps on the varieties.** We show that if $\varphi : \mathcal{M}_V \to \mathcal{M}_W$ is an isomorphism, then the induced map $W \to V$ is bi-Lipshitz with respect to the pseudohyperbolic distance. This gives a useful necessary condition, other than that V be biholomorphic to W, for the algebras $\mathcal{M}_V$ and $\mathcal{M}_W$ to be isomorphic.

2.4. **An in-depth study of a family of discs embedded in the ball.** We consider a family of embeddings $f : \mathbb{D} \to \mathbb{B}_\infty$ of the form $f(z) = (b_1 z, b_2 z^2, b_3 z^3, \ldots)$ (where $b_1 \neq 0$). Among other things, we find that these embeddings give rise to an uncountable family of mutually non-isomorphic algebras $\mathcal{M}_{f(\mathbb{D})}$, and we characterize which ones are isomorphic to $H^\infty(\mathbb{D})$. We find that $\mathcal{M}_{f(\mathbb{D})} \cong H^\infty$ if and only if $\sum |b_n|^2 = 1$ and $\sum n|b_n|^2 < \infty$.

3. Arveson's conjecture on essential normality

Versions of Arveson's conjecture on essential normality of graded quotients of the Drury-Arveson Hilbert module have also drawn my attention in last five years, see [11, 15].

Let $S = (S_1, \ldots, S_d)$ denote the compression of the d-shift to the complement of a homogeneous ideal I of $\mathbb{C}[z_1, \ldots, z_d]$. Arveson conjectured that S is essentially normal. In the recent preprint [12], joint work with Matt Kennedy, we establish new results supporting this conjecture, and connect the notion of essential normality to the theory of the C*-envelope and the noncommutative Choquet boundary (some of these results might be touched on in a talk by Matt Kennedy). These results are different from most results on this problem (e.g., [5, 10]) in that they supply partial results for all homogeneous ideals, rather than full results for some subclass of homogeneous ideals.

Let us denote by $\mathcal{O}_I$ the C*-algebra generated by the images of $S_1, \dots, S_d$ in the Calkin algebra. If we denote by $\mathcal{Z}(I)$ the variety corresponding to I, then Arveson's conjecture implies that $\mathcal{O}_I$ is an algebra of continuous functions, and one can show then that $\mathcal{O}_I = C(\mathcal{Z}(I) \cap \partial\mathbb{B}_d)$.

First thing, we prove that the space of multiplicative linear functional of $\mathcal{O}_I$ is equal to $\mathcal{Z}(I)\cap\partial\mathbb{B}_d$, as the conjecture predicts. We then use this information to obtain further results.

We show that the unital norm closed algebra $\mathcal{B}_I$ generated by $S_1, \dots, S_d$ modulo the compact operators is completely isometrically isomorphic to the uniform algebra generated by polynomials on $\mathcal{Z}(I) \cap \overline{\mathbb{B}_d}$. Consequently, the essential norm of an element in $\mathcal{B}_I$ is equal to the sup norm of its Gelfand transform, and the C*-envelope of $\mathcal{B}_I$ is identified as the algebra of continuous functions on $\mathcal{Z}(I) \cap \partial\mathbb{B}_d$. As a consequence, $C^*_{env}(\mathcal{B}_I)$ is a complete invariant of the topology of the variety determined by I.

The identification of the C*-envelope of $\mathcal{B}_I$ suggests a new approach to the problem of essential normality. In fact, by our results Arveson's conjecture is equivalent to the Shilov ideal of $\mathcal{B}_I$ in $C^*(\mathcal{B}_I)$ being trivial, for all I. Motivated by this, we prove that the tuple S is essentially normal if and only if it is hyperrigid as the generating set of a C*-algebra.

References

1. J. Agler and J.E. McCarthy, Distinguished varieties, *Acta Math.* 194 (2005), no. 2, 133–153.
2. J. Agler and N.J. Young, A commutant lifting theorem for a domain in $\mathbb{C}^2$ and spectral interpolation, *J. Funct. Anal.* 161 (1999), 452–477.
3. D. Alpay, M. Putinar and V. Vinnikov, A Hilbert space approach to bounded analytic extension in the ball, *Commun. Pure Appl. Anal.* 2 (2003), 139–145.
4. N. Arcozzi, R. Rochberg and E. Sawyer, Carleson measures for the Drury-Arveson Hardy space and other Besov-Sobolev spaces on complex balls, *Adv. Math.* 218 (2008), 1107–1180.
5. Wm. B. Arveson, p-Summable commutators in dimension d, *J. Operator Theory* 54 (2005), no. 1, 101–117.
6. W.B. Arveson, The noncommutative Choquet boundary II: hyperrigidity, *Israel J. of Math.* 184 (2011), no. 1, 349–385.
7. K.R. Davidson, M. Hartz and O.M. Shalit, Multipliers of embedded discs, *Complex Anal. Oper. Theory*, to appear.
8. K.R. Davidson, C. Ramsey and O.M. Shalit, The isomorphism problem for some universal operator algebras, *Adv. Math.* 228 (2011), 167–218.
9. K.R. Davidson, C. Ramsey and O.M. Shalit, Operator algebras for analytic varieties, *Trans. Amer. Math. Soc.*, to appear.
10. K. Guo, K. Wang, Essentially normal Hilbert modules and K-homology, *Math. Ann.* 340 (2008), 907–934.
11. M. Kennedy and O.M. Shalit, Essential normality and the decomposability of algebraic varieties, *New York J. Math.*, 18 (2012) 877–890.
12. M. Kennedy and O.M. Shalit, Essential normality, essential norms and hyperrigidity, *preprint*, arXiv:1309.3737 [math.OA].
13. M. Kerr, J.E. McCarthy and O.M. Shalit, On the isomorphism question for complete multiplier algebras, *Integral Eqtns. Operator Theory* 76 (2013), 39–53.
14. S. Pal and O.M. Shalit, Spectral sets and distinguished varieties in the symmetrized bidisc, *J. Funct. Anal.*, to appear.
15. O.M. Shalit, Stable polynomial division and essential normality of graded Hilbert modules, *J. Lond. Math. Soc.*, 83 (2011), no. 2, 273–289.