The isomorphism problem for complete Pick algebras

Orr Shalit

Ben-Gurion University of the Negev

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Based on joint works with Davidson, Hartz, Kerr, McCarthy and Ramsey.

Other contributors: Alpay, Arcozzi, Putinar, Rochberg, Sawyer, Vinnikov...

A long long time ago...

A corollary of Gelfand's theory

Theorem (1940s)

Let X and Y be compact Hausdorff topological spaces.
If $C(X)$ is isomorphic to $C(Y)$ then $X \cong Y$.

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In fact, $\varphi : C(X) \to C(Y)$ is an isomorphism, then there exists a homeomorphism $\alpha : Y \to X$ such that

$$\varphi(f) = f \circ \alpha \ , \ f \in C(X).$$

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Key: X is the maximal ideal space of $C(X)$

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Now compute:

$$\varphi(f)(y) = \rho_y(\varphi(f)) = \varphi^*(\rho_y)(f) = f(\alpha(y)).$$

Thus $\varphi(f) = f \circ \alpha$. (The converse is obvious.)

A Theorem of Bers

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Proof: $\mathfrak{M}(H(U))$ is more complicated than U. However

$$U \cong \textbf{principal} \text{ maximal ideals of } H(U) \cong \text{ complex homomorphisms,}$$

and the rest is similar.

And now **our** story begins.

A classical interpolation theorem

Theorem (Pick 1916, Nevanlinna 1919)

Let $z_1, \dots, z_n \in \mathbb{D}$ and $w_1, \dots, w_n \in \mathbb{C}$. There exists $f \in H^\infty$ with

$$f(z_i) = w_i \quad \text{for } 1 \leq i \leq n \quad \text{and} \quad ||f||_\infty \leq 1$$

if and only if the matrix

$$\left(\frac{1 - w_i \overline{w_j}}{1 - z_i \overline{z_j}}\right)_{i,j=1}^n$$

is positive definite.

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The function

$$K(z, w) = \frac{1}{1 - z\overline{w}}$$

is the reproducing kernel of the Hardy space H^2, and

$$H^\infty = \mathrm{Mult}(H^2) := \{f \in H(\mathbb{D}) : M_f h := fh \in H^2 \text{ for all } h \in H^2\}.$$

Pick and complete Pick spaces

Let $\mathcal{H}$ be a Hilbert function space on a set X with kernel K. Given $z_1, \ldots, z_n \in X$ and $w_1, \ldots, w_n \in M_d(\mathbb{C})$, does there exist $f \in \text{Mult}(\mathcal{H}) \otimes M_d$ with $||f||_{\text{Mult}(\mathcal{H}) \otimes M_d} \leq 1$ and

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Definition

If this condition is sufficient, $\mathcal{H}$ is called a complete Pick space. If $\mathcal{H}$ is a complete Pick space then $\mathrm{Mult}(\mathcal{H})$ is called a complete Pick algebra.

Drury-Arveson space and the algebras $\mathcal{M}_V$

Let $d \in \mathbb{N} \cup \{\infty\}$. The Drury-Arveson space H^2_d is the Hilbert function space on $\mathbb{B}_d$ with kernel

$$k(z,w) = k_w(z) = \frac{1}{1 - \langle z, w \rangle} \quad (z, w \in \mathbb{B}_d).$$

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The Drury-Arveson space H^2_d is a complete Pick space.
Moreover, every multiplier algebra of an irreducible complete Pick space can be identified with

$$\mathcal{M}_X := \mathrm{Mult}(H^2_d)\big|_X$$

for some subset $X \subset \mathbb{B}_d$ (possibly $d = \infty$).

* irreducible - the kernel K satisfies $K(x,y) \neq 0$ for all x, y.

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The set X determines a variety $V = V(I(X))$ by

$$V = \{z \in \mathbb{B}_d : f(z) = 0 \text{ for all } f \in \mathrm{Mult}(H^2_d) \text{ with } f|_X = 0\}$$

and $\mathcal{M}_V \simeq \mathcal{M}_X$ (Davidson-Ramsey-S).

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$\mathcal{M}_V$ is an operator algebra (follows from Pick property) and weak$*$ closed

$\mathcal{M}_V = \mathrm{Mult}(\mathcal{H}_V)$, where

$$\mathcal{H}_V = \overline{\mathrm{span}}\{k_\lambda : \lambda \in V\}.$$

The isomorphism problem for $\mathcal{M}_V$

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Problem A

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More concretely:

Problem B

Let $V, W \subset \mathbb{B}_d$ be two varieties. When are $\mathcal{M}_V$ and $\mathcal{M}_W$ isomorphic? Isometrically isomorphic? Completely Isometrically isomorphic? Similar/unitarily equivalent?

(Note: these algebras are semisimple, so algebraic isomorphisms are automatically bounded.)

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For every $\lambda \in V$, there exists $\rho_\lambda \in \pi^{-1}(\lambda)$ given by

$$\rho_\lambda(f) = f(\lambda).$$

We identify $V \subset \mathfrak{M}(\mathcal{M}_V)$, and consider V as the **visible** part of the spectrum.

The maximal ideal space $\mathcal{M}_V$

Theorem (Davidson-Pitts 1998, Davdison-Ramsey-S)

There is a continuous projection $\pi : \mathfrak{M}(\mathcal{M}_V) \to \overline{\mathbb{B}_d}$ *given by*

$$\pi(\rho) = (\rho(z_1), \rho(z_2), \ldots) \ , \ \ \rho \in \mathfrak{M}(\mathcal{M}_V).$$

For each $\lambda \in V$ *there is a unique weak$*$-continuous character* $\rho_\lambda \in \pi^{-1}(\lambda)$ *given by* $\rho_\lambda(f) = f(\lambda)$. *Every weak$*$-continuous character arises this way.*

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If $d < \infty$*, then*

$$\pi(\mathfrak{M}(\mathcal{M}_V)) \cap \mathbb{B}_d = V, \tag{1}$$

$$\pi^{-1}(\lambda) = \{\rho_\lambda\}, \text{ for all } \lambda \in V. \tag{2}$$

If $d = \infty$, then (1) or (2) may fail.

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Theorem (Davidson-Ramsey-S)

Let $V, W \subset \mathbb{B}_d$ be varieties. The following assertions are equivalent:

(i) *$\mathcal{M}_V$ and $\mathcal{M}_W$ are unitarily equivalent.*

(ii) *$\mathcal{M}_V$ and $\mathcal{M}_W$ are completely isometrically isomorphic.*

(iii) *There exists a conformal automorphism $\alpha \in \mathrm{Aut}(\mathbb{B}_d)$ such that*

$$\alpha(W) = V.$$

If $d < \infty$, this is equivalent to

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Proof:

(i) $\Rightarrow$ (ii) is immediate.

Proof (cont'd):

(ii) $\Rightarrow$ (iii): **IF** φ^* maps point evaluation at λ to point evaluation at μ, then

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Thus $\mu = (\varphi(z_1)(\lambda), \ldots, \varphi(z_d)(\lambda)) = \alpha(\lambda)$, where $\alpha = (\varphi(z_1), \ldots, \varphi(z_d))$.

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To show that φ^* preserves point evaluations ($\equiv$ weak$*$ continuous characters) we use that the algebras both have **isometric strongly unique preduals** (Kennedy-Yang).

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(iii) $\Rightarrow$ (i): We write down a unitary

$$Uk_\lambda = c_\lambda k_{\alpha(\lambda)}$$

and this unitary implements an isomorphism between $\mathcal{M}_V$ and $\mathcal{M}_W$.

Algebraic isomorphism for homogeneous varieties

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Theorem (Davidson-Ramsey-S 2011, Hartz 2012)

Let $V, W \subset \mathbb{B}_d$ be homogeneous varieties with $d < \infty$. Then the following are equivalent:

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(iv) *There is an invertible linear map on $\mathbb{C}^d$ which maps V onto W.*

* - A homogeneous variety is the common vanishing locus of homogeneous polynomials.

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(iii) $\Rightarrow$ (iv): Use elementary **several complex variables**.
(iv) $\Rightarrow$ (i): If $A : V \to W$, define $T : k_\lambda \mapsto k_{A\lambda}$.
One shows (very difficult!) that T extends to a bounded isomorphism $\mathcal{H}_V \to \mathcal{H}_W$.

Algebraic isomorphism $\Rightarrow$ biholomorphic equivalence

Theorem (Davidson-Ramsey-S)

Let $V, W \subset \mathbb{B}_d$ be varieties in $\mathbb{B}_d$ with $d < \infty$ which are the union of finitely many irreducible varieties and a discrete variety. If $\mathcal{M}_V$ and $\mathcal{M}_W$ are algebraically isomorphic, then V and W are biholomorphic.

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Questions

1. **What about the converse?**
2. Are the technical conditions (finite union of irreducibles, $d < \infty$) really needed?

Embedded discs

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Question

Is $\mathcal{M}_V$ always isomorphic to H^∞?

A positive result for discs

Theorem (Alpay-Putinar-Vinnikov 2003)

Let $\alpha : \mathbb{D} \to V \subset \mathbb{B}_d$ *be a biholomorphism,* $d < \infty$. *If*

(1) α *extends to an injective* C^2 *function on* $\overline{\mathbb{D}}$,

(2) $\alpha'(z) \neq 0$ *for* $z \in \overline{\mathbb{D}}$,

(3) $||\alpha(z)|| = 1$ *if and only if* $|z| = 1$, *and*

(4) $\langle \alpha(z), \alpha'(z) \rangle \neq 0$ *when* $|z| = 1$.

Then $\mathcal{M}_V$ *is isomorphic to* H^∞. *In fact* $\mathcal{M}_V = H^\infty(V)$.

Extended to planar domains (in place of $\mathbb{D}$) by Arcozzi-Rochberg-Sawyer, and to finite (open) Riemann surfaces by Kerr-McCarthy-S.

A positive result for discs

Theorem (Alpay-Putinar-Vinnikov 2003)

Let $\alpha : \mathbb{D} \to V \subset \mathbb{B}_d$ *be a biholomorphism,* $d < \infty$*. If*

(1) α *extends to an injective* C^2 *function on* $\overline{\mathbb{D}}$*,*

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Proposition (A Hopf type lemma, Davidson-Hartz-S)

Condition (4) is always satisfied for C^1 proper embeddings.

An application

Let $\alpha : \mathbb{D} \to V \subset \mathbb{B}_d$ be a biholomorphism, $d < \infty$. By [APV03]

$$H^\infty(V) = \mathcal{M}_V := \mathrm{Mult}(H^2_d)\big|_V$$

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Corollary (Alpay-Putinar-Vinnikov 2003)

Let $\alpha : \mathbb{D} \to V \subset \mathbb{B}_d$ be a biholomorphism as in the theorem. Then for every $f \in H^\infty(V)$ there exists $\tilde{f} \in \mathrm{Mult}(H^2_d)$ such that $\tilde{f}\big|_V = f$.

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One also has control on the multiplier norm of $\tilde{f}$.
Since $\mathrm{Mult}(H^2_d) \subsetneq H^\infty(\mathbb{B}_d)$, this is sharpening of Henkin's extension theorem in the one dimensional case.

A negative result for discs

Theorem (Davidson-Hartz-S)

There exists a rational function α mapping $\mathbb{D}$ onto $V \subset \mathbb{B}_2$ with poles outside $\overline{\mathbb{D}}$ as in the Alpay-Putinar-Vinnikov theorem, except for $\alpha(1) = \alpha(-1)$.
In this case, $\mathcal{M}_V$ is not isomorphic to H^∞, and $\alpha^{-1} \notin \mathcal{M}_V$.

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Proof: We show that if $\varphi : \mathcal{M}_V \to \mathcal{M}_W$ is an isomorphism, then φ^* is bi-Lipschitz w.r.t. pseudo-hyperbolic distance, and such a map α can not be bi-Lipschitz.

A stronger notion of biholomorphism

A multiplier biholomorphism is a biholomorphism $\alpha : V \to W$ such that the coordinates of α are in $\mathcal{M}_V$, and the coordinates of α^{-1} are in $\mathcal{M}_W$.

Theorem (Davidson-Ramsey-S)

Let $V, W \subset \mathbb{B}_d$ be varieties in $\mathbb{B}_d$ with $d < \infty$ which are the union of finitely many irreducible varieties and a discrete variety. If $\mathcal{M}_V$ and $\mathcal{M}_W$ are algebraically isomorphic, then V and W are multiplier biholomorphic.

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Example (Davidson-Hartz-S)

Every Blaschke sequence $V \subset \mathbb{D}$ is multiplier biholomorphic to an interpolating sequence W.
However, $\mathcal{M}_W \simeq \ell^\infty \not\simeq \mathcal{M}_V$ if V is not interpolating.

What happens for embedded discs?

Discs in the infinite ball

Consider embeddings

$$\alpha : \mathbb{D} \to \mathbb{B}_\infty, \quad z \mapsto (b_1 z, b_2 z^2, b_3 z^3, \ldots),$$

where $(b_n) \in \ell^2$ with $||(b_n)||_2 = 1$ and $b_1 \neq 0$. Then $V = \alpha(\mathbb{D})$ is a variety. Any two varieties of this type are multiplier biholomorphic.

A continuum of multiplier algebras

For $-1 \leq s \leq 0$, let $\mathcal{H}_s$ be the Hilbert function space on $\mathbb{D}$ with kernel

$$K^s(z,w) = \sum_{n=0}^{\infty}(n+1)^s(z\overline{w})^n \quad (z,w \in \mathbb{D}).$$

Then $\mathcal{H}_0 = H^2$ and $\mathcal{H}_{-1}$ is the Dirichlet space.

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$$\alpha_s : \mathbb{D} \to V_s \subset \mathbb{B}_\infty, \quad \alpha_s(z) = (b_{1,s}z, b_{2,s}z^2, b_{3,s}z^3, \ldots)$$

such that $\mathcal{M}_{V_s} \simeq \mathrm{Mult}(\mathcal{H}_s)$.
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Theorem (Davidson-Hartz-S)

If $s \neq t \neq 0$, *then* $\mathcal{M}_{V_s} \not\simeq \mathcal{M}_{V_t} \not\simeq H^\infty$.

Open problem

Recall

If $\mathcal{M}_V$ and $\mathcal{M}_W$ isomorphic then V and W are multiplier biholomorphic.

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Let $V, W \in \mathbb{B}_d$ with $d < \infty$ be irreducible varieties which are multiplier biholomorphic. Are $\mathcal{M}_V$ and $\mathcal{M}_W$ isomorphic?

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Question

Let $V, W \in \mathbb{B}_d$ with $d < \infty$ be discs which are multiplier biholomorphic. Are $\mathcal{M}_V$ and $\mathcal{M}_W$ isomorphic?

Thank you!