

The isomorphism problem for complete Pick algebras

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Based on joint works with Davidson, Hartz, Kerr, McCarthy and Ramsey.

Other contributors: Alpay, Arcozzi, Putinar, Rochberg, Sawyer, Vinnikov...

A long long time ago...

A corollary of Gelfand's theory

Theorem (1940s)

Let X and Y be compact Hausdorff topological spaces.

If $C(X)$ is isomorphic to $C(Y)$ then $X \cong Y$.

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In fact, $\varphi : C(X) \rightarrow C(Y)$ is an isomorphism, then there exists a homeomorphism $\alpha : Y \rightarrow X$ such that

$$\varphi(f) = f \circ \alpha, \quad f \in C(X).$$

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Key: X is the maximal ideal space of $C(X)$

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Now compute:

$$\varphi(f)(y) = \rho_y(\varphi(f)) = \varphi^*(\rho_y)(f) = f(\alpha(y)).$$

Thus $\varphi(f) = f \circ \alpha$. (The converse is obvious.)

A Theorem of Bers

Theorem (Bers 1948)

*Let U and V be open subsets of the plane \mathbb{C} .
If the algebras of holomorphic functions $H(U)$ and $H(V)$ are isomorphic
then U and V are biholomorphic.*

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biholomorphism $\alpha : V \rightarrow U$ such that*

$$\varphi(f) = f \circ \alpha, \quad f \in H(U).$$

Proof: $\mathfrak{M}(H(U))$ is more complicated than U . However

$U \cong$ **principal** maximal ideals of $H(U) \cong$ complex homomorphisms,
and the rest is similar.

And now **our** story begins.

A classical interpolation theorem

Theorem (Pick 1916, Nevanlinna 1919)

Let $z_1, \dots, z_n \in \mathbb{D}$ and $w_1, \dots, w_n \in \mathbb{C}$. There exists $f \in H^\infty$ with

$$f(z_i) = w_i \quad \text{for } 1 \leq i \leq n \quad \text{and} \quad \|f\|_\infty \leq 1$$

if and only if the matrix

$$\left(\frac{1 - w_i \bar{w}_j}{1 - z_i \bar{z}_j} \right)_{i,j=1}^n$$

is positive definite.

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The function

$$K(z, w) = \frac{1}{1 - z\bar{w}}$$

is the **reproducing kernel** of the Hardy space H^2 , and

$$H^\infty = \text{Mult}(H^2) := \{f \in H(\mathbb{D}) : M_f h := fh \in H^2 \text{ for all } h \in H^2\}.$$

Pick and complete Pick spaces

Let \mathcal{H} be a Hilbert function space on a set X with kernel K . Given $z_1, \dots, z_n \in X$ and $w_1, \dots, w_n \in M_d(\mathbb{C})$, does there exist $f \in \text{Mult}(\mathcal{H}) \otimes M_d$ with $\|f\|_{\text{Mult}(\mathcal{H}) \otimes M_d} \leq 1$ and

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Definition

If this condition is sufficient, \mathcal{H} is called a **complete Pick space**. If \mathcal{H} is a complete Pick space then $\text{Mult}(\mathcal{H})$ is called a **complete Pick algebra**.

Drury-Arveson space and the algebras \mathcal{M}_V

Let $d \in \mathbb{N} \cup \{\infty\}$. The **Drury-Arveson space** H_d^2 is the Hilbert function space on \mathbb{B}_d with kernel

$$k(z, w) = k_w(z) = \frac{1}{1 - \langle z, w \rangle} \quad (z, w \in \mathbb{B}_d).$$

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*Moreover, every multiplier algebra of an **irreducible** complete Pick space can be identified with*

$$\mathcal{M}_X := \text{Mult}(H_d^2)|_X$$

for some subset $X \subset \mathbb{B}_d$ (possibly $d = \infty$).

* **irreducible** - the kernel K satisfies $K(x, y) \neq 0$ for all x, y .

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The set X determines a **variety** $V = V(I(X))$ by

$$V = \{z \in \mathbb{B}_d : f(z) = 0 \text{ for all } f \in \text{Mult}(H_d^2) \text{ with } f|_X = 0\}$$

and $\mathcal{M}_V \simeq \mathcal{M}_X$ (Davidson-Ramsey-S).

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\mathcal{M}_V is an operator algebra (follows from Pick property) and weak* closed

$\mathcal{M}_V = \text{Mult}(\mathcal{H}_V)$, where

$$\mathcal{H}_V = \overline{\text{span}}\{k_\lambda : \lambda \in V\}.$$

The isomorphism problem for \mathcal{M}_V

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Problem A

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More concretely:

Problem B

Let $V, W \subset \mathbb{B}_d$ be two varieties. When are \mathcal{M}_V and \mathcal{M}_W isomorphic? Isometrically isomorphic? Completely Isometrically isomorphic? Similar/unitarily equivalent?

(Note: these algebras are semisimple, so algebraic isomorphisms are automatically bounded.)

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For every $\lambda \in V$, there exists $\rho_\lambda \in \pi^{-1}(\lambda)$ given by

$$\rho_\lambda(f) = f(\lambda).$$

We identify $V \subset \mathfrak{M}(\mathcal{M}_V)$, and consider V as the **visible** part of the spectrum.

The maximal ideal space \mathcal{M}_V

Theorem (Davidson-Pitts 1998, Davidson-Ramsey-S)

There is a continuous projection $\pi : \mathfrak{M}(\mathcal{M}_V) \rightarrow \overline{\mathbb{B}_d}$ given by

$$\pi(\rho) = (\rho(z_1), \rho(z_2), \dots) \quad , \quad \rho \in \mathfrak{M}(\mathcal{M}_V).$$

For each $\lambda \in V$ there is a unique weak-continuous character $\rho_\lambda \in \pi^{-1}(\lambda)$ given by $\rho_\lambda(f) = f(\lambda)$. Every weak*-continuous character arises this way.*

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If $d < \infty$, then

$$\pi(\mathfrak{M}(\mathcal{M}_V)) \cap \mathbb{B}_d = V, \tag{1}$$

$$\pi^{-1}(\lambda) = \{\rho_\lambda\}, \text{ for all } \lambda \in V. \tag{2}$$

If $d = \infty$, then (1) or (2) may fail.

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Theorem (Davidson-Ramsey-S)

Let $V, W \subset \mathbb{B}_d$ be varieties. The following assertions are equivalent:

- (i) \mathcal{M}_V and \mathcal{M}_W are unitarily equivalent.
- (ii) \mathcal{M}_V and \mathcal{M}_W are completely isometrically isomorphic.
- (iii) There exists a conformal automorphism $\alpha \in \text{Aut}(\mathbb{B}_d)$ such that

$$\alpha(W) = V.$$

If $d < \infty$, this is equivalent to

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Proof:

- (i) \Rightarrow (ii) is immediate.

Proof (cont'd):

(ii) \Rightarrow (iii): **IF** φ^* maps point evaluation at λ to point evaluation at μ , then

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(iii) \Rightarrow (i): We write down a unitary

$$Uk_\lambda = c_\lambda k_{\alpha(\lambda)}$$

and this unitary implements an isomorphism between \mathcal{M}_V and \mathcal{M}_W .

Algebraic isomorphism for homogeneous varieties

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Let $V, W \subset \mathbb{B}_d$ be homogeneous varieties with $d < \infty$. Then the following are equivalent:

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* - A **homogeneous variety** is the common vanishing locus of homogeneous polynomials.

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(iv) \Rightarrow (i): If $A : V \rightarrow W$, define $T : k_\lambda \mapsto k_{A\lambda}$.

One shows (very difficult!) that T extends to a bounded isomorphism $\mathcal{H}_V \rightarrow \mathcal{H}_W$.

Algebraic isomorphism \Rightarrow biholomorphic equivalence

Theorem (Davidson-Ramsey-S)

Let $V, W \subset \mathbb{B}_d$ be varieties in \mathbb{B}_d with $d < \infty$ which are the union of finitely many irreducible varieties and a discrete variety. If \mathcal{M}_V and \mathcal{M}_W are algebraically isomorphic, then V and W are biholomorphic.

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Questions

1. **What about the converse?**
2. Are the technical conditions (finite union of irreducibles, $d < \infty$) really needed?

Embedded discs

Consider varieties which are biholomorphic to the unit disc \mathbb{D} , say

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Question

Is \mathcal{M}_V always isomorphic to H^∞ ?

A positive result for discs

Theorem (Alpay-Putinar-Vinnikov 2003)

Let $\alpha : \mathbb{D} \rightarrow V \subset \mathbb{B}_d$ be a biholomorphism, $d < \infty$. If

- (1) α extends to an injective C^2 function on $\overline{\mathbb{D}}$,
- (2) $\alpha'(z) \neq 0$ for $z \in \overline{\mathbb{D}}$,
- (3) $\|\alpha(z)\| = 1$ if and only if $|z| = 1$, and
- (4) $\langle \alpha(z), \alpha'(z) \rangle \neq 0$ when $|z| = 1$.

Then \mathcal{M}_V is isomorphic to H^∞ . In fact $\mathcal{M}_V = H^\infty(V)$.

Extended to planar domains (in place of \mathbb{D}) by Arcozzi-Rochberg-Sawyer, and to finite (open) Riemann surfaces by Kerr-McCarthy-S.

A positive result for discs

Theorem (Alpay-Putinar-Vinnikov 2003)

Let $\alpha : \mathbb{D} \rightarrow V \subset \mathbb{B}_d$ be a biholomorphism, $d < \infty$. If

- (1) α extends to an injective C^2 function on $\overline{\mathbb{D}}$,
- (2) $\alpha'(z) \neq 0$ for $z \in \overline{\mathbb{D}}$,
- (3) $\|\alpha(z)\| = 1$ if and only if $|z| = 1$, and
- (4) $\langle \alpha(z), \alpha'(z) \rangle \neq 0$ when $|z| = 1$.

Then \mathcal{M}_V is isomorphic to H^∞ . In fact $\mathcal{M}_V = H^\infty(V)$.

Extended to planar domains (in place of \mathbb{D}) by Arcozzi-Rochberg-Sawyer, and to finite (open) Riemann surfaces by Kerr-McCarthy-S.

Proposition (A Hopf type lemma, Davidson-Hartz-S)

Condition (4) is always satisfied for C^1 proper embeddings.

An application

Let $\alpha : \mathbb{D} \rightarrow V \subset \mathbb{B}_d$ be a biholomorphism, $d < \infty$. By [APV03]

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Corollary (Alpay-Putinar-Vinnikov 2003)

Let $\alpha : \mathbb{D} \rightarrow V \subset \mathbb{B}_d$ be a biholomorphism as in the theorem. Then for every $f \in H^\infty(V)$ there exists $\tilde{f} \in \text{Mult}(H_d^2)$ such that $\tilde{f}|_V = f$.

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One also has control on the multiplier norm of \tilde{f} .

Since $\text{Mult}(H_d^2) \subsetneq H^\infty(\mathbb{B}_d)$, this is sharpening of Henkin's extension theorem in the one dimensional case.

A negative result for discs

Theorem (Davidson-Hartz-S)

There exists a rational function α mapping \mathbb{D} onto $V \subset \mathbb{B}_2$ with poles outside $\overline{\mathbb{D}}$ as in the Alpay-Putinar-Vinnikov theorem, except for $\alpha(1) = \alpha(-1)$.

In this case, \mathcal{M}_V is not isomorphic to H^∞ , and $\alpha^{-1} \notin \mathcal{M}_V$.

(So the answer to the question whether $\mathcal{M}_V \simeq H^\infty$ (for V a disc) is **no**, even when $d < \infty$.)

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Proof: We show that if $\varphi : \mathcal{M}_V \rightarrow \mathcal{M}_W$ is an isomorphism, then φ^* is bi-Lipschitz w.r.t. pseudo-hyperbolic distance, and such a map α can not be bi-Lipschitz.

A stronger notion of biholomorphism

A **multiplier biholomorphism** is a biholomorphism $\alpha : V \rightarrow W$ such that the coordinates of α are in \mathcal{M}_V , and the coordinates of α^{-1} are in \mathcal{M}_W .

Theorem (Davidson-Ramsey-S)

*Let $V, W \subset \mathbb{B}_d$ be varieties in \mathbb{B}_d with $d < \infty$ which are the union of finitely many irreducible varieties and a discrete variety. If \mathcal{M}_V and \mathcal{M}_W are algebraically isomorphic, then V and W are **multiplier biholomorphic**.*

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NO!

Example (Davidson-Hartz-S)

Every Blaschke sequence $V \subset \mathbb{D}$ is multiplier biholomorphic to an interpolating sequence W .

However, $\mathcal{M}_W \simeq \ell^\infty \not\simeq \mathcal{M}_V$ if V is not interpolating.

What happens for embedded discs?

Discs in the infinite ball

Consider embeddings

$$\alpha : \mathbb{D} \rightarrow \mathbb{B}_\infty, \quad z \mapsto (b_1 z, b_2 z^2, b_3 z^3, \dots),$$

where $(b_n) \in \ell^2$ with $\|(b_n)\|_2 = 1$ and $b_1 \neq 0$. Then $V = \alpha(\mathbb{D})$ is a variety. Any two varieties of this type are multiplier biholomorphic.

A continuum of multiplier algebras

For $-1 \leq s \leq 0$, let \mathcal{H}_s be the Hilbert function space on \mathbb{D} with kernel

$$K^s(z, w) = \sum_{n=0}^{\infty} (n+1)^s (z\bar{w})^n \quad (z, w \in \mathbb{D}).$$

Then $\mathcal{H}_0 = H^2$ and \mathcal{H}_{-1} is the Dirichlet space.

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There are embeddings

$$\alpha_s : \mathbb{D} \rightarrow V_s \subset \mathbb{B}_\infty, \quad \alpha_s(z) = (b_{1,s}z, b_{2,s}z^2, b_{3,s}z^3, \dots)$$

such that $\mathcal{M}_{V_s} \simeq \text{Mult}(\mathcal{H}_s)$.

For all $s, t \in [-1, 0]$, V_s and V_t are multiplier biholomorphic to each other and to \mathbb{D} .

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Theorem (Davidson-Hartz-S)

If $s \neq t \neq 0$, then $\mathcal{M}_{V_s} \not\cong \mathcal{M}_{V_t} \not\cong H^\infty$.

Open problem

Recall

If \mathcal{M}_V and \mathcal{M}_W isomorphic then V and W are multiplier biholomorphic.

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Let $V, W \in \mathbb{B}_d$ with $d < \infty$ be **irreducible** varieties which are multiplier biholomorphic. Are \mathcal{M}_V and \mathcal{M}_W isomorphic?

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Question

Let $V, W \in \mathbb{B}_d$ with $d < \infty$ be **discs** which are multiplier biholomorphic. Are \mathcal{M}_V and \mathcal{M}_W isomorphic?

Thank you!