OPERATOR THEORY AND FUNCTION THEORY IN DRURY-ARVESON SPACE AND ITS QUOTIENTS

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Abstract. The Drury-Arveson space H^2_d, also known as symmetric Fock space or the d-shift space, is a Hilbert function space that has a natural d-tuple of operators acting on it, which gives it the structure of a Hilbert module. This survey aims to introduce the Drury-Arveson space, to give a panoramic view of the main operator theoretic and function theoretic aspects of this space, and to describe the universal role that it plays in multivariable operator theory and in Pick interpolation theory.

Contents

1. Introduction

The *Drury-Arveson space* is a Hilbert function space which plays a universal role in operator theory as well as function theory. This space, denoted H^2_d (or sometimes $\mathcal{F}_+(E)$), and also known as the *d-shift space*, *Arveson's Hardy space* or the *symmetric Fock space*, has been the object of intensive study in the last fifteen years or so. Arguably, it is the subject of so much interest because it is the correct generalization of the classical Hardy space $H^2(\mathbb{D})$ from one variable to several. The goal of this survey is to collect together various important features of H^2_d, with detailed references and sometimes proofs, so as to serve as a convenient reference for researchers working with this space.

Of course, a Hilbert space is a Hilbert space, and any two are isomorphic. Thus, when one sets out to study the Drury-Arveson space one is in fact interested in a certain concrete realization of Hilbert space which carries some additional structure. The additional structures are of two kinds: operator theoretic or function theoretic. For the operator theorist, the object of interest is the space H^2_d together with a particular d-tuple $S = (S_1, \ldots, S_d)$ of commuting operators called the d-shift; in other words, the object of interest is a *Hilbert module* over the algebra $\mathbb{C}[z]$ of polynomials in d variables. The function theorist would rather view H^2_d as a Hilbert space comprised of functions on the unit ball $\mathbb{B}_d$ of $\mathbb{C}^d$, in which point evaluation is a bounded functional — in other words: a *Hilbert function space*.

There are many Hilbert modules and many Hilbert function spaces that one may study. Many of the results presented below have versions that work in other spaces. This survey focuses on the results in Drury-Arveson space for three reasons. First, as is explained below, H^2_d is a universal object both as a Hilbert module and as Hilbert functions space, and result about H^2_d have consequences on other spaces of interest. Second, H^2_d is an interesting object of study in itself: being a natural analogue of $H^2(\mathbb{D})$ it enjoys several remarkable properties, and it could be useful to have an exposition which treats various facets of this space. Third, the study of H^2_d is now quite developed, and can serve as a model for a theory in which multivariable operator theory and function theory are studied together.

Most results are presented below without proof, but with detailed references. When a proof is presented it is usually because the result and/or the proof are of special importance. Sometimes a proof is also provided for a piece of folklore for which a convenient reference is lacking.

2. Notation and terminology

2.1. **Basic notation.** Let d be an integer or ∞ (the symbol ∞ will always stand for a countable infinity). $\mathbb{C}^d$ denotes d-dimensional complex Hilbert space. $\mathbb{B}_d$ denotes the (open) unit ball in $\mathbb{C}^d$. The unit disc $\mathbb{B}_1$ is also denoted $\mathbb{D}$. It has become a convenient notational convention in the field to treat d as a finite integer even when it is not. Some of the results are valid (or are known to be valid) only in the case of $d < \infty$, and these cases will be pointed out below.

Let H be a Hilbert space. The identity operator on H is denoted by I_H or I. If M is a closed subspace of H then P_M always denotes the orthogonal projection from H onto M. If $\mathcal{S}$ is a subset of H, then $[\mathcal{S}]$ denotes the closed subspace spanned by $\mathcal{S}$. All operators below are assumed to be bounded operators on a separable Hilbert space.

If $z_1, \ldots, z_d$ are d commuting variables, then let $z = (z_1, \ldots, z_d)$ and write z^α for the product $z_1^{\alpha_1} \cdots z_d^{\alpha_d}$ for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$. The algebra of polynomials in d commuting variables is denoted $\mathbb{C}[z_1, \ldots, z_d]$ or $\mathbb{C}[z]$ (this has an obvious interpretation also when $d = \infty$). The symbols $\alpha!$ and $|\alpha|$ are abbreviations for $\alpha_1! \cdots \alpha_d!$ and $|\alpha| = \alpha_1 + \ldots + \alpha_d$, respectively.

For the purposes of this survey, a function $f : \mathbb{B}_d \to \mathbb{C}$ is said to be *analytic* if it can be expressed as an absolutely convergent power series $f(z) = \sum_{\alpha \in \mathbb{N}^d} c_\alpha z^\alpha$ (when $d < \infty$ this is equivalent to the usual local definition). $\mathcal{O}(\mathbb{B}_d)$ will denote the analytic functions on $\mathbb{B}_d$.

2.2. **Tuples of operators.** A d-tuple of operators on a Hilbert space H is denoted $T = (T_1, \ldots, T_d)$. If A and B are d-tuples on two Hilbert spaces H and K, and $U : H \to K$ is a unitary such that $UA_iU^* = B_i$ for all $i = 1, \ldots, d$, then one says that A and B are *unitarily equivalent*, and one writes $UAU^* = B$. Similarly, one writes T^* for the tuple $(T_1^*, \ldots, T_d^*)$, and so forth.

2.3. **Commuting and essentially normal tuples.** Let T be a d-tuple in $B(H)$. T is said to be *commuting* if $[T_i, T_j] := T_iT_j - T_jT_i = 0$ for all i, j. If T is a commuting contraction then for every $p \in \mathbb{C}[z]$ one may evaluate $p(T)$; for example, $T^\alpha = T_1^{\alpha_1} \cdots T_d^{\alpha_d}$.

A commuting tuple T is said to be *normal* if $[T_i, T_j^*] = 0$ for all i, j, and *essentially normal* if $[T_i, T_j^*]$ is compact for all i, j. Finally, a commuting tuple T that satisfies $\mathrm{trace}|[T_i, T_j^*]|^p < \infty$ is said to be *p-essentially normal*.

T is said to be *subnormal* if there is a Hilbert space $K \supseteq H$ and a normal d-tuple N on K such that $T = N\big|_H$.

2.4. **Row contractions and d-contractions.** T is said to be a *row contraction* if $\sum_{i=1}^d T_iT_i^* \leq I_H$ (when $d = \infty$ it is assumed that the partial sums are bounded by I, and hence that the sum converges in the strong operator topology to a positive operator less than the identity). Equivalently, this means that the row operator

$$\begin{bmatrix} T_1 & T_2 & \cdots & T_d \end{bmatrix} : \underbrace{H \oplus \cdots \oplus H}_{d \text{ times}} \to H$$

is a contraction. With every row contraction one associates a completely positive map $\Theta_T : B(H) \to B(H)$ given by $\Theta_T(A) = \sum_{i=1}^d T_iAT_i^*$. Note that when $d = \infty$ the assumption $\sum T_iT_i^* \leq I$ ensures that $\sum_{i=1}^d T_iAT_i^*$ indeed converges in the strong operator topology. T is said to be *pure* if $\Theta_T^n(I) \longrightarrow_{n\to\infty} 0$ in the strong operator topology. A commuting row contraction $T = (T_1, \ldots, T_d)$ is also called a *d-contraction*.

2.5. **Defect operator and defect space.** The *defect operator* of a row contraction T is the operator $\Delta_T = \sqrt{I - \Theta_T(I)}$, and the *defect space* is $\mathcal{D}_T = \overline{\Delta_T H}$. The *rank* of T is defined to be the dimension of the the defect space, $\mathrm{rank}(T) = \dim(\mathcal{D}_T)$. When no confusion may arise the notation $\Delta = \Delta_T$ is used.

2.6. **Hilbert modules.** A popular and fruitful point of view for studying commuting operators on Hilbert space is that of *Hilbert modules* [55] (see the chapter on Hilbert modules by J. Sarker in this Handbook). If T is a commuting d-tuple on H, then T induces on H the structure of a Hilbert module via

$$p \cdot h = p(T)h \ , \ p \in \mathbb{C}[z], h \in H.$$

A Hilbert module is said to be *pure/contractive/of finite rank/essenitally normal/etc.*, if T is pure/a row contraction/of finite rank/essenitally normal/etc., respectively. In [55] Douglas and Paulsen put emphasis on Hilbert modules over *function* algebras, but Arveson [24] has found it useful to consider Hilbert modules over $\mathbb{C}[z]$. In general there is a big difference between these approaches, but by Section 6 below every pure contractive Hilbert module over $\mathbb{C}[z]$ is in fact a Hilbert module over a certain natural, canonical algebra of functions.

2.7. **Hilbert function spaces.** A *Hilbert function space* is a Hilbert space H consisting of functions on some space X, such that for every $x \in X$ the point evaluation $f \mapsto f(x)$ is bounded linear functional on H (such spaces are also commonly referred to as *reproducing kernel Hilbert spaces*). The reader is referred to [3] as a reference for Hilbert function spaces.

3. Drury-Arveson space as a function space

The Drury-Arveson space is named after S. Drury, who basically introduced it into multivariable operator theory [59], and after W. Arveson, who has brought this space to the center of the stage [20]. Though not all researchers prefer to use this name for this space, everybody understands what is meant by it.

3.1. **H^2_d as a graded completion of the polynomials.** The most elementary definition of the Drury-Arveson space H^2_d is as a graded completion of the polynomials [24]. Define an inner product on $\mathbb{C}[z]$ by setting

$$\langle z^\alpha, z^\beta \rangle = 0 \ , \quad \text{if } \alpha \neq \beta, \tag{3.1.1}$$

and

$$\langle z^\alpha, z^\alpha \rangle = \frac{\alpha!}{|\alpha|!}. \tag{3.1.2}$$

The condition (3.1.1) may seem natural, but the choice of weights (3.1.2) might appear arbitrary at this point; see Section 4.8. The completion of $\mathbb{C}[z]$ with respect to this inner product is denoted by H^2_d. It is clear that H^2_d can be identified with the space of holomorphic functions $f : \mathbb{B}_d \to \mathbb{C}$ which have a power series $f(z) = \sum_\alpha c_\alpha z^\alpha$ such that

$$\|f\|^2_{H^2_d} \equiv \|f\|^2 := \sum_\alpha |c_\alpha|^2 \frac{\alpha!}{|\alpha|!} < \infty.$$

3.2. **H^2_d as a Hilbert function space.** The space H^2_d turns out to be the Hilbert function space on $\mathbb{B}_d$ determined by the kernel

$$k(z,w) = k_w(z) = \frac{1}{1-\langle z,w\rangle}. \tag{3.2.1}$$

Indeed, for $|w| < 1$, $k_w(z) = \sum_{n=0}^\infty \langle z,w\rangle^n = \sum_{n=0}^\infty \sum_{|\alpha|=n} \frac{|\alpha|!}{\alpha!}\overline{w}^\alpha z^\alpha$ is clearly in H^2_d, and

$$f(w) = \sum_\alpha c_\alpha w^\alpha = \sum_\alpha c_\alpha \frac{|\alpha|!}{\alpha!} w^\alpha \langle z^\alpha, z^\alpha\rangle = \langle f, k_w\rangle. \tag{3.2.2}$$

This shows that point evaluation is a bounded functional on H^2_d, so H^2_d is a Hilbert function space [3], and it has the kernel (3.2.1). Since the only function that is orthogonal to all the kernel function k_w is the zero function, $\mathrm{span}\{k_w : w \in \mathbb{B}_d\}$ is dense in H^2_d. When $d = 1$, $H^2_d = H^2(\mathbb{D})$ is the usual Hardy space on the unit disc (see [70] for a thorough treatment of $H^2(\mathbb{D})$).

3.3. **The multiplier algebra of H^2_d.** As every Hilbert function space, H^2_d comes along with its *multiplier algebra*

$$\mathcal{M}_d := \mathrm{Mult}(H^2_d) = \{f : \mathbb{B}_d \to \mathbb{C} \big| fh \in H^2_d \text{ for all } h \in H^2_d\}.$$

To every multiplier $f \in \mathcal{M}_d$ there is associated a *multiplication operator* $M_f : h \mapsto fh$. Standard arguments (see [3]) show that M_f is bounded and that

$$\|f\|_\infty \leq \|M_f\|. \tag{3.3.1}$$

The *multiplier norm* of $f \in \mathcal{M}_d$ is given by

$$\|f\|_{\mathcal{M}_d} = \|M_f\|, \tag{3.3.2}$$

and this norm gives $\mathcal{M}_d$ the structure of an operator algebra. On the other hand, $\mathcal{M}_d$ is also an algebra of analytic functions contractively contained in $H^\infty(\mathbb{B}_d)$. It will be shown below that $\mathcal{M}_d$ is strictly contained in H^∞, that the supremum norm is not comparable with the multiplier norm, and hence that $\mathcal{M}_d$ is not a *function algebra*.

A trivial but useful observation is that since $1 \in H^2_d$, one immediately obtains $\mathcal{M}_d \subset H^2_d$, as spaces of functions.

3.4. **The d-shift.** The most natural d-tuple of operators occurring in the setting of H^2_d is the d-shift, given by $M_z = (M_{z_1}, \ldots, M_{z_d})$, where $z_1, \ldots, z_d$ are the coordinate functions in $\mathbb{C}^d$; thus

$$(M_{z_i}h)(z) = z_i h(z) \ , \ i = 1, \ldots, d, \quad h \in H^2_d. \tag{3.4.1}$$

It is straightforward that multiplication by every coordinate function is a bounded operator, hence the coordinate functions are all in $\mathcal{M}_d$. In fact, by 4.6 and 4.8 below, M_z is a pure row contraction. Consequently, $\mathbb{C}[z] \subseteq \mathcal{M}_d$.

When $d = 1$ then the d-shift is nothing but the unilateral shift on $H^2(\mathbb{D})$.

3.5. **Homogeneous decomposition of functions.** Every $f \in \mathcal{O}(\mathbb{B}_d)$ has a Taylor series $f(z) = \sum_\alpha a_\alpha z^\alpha$ convergent in $\mathbb{B}_d$, so in particular f has a decomposition into its homogeneous parts:

$$f(z) = \sum_{n=0}^{\infty} f_n(z), \tag{3.5.1}$$

where $f_n(z) = \sum_{|\alpha|=n} a_\alpha z^\alpha$, and the series (3.5.1) converges uniformly on compact subsets of the ball. When f happens to be in one of the function spaces studied here then more can be said.

For $f \in H^2_d$ the homogeneous components f_n are all orthogonal one to another, the series (3.5.1) converges in norm and $\|f\|^2 = \sum \|f_n\|^2$. If $f \in \mathcal{M}_d$ this is still true since $\mathcal{M}_d \subset H^2_d$, but understanding (3.5.1) in terms of the structure of $\mathcal{M}_d$ is a more delicate task. The series does not necessarily converge in norm (as can be seen by considering the case $d = 1$).

Recall that (3.3.2) allows one to consider $\mathcal{M}_d$ as an algebra of operators on H^2_d. For $t \in \mathbb{R}$, let U_t be the unitary on H^2_d sending $h(z)$ to $h(e^{it}z)$, and let γ_t be the automorphism on $B(H^2_d)$ implemented by U_t . A computation shows that $\mathcal{M}_d$ is stable under γ and that $\gamma_t(f)(z) = f(e^{it}z)$ for $f \in \mathcal{M}_d$.

Lemma 3.5.2. *For all $n = 0, 1, \ldots,$, the integral*

$$\frac{1}{2\pi}\int_0^{2\pi} \gamma_t(f) e^{-int} dt$$

converges in the strong operator topology to f_n. In particular, $\|f_n\|_{\mathcal{M}_d} \leq \|f\|_{\mathcal{M}_d}$.

For $r \in (0, 1)$, the function $f_r(z) := f(rz)$ has homogeneous decomposition

$$f_r(z) = \sum_{n=0}^{\infty} r^n f_n(z), \tag{3.5.3}$$

and this series converges absolutely in the multiplier norm, by the lemma. Rewrite

$$f_r = \frac{1}{2\pi}\int_0^{2\pi} \gamma_t(f) P_r(t) dt, \tag{3.5.4}$$

where $P_r(t)$ denotes the Poisson kernel on the disc. By well known techniques of harmonic analysis, one has the following theorem.

Theorem 3.5.5. *Let $f \in \mathcal{M}_d$, and for all $r \in (0, 1)$ denote $f_r(z) = f(rz)$. Then $f_r \in \mathcal{M}_d$, $\|f_r\|_{\mathcal{M}_d} \leq \|f\|_{\mathcal{M}_d}$, and the the series (3.5.1) is Poisson summable to f in the strong operator topology: $\lim_{r\to 1} f_r = f$.*

3.6. **The structure of $\mathcal{M}_d$.** Denote by $\overline{\text{alg}}^{\text{WOT}}(M_z)$ the unital weak-operator topology (WOT) closed operator algebra generated by the d-shift. The d-shift generates $\mathcal{M}_d$ in the sense of the following theorem.

Theorem 3.6.1. *The unital* WOT*-closed algebra generated by M_z is equal to $\{M_f : f \in \mathcal{M}_d\}$.*

The following lemma is required for the proof of the theorem.

Lemma 3.6.2. *Let $\{f_\alpha\}$ be a bounded net in $\mathcal{M}_d$ that is bounded in the multiplier norm. If $f \in \mathcal{M}_d$, then M_{f_α} converges to M_f in the weak-operator topology if and only if $f_\alpha(z) \to f(z)$ for all $z \in \mathbb{B}_d$.*

Proof. See, e.g., [50, Lemma 11.10]. ∎

Proof of Theorem 3.6.1. Every multiplier algebra is WOT-closed, so $\overline{\mathrm{alg}}^{\text{WOT}}(M_z)$ is contained in $\{M_f : f \in \mathcal{M}_d\}$. Let $f \in \mathcal{M}_d$. For $r \in (0,1)$, define $f_r(z) = f(rz)$. Then by (3.5.3) M_{f_r} is in the norm closed algebra generated by M_z. By 3.5.5 the net $\{f_r\}_{r\in(0,1)}$ is bounded by $\|f\|$. Since $f_r \to f$ pointwise, the lemma implies the WOT-convergence $M_{f_r} \to M_f$. ∎

The above theorem allows one to make the identification

$$\mathcal{M}_d = \overline{\mathrm{alg}}^{\text{WOT}}(M_z). \tag{3.6.3}$$

3.7. **The strict containment $\mathcal{M}_d \subsetneq H^\infty(\mathbb{B}_d)$.** When $d = 1$, $H^2_d = H^2(\mathbb{D})$ is the usual Hardy space, its multiplier algebra is equal to H^∞, and the multiplier norm of a multiplier f is equal to $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$. When $d > 1$ this is no longer true.

Theorem 3.7.1. *For $d > 1$ the norms $\|\cdot\|_\infty$ and $\|\cdot\|_{\mathcal{M}_d}$ are not comparable on $\mathcal{M}_d$, there is a strict containment*

$$\mathcal{M}_d \subsetneq H^\infty(\mathbb{B}_d), \tag{3.7.2}$$

and the d-tuple M_z is not subnormal.

Proof. If $f \in \mathcal{M}_d$ and $\lambda \in \mathbb{B}_d$, then for all $h \in H^2_d$

$$\langle h, M_f^* k_\lambda\rangle = f(\lambda)h(\lambda) = \langle h, \overline{f(\lambda)} k_\lambda\rangle.$$

Thus $\overline{f(\lambda)}$ is an eigenvalue of M_f^* and in particular $|\overline{f(\lambda)}| \leq \|M_f\|$. It follows that f is bounded on $\mathbb{B}_d$ and that $\sup_{\mathbb{B}_d} |f| \leq \|M_f\|$ (this argument works for any multiplier algebra). Since $1 \in H^2_d$ it follows that $f = f \cdot 1$ is analytic, thus $\mathcal{M}_d \subseteq H^\infty(\mathbb{B}_d)$.

For the strictness of the containment it suffices to consider the case $d < \infty$. Direct computations show that for a suitable choice of constants $a_1, a_2, \ldots$, the functions

$$f_N(z) := \sum_{n=0}^{N} a_n (z_1 \cdots z_d)^n$$

satisfy $\|f_N\|_\infty \leq 1$ while $\|f_N\|_{\mathcal{M}_d} \to \infty$. Moreover, the limit $f := \lim_{N\to\infty} f_N$ exists uniformly, and serves as an explicit example of a function that is in the "ball algebra" $A(\mathbb{B}_d)$ (that is, the algebra of continuous functions on the closed ball which are analytic on the interior), but is not in $\mathcal{M}_d$.

That M_z is not subnormal follows from the incomparability of the norms; see Section 3 in [20] for full details (see also Section 2 in [47] for a slightly different derivation of the first parts of the theorem). ∎

3.8. **Vector valued H^2_d and operator valued multipliers.** Let K be a Hilbert space. The Hilbert space tensor product $H^2_d \otimes K$ can be considered as the space of all holomorphic functions $f : \mathbb{B}_d \to K$ with Taylor series $f(z) = \sum_\alpha a_\alpha z^\alpha$, where the coefficients a_α are all in K and

$$\sum_\alpha \frac{\alpha!}{|\alpha|!} \|a_\alpha\|^2 < \infty.$$

Let K_1 and K_2 be two Hilbert spaces, and let $\Phi : \mathbb{B}_d \to B(K_1, K_2)$ be an operator valued function. For $h \in H^2_d \otimes K_1$, define $M_\Phi h$ to be the function $\mathbb{B}_d \to K_1$ given by

$$M_\Phi h(z) = \Phi(z)h(z) \ , \ z \in \mathbb{B}_d.$$

Denote by $\mathcal{M}_d(K_1, K_2)$ the space of all Φ for which $M_\Phi h \in H^2_d \otimes K_2$ for all $h \in H^2_d \otimes K_1$ (another common notation is $\mathrm{Mult}(H^2_d \otimes K_1, H^2_d \otimes K_2)$). An element Φ of $\mathcal{M}_d(K_1, K_2)$ is said to be a *multiplier*, and in this case M_Φ (which can be shown to be bounded) is called a *multiplication operator*. If $K_1 = K_2 = K$ then $\mathcal{M}_d(K_1, K_2)$ is abbreviated to $\mathcal{M}_d(K)$. The space $\mathcal{M}_d(K_1, K_2)$ is endowed with the norm $\|\Phi\| = \|M_\Phi\|$.

The following characterization of multipliers, which is useful also in the scalar case, holds in any Hilbert function space (the proof is straightforward, see [3, Theorem 2.41]).

Theorem 3.8.1. *Let $\Phi : \mathbb{B}_d \to B(K_1, K_2)$ be an operator valued function. If Φ is a multiplier then*

$$M^*_\Phi(k_\lambda \otimes v) = k_\lambda \otimes \Phi(\lambda)^* v \tag{3.8.2}$$

for all $\lambda \in \mathbb{B}_d$ and $v \in K_2$. Conversely, if $\Phi : \mathbb{B}_d \to B(K_1, K_2)$ and the mapping $k_\lambda \otimes v \mapsto k_\lambda \otimes \Phi(\lambda)^* v$ extends to a bounded operator $T \in B(H^2_d \otimes K_2, H^2_d \otimes K_1)$, then $\Phi \in \mathcal{M}_d(K_1, K_2)$ and $T = M^*_\Phi$.

It is immediate from (3.8.2) that any multiplier Φ is bounded (in the sense that there is $M > 0$ such that $\|\Phi(z)\| \leq M$ for all $z \in \mathbb{B}_d$) and holomorphic (in the sense that for all $u \in K_1, v \in K_2$ the function $z \mapsto \langle \Phi(z)u, v\rangle$ is holomorphic in the ball).

The following theorem, due to J. Ball, T. Trent and V. Vinnikov, provides a characterization of multipliers in $\mathcal{M}_d(K_1, K_2)$, which is specific to the setting of H^2_d. For a proof and additional characterizations, see [33, Section 2] (see also [8, 63]).

Theorem 3.8.3 ([33], Theorem 2.1; [63], Theorem 1.3)**.** *Let $\Phi : \mathbb{B}_d \to B(H^2_d \otimes K_1, H^2_d \otimes K_2)$. Then the following statements are equivalent:*

1. $\Phi \in \mathcal{M}_d(K_1, K_2)$ *with* $\|\Phi\| \leq 1$.
2. *The kernel*

$$K_\Phi(z, w) = \frac{I - \Phi(z)\Phi(w)^*}{1 - \langle z, w\rangle} \tag{3.8.4}$$

is a positive sesqui-analytic $B(K_2)$ valued kernel on $\mathbb{B}_d \times \mathbb{B}_d$; i.e., there is an auxiliary Hilbert space H and a holomorphic $B(H, K_2)$-valued function Ψ on $\mathbb{B}_d$ such that for all $z, d \in \mathbb{B}_d$,

$$K_\Phi(z, w) = \Psi(z)\Psi(w)^*. \tag{3.8.5}$$

3. *There exists an auxiliary Hilbert space H and a unitary operator*

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} H \\ K_1 \end{pmatrix} \to \begin{pmatrix} \oplus_1^d H \\ K_2 \end{pmatrix} \tag{3.8.6}$$

such that

$$\Phi(z) = D + C(I - Z(z)A)^{-1} Z(z) B, \tag{3.8.7}$$

where $Z(z) = \begin{bmatrix} z_1 I_H & \cdots & z_d I_H \end{bmatrix} : \oplus_1^d H \to H$.

The formula (3.8.7) is referred to as *the realization formula*. Sometimes, U is said to be a *unitary colligation*, and Φ is called the associated *transfer function*. The papers [28, 29, 30] of J.A. Ball, V. Bolotnikov and Q. Fang provide more details on the connections of the transfer function with systems theory in the context of Drury-Arveson space.

3.9. **The commutant of $\mathcal{M}_d$.** The *commutant* of an operator algebra $\mathcal{B} \subseteq B(H)$ is defined to be

$$\mathcal{B}' = \{a \in B(H) : \forall b \in B.\, ab = ba\}.$$

A standard argument shows that $\mathcal{M}_d$ is its own commutant:

$$\mathcal{M}_d' = \mathcal{M}_d. \tag{3.9.1}$$

More generally, one has the following, which is a special case of the commutant lifting theorem (Theorem 6.10.1 below).

Theorem 3.9.2. *Let K_1, K_2 be Hilbert space, and let $X \in B(K_1, K_2)$ such that*

$$X(M_f \otimes I_{K_1}) = (M_f \otimes I_{K_2})X, \tag{3.9.3}$$

for all $f \in \mathcal{M}_d$. Then there exists $\Phi \in \mathcal{M}_d(K_1, K_2)$ such that $X = M_\Phi$.

3.10. **H^2_d as a Besov-Sobolev space.** The Drury-Arveson space also fits into a family of function spaces which have been of interest in harmonic analysis (see, e.g., [13, 43, 121]). In this subsection it is assumed that $d < \infty$.

For an analytic function $f \in \mathcal{O}(\mathbb{B}_d)$, *the radial derivative of f* is defined to be $Rf = \sum_{i=1}^d z_i \frac{\partial f}{\partial z_i}$. It is useful to note that if f is a homogeneous polynomial of degree n, then $Rf = nf$.

Let $\sigma \geq 0$, $p \in [1, \infty)$, and let m be an integer strictly greater than $d/p - \sigma$. For every $f \in \mathcal{O}(\mathbb{B}_d)$, one can consider the norm $\|f\|_{m,\sigma,p}$ defined by

$$\|f\|^p_{m,\sigma,p} = \sum_{|\alpha|<m} \left| \frac{\partial^\alpha f}{\partial z^\alpha}(0) \right|^p + \int_{\mathbb{B}_d} |R^m f(z)|^p (1 - |z|^2)^{p(m+\sigma)-d-1} d\lambda(z),$$

where λ is Lebesgue measure on the ball. It turns out that choosing different $m > d/p - \sigma$ results in equivalent norms. One defines *the analytic Besov-Sobolev spaces $B^\sigma_p(\mathbb{B}_d)$* as

$$B^\sigma_p(\mathbb{B}_d) = \{f \in \mathcal{O}(\mathbb{B}_d) : \|f\|_{m,\sigma,p} < \infty\}.$$

When $p = 2$ one obtains a family of Hilbert function spaces, which — up to a modification to an equivalent norm — have reproducing kernel (for $\sigma > 0$)

$$k^\sigma(z, w) = \frac{1}{(1 - \langle z, w\rangle)^{2\sigma}}.$$

The proof of this is straightforward, using basic integral formulas on the ball (available in [106, Section 1.4] or [123, Section 1.3]) and the fact that a the reproducing kernel in a Hilbert function space is given by $\sum e_k(z)\overline{e_k(w)}$, where $\{e_k\}_{k=1}^\infty$ is any orthonormal basis. In particular this scale of spaces contains the Bergman space $L^2_a(\mathbb{B}_d)$ ($\sigma = d/2$) and the Hardy space $H^2(\mathbb{B}_d)$ ($\sigma = (d+1)/2$). For $p = 2$ and $\sigma = 1/2$ one gets the Drury-Arveson space.

Theorem 3.10.1. *Fix an integer $m > (d-1)/2$. For $f \in \mathcal{O}(\mathbb{B}_d)$ the following are equivalent:*

(1) $f \in H^2_d$.

(2) $R^{(d-1)/2}f \in H^2(\mathbb{B}_d)$ *(the Hardy space of the ball).*
(3) $\|f\|_{m,1/2,2} < \infty$.
(4) $|||f||| < \infty$, *where*

$$|||f|||^2 = \sum_{|\alpha|<m} \left|\frac{\partial^\alpha f}{\partial z^\alpha}(0)\right|^2 + \sum_{|\alpha|=m} \int_{\mathbb{B}_d} \left|\frac{\partial^\alpha f}{\partial z^\alpha}(z)\right|^2 (1-|z|^2)^{2m-d} d\lambda(z).$$

Moreover, the norms $\|\cdot\|_{m,1/2,2}$, $|||\cdot|||$ *and* $\|\cdot\|_{H^2_d}$ *are equivalent.*

Theorem 3.10.1 appears as Theorem 1 in [42] (one should beware that the same paper included another characterization of Drury-Arveson space [42, Theorem 2], but unfortunately that other result (which will not be stated here) is incorrect — see [64]). In [42] the result was stated only for the smallest integer m satisfying $m > (d-1)/2$, but the proof of the theorem — which boils down to calculations of the various integrals defining the norms (using formulas from [106, Section 1.4] or [123, Section 1.3]) — works for all $m > (d-1)/2$.

4. Drury-Arveson space as symmetric Fock space

A crucial fact is that the Hilbert function space H^2_d can be identified with the familiar symmetric Fock space. This identification (essentially contained in [59], but most clearly explained in [20]) accounts for the universal properties of H^2_d, and among other things also explains the significance of the choice of weights (3.1.2).

4.1. **Full Fock space.** Let E be a d-dimensional Hilbert space. The *full Fock space* is the space

$$\mathcal{F}(E) = \mathbb{C} \oplus E \oplus E^{\otimes 2} \oplus E^{\otimes 3} \oplus \dots$$

4.2. **The noncommutative d-shift.** Fix a basis $\{e_1, \dots, e_d\}$ of E. On $\mathcal{F}(E)$ define $L = (L_1, \dots, L_d)$ by

$$L_i x_1 \otimes \cdots \otimes x_n = e_i \otimes x_1 \otimes \cdots \otimes x_n.$$

L is called the *noncommutative d-shift*. The tuple L is easily seen to be a *row isometry*, meaning that the row operator $\begin{bmatrix} L_1 & L_2 & \cdots & L_d \end{bmatrix}$ from the direct sum of $\mathcal{F}(E)$ with itself d times into $\mathcal{F}(E)$ is an isometry; equivalently, this means that $L_1, \dots, L_d$ are isometries with pairwise orthogonal ranges. The tuple L plays a central role in noncommutative multivariable operator theory, see, e.g., [15, 47, 48, 49, 96, 97, 100]. The noncommutative d-shift is a universal row contraction, see Section 6.5 below.

The construction does not depend on the choice of the space E or the orthonormal basis, and henceforth $\mathcal{F}(E)$ will be sometimes denoted $\mathcal{F}_d$, understanding that some choice has been made.

4.3. **The noncommutative analytic Toeplitz algebra $\mathcal{L}_d$.** The noncommutative analytic Toeplitz algebra $\mathcal{L}_d$ is defined to be $\overline{\text{alg}}^{\text{WOT}}(L)$. This algebra was introduced by G. Popescu in [97], where it was shown that it is the same as the noncommutative multiplier algebra of the full Fock space. $\mathcal{L}_d$ is also referred to as the *left regular representation free semigroup algebra*, and plays a fundamental role in the theory of *free semigroup algebras* (see the survey [44]).

Since $\mathcal{L}_d$ is WOT-closed, it is also weak-$*$ closed as a subspace of $B(\mathcal{F}_d)$, the latter considered as the dual space of the trace class operators on $\mathcal{F}_d$. Thus it is a *dual algebra*, that is, an operator algebra that is also the dual space of a Banach space.

One then has a weak-$*$ topology on $\mathcal{L}_d$, and weak-$*$ continuous functionals come into play. The following factorization property for weak-$*$ functionals has proved very useful [34].

Definition 4.3.1. *Let $\mathcal{B} \subseteq B(H)$ be a dual algebra, and denote by $\mathcal{B}_*$ its predual. $\mathcal{B}$ is said to have* property $\mathbb{A}_1$ *if for every $\rho \in \mathcal{B}_*$ there exist $g, h \in H$ such that*

$$\rho(b) = \langle bg, h \rangle \ , \ b \in \mathcal{B}.$$

If, for every $\epsilon > 0$, g and h can be chosen to satisfy $\|g\|\|h\| < (1+\epsilon)\|\rho\|$, then $\mathcal{B}$ is said to have property $\mathbb{A}_1(1)$.

Theorem 4.3.2 ([49], Theorem 2.10)**.** *$\mathcal{L}_d$ has property $\mathbb{A}_1(1)$.*

Corollary 4.3.3. *The weak-$*$ and* WOT*-topologies on $\mathcal{L}_d$ coincide.*

4.4. **Quotients of $\mathcal{L}_d$.** The following theorem is a collection of results from [15, Section 4] and [47, Section 2].

Theorem 4.4.1. *Fix a* WOT*-closed two sided ideal J and denote $N = [J\mathcal{F}_d]^\perp$. Put $B = P_N L P_N$. Then the map $\pi : A \mapsto P_N A P_N$ is a homomorphism from the algebra $\mathcal{L}_d$ onto $P_N \mathcal{L}_d P_N$ which annihilates J. Moreover:*

1. *$P_N \mathcal{L}_d P_N = \overline{\mathrm{alg}}^{\mathrm{WOT}}(B)$ — the unital* WOT*-closed algebra generated by B.*
2. *$P_N \mathcal{L}_d P_N$ has property $\mathbb{A}_1(1)$.*
3. *$P_N \mathcal{L}_d P_N = (P_N \mathcal{L}_d P_N)''$.*
4. *π promotes to a natural completely isometric isomorphism and weak-$*$ homeomorphism $\mathcal{L}_d/J$ onto $P_N \mathcal{L}_d P_N$.*

Additional information on the predual structure of a quotient algebra $\mathcal{L}_d/J$ (where J is a weak-$*$ closed two sided ideal in $\mathcal{L}_d$) is contained in the following two theorems of M. Kennedy and D. Yang. A linear functional $\phi \in \mathcal{L}_d^*$ is said to be *absolutely continuous* if it is weak-$*$ continuous, and *singular* if it is far from being weak-$*$ continuous in a precise sense (see [82]).

Theorem 4.4.2 (Non-selfadjoint Lebesgue decomposition. [82], Theorem 1.1)**.** *Let J be a weak-$*$ closed two sided ideal in $\mathcal{L}_d$, and let ϕ be a bounded linear functional on $\mathcal{L}_d/J$. Then there exists a unique absolutely continuous ϕ_a and a unique singular ϕ_s such that*

$$\phi = \phi_a + \phi_s,$$

and

$$\|\phi\| \leq \|\phi_a\| + \|\phi_s\| \leq \sqrt{2}\|\phi\|.$$

The constant $\sqrt{2}$ appearing in the above theorem sis sharp [82]. From the above theorem Kennedy and Yang deduced that every quotient of $\mathcal{L}_d$ by a weak-$*$ closed two sided ideal has a unique predual.

Theorem 4.4.3 ([82], Theorem 1.4)**.** *For every weak-$*$ closed two sided ideal J in $\mathcal{L}_d$, the quotient $\mathcal{L}_d/J$ has a strongly unique predual.*

4.5. **Symmetric Fock space.** For every permutation σ on n elements, one defines a unitary operator U_σ on $E^{\otimes n}$ by

$$U_\sigma(x_1 \otimes \cdots \otimes x_n) = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

The *nth-fold symmetric tensor product of* E, denoted E^n, is defined to be the subspace of $E^{\otimes n}$ which consists of the vectors fixed under the unitaries U_σ for all σ. The *symmetric Fock space* is the subspace of $\mathcal{F}(E)$ given by

$$\mathcal{F}_+(E) = \mathbb{C} \oplus E \oplus E^2 \oplus E^3 \oplus \dots .$$

If $x_1 \in E^{n_1}, \dots x_k \in E^{n_k}$, write $x_1 x_2 \cdots x_k$ for the projection of $x_1 \otimes x_2 \otimes \cdots \otimes x_k$ into $E^{n_1+\dots+n_k}$. Letting $\{e_1, \dots, e_d\}$ be an orthonormal basis for E, e^α is shorthand for $e_1^{\alpha_1} \cdots e_d^{\alpha_d}$ for all $\alpha \in \mathbb{N}^d$. A computation shows that $\{e^\alpha\}_{|\alpha|=n}$ is an orthogonal basis for E^n and that

$$\|e^\alpha\|^2 = \frac{\alpha!}{|\alpha|!}. \tag{4.5.1}$$

The space $\mathcal{F}_+(E)$ is not invariant under the noncommutative d-shift L, but it is *co-invariant*, meaning that $L_i^* \mathcal{F}_+(E) \subseteq \mathcal{F}_+(E)$ for all i.

4.6. **The d-shift.** The *(commutative) d-shift* is the d tuple $S = (S_1, \dots, S_d)$ of operators given by compressing the noncommutative d-shift to $\mathcal{F}(E)$. Thus, for all n and all $x \in E^n$

$$S_i x = e_i x \quad , \quad i = 1, \dots, d. \tag{4.6.1}$$

It is straightforward to check that the d-shift has the following properties:

(1) S is commuting, i.e., $S_i S_j - S_j S_i = 0$.
(2) $\sum_{i=1}^d S_i S_i^* = I - P_{\mathbb{C}}$, and in particular S is a row contraction.
(3) S is pure.

Many results on the d-shift can be obtained by "compressing theorems" about the noncommutative d-shift down to $\mathcal{F}_+(E)$; see, e.g., [46, 47, 51, 100], the proof of Theorem 7.2.4 or Sections 4.9 and 8.2 below. This is a powerful technique, due to the availability of strong results for the noncommutative d-shift, e.g., [48, 49, 96, 97] or more generally [91]. Another advantage of this technique is that it allows to obtain similar results for a very large class of Hilbert modules by compressing the noncommutative d-shift to other co-invariant spaces; see [100, 109].

4.7. **Essential normality of the d-shift.** Let N be the unbounded operator N on H_d^2 with domain $\mathbb{C}[z]$ defined by $Nh = nh$ for $h \in E^n$. N is usually referred to in this context as the *number operator* (it is equal to the restriction of the radial derivative R from 3.10). A straightforward computation (see [20, Proposition 5.3]) shows that

$$[S_i^*, S_j] = S_i^* S_j - S_j S_i^* = (1+N)^{-1}(\delta_{ij} 1 - S_j S_i^*). \tag{4.7.1}$$

It follows readily that if $d < \infty$ then S is p-essentially normal for all $p > d$ (but not for $p = d$). In particular $[S_i, S_j^*]$ is compact when $d < \infty$. It is not compact when $d = \infty$.

4.8. **Identification of H_d^2 with symmetric Fock space.** Fix $d \in \{1, 2, \dots, \infty\}$ and let E be a d-dimensional Hilbert space with orthonormal basis $\{e_n\}_n$. Define $V : \mathbb{C}[z_1, \dots, z_d] \to \mathcal{F}_+(E)$ by

$$V\left(\sum_\alpha c_\alpha z^\alpha\right) = \sum_\alpha c_\alpha e^\alpha.$$

By equations (3.1.2) and (4.5.1) V extends to a unitary from H^2_d onto $\mathcal{F}_+(E)$. All separable infinite dimensional Hilbert spaces are isomorphic, the important feature here is that

$$VM_zV^* = S.$$

Alternatively, there is also an anti-unitary identification of these two spaces. Every $f \in H^2_d$ can be written in a unique way as

$$f(z) = \sum \langle z^n, \xi_n \rangle,$$

where z^n denotes the nth symmetric product of $z \in \mathbb{C}^d$ with itself, $\xi_n \in (\mathbb{C}^d)^n$, and $\sum_n \|\xi_n\|^2 < \infty$ (see [20, Section 1]). Then the map $J : H^2_d \to \mathcal{F}_+(E)$ given by $Jf = \sum_n \xi_n$ is an anti-unitary and $JM_zJ^{-1} = S$.

Because of the above identification, the notation S is also used for the tuple M_z acting on H^2_d. It is safe to switch from $\mathcal{F}_+(E)$ to H^2_d and back, as convenient. Together with this identification, the results of Section 3.6 allow one to identify between $\mathcal{M}_d$ and the unital WOT-closed algebra generated by S.

4.9. **Identification of $\mathcal{M}_d$ with the compression of $\mathcal{L}_d$.** The *antisymmetric Fock space (over E)* is defined to be $\mathcal{F}_-(E) = \mathcal{F}(E) \ominus \mathcal{F}_+(E)$. By 4.4 and 4.8 $\mathcal{M}_d$ can be identified with the compression of $\mathcal{L}_d$ to $\mathcal{F}_+(E)$, or as the quotient of $\mathcal{L}_d$ by the two sided WOT-closed commutator ideal corresponding to $\mathcal{F}_-(E)$. From 4.4 the following theorem follows.

Theorem 4.9.1. *$\mathcal{M}_d$ is a dual algebra which has property $\mathbb{A}_1(1)$ and a strongly unique predual. In particular, the weak-$*$ and weak operator topologies on $\mathcal{M}_d$ coincide. The same holds for quotients of $\mathcal{M}_d$ by weak-$*$ closed ideals.*

4.10. **Subproduct systems.** The commutative and noncommutative d-shifts were defined above in a way which might make it seems to depend on the choice of an orthonormal basis in a d-dimensional space E (and, in the function space picture, on a choice of coordinate system in $\mathbb{C}^d$). Of course, there is an obvious theorem stating that the same structure is obtained regardless of the choice of basis (see, e.g., [20]). Alternatively, a coordinate free definition of the shift is given by viewing it as a representation of a *subproduct system*; see [109] for details.

5. Operator algebras associated to the d-shift

5.1. **The norm closed algebra and the Toeplitz algebra.** Let $\mathcal{A}_d$ be the norm closed algebra generated by S on H^2_d. This algebra is sometimes referred to as the "algebra of continuous multipliers", but this terminology is misleading — see 5.2.1 below. The *Toeplitz algebra* $\mathcal{T}_d$ is defined to be the unital C*-algebra generated by S, that is,

$$\mathcal{T}_d = C^*(\mathcal{A}_d) = C^*(1, S). \tag{5.1.1}$$

From 4.6 and 4.7 the following theorem follows (for proof see [20, Theorem 5.7]).

Theorem 5.1.2. *Fix $d < \infty$ and denote the compact operators on H^2_d by $\mathcal{K}$. Then $\mathcal{K} \subset \mathcal{A}_d$, and*

$$\mathcal{T}_d/\mathcal{K} \cong C(\partial\mathbb{B}_d). \tag{5.1.3}$$

Thus, there exists an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T}_d \longrightarrow C(\partial\mathbb{B}_d) \longrightarrow 0. \tag{5.1.4}$$

The isomorphism (5.1.3) is the natural one given by sending the image of S_i in the quotient to the coordinate function z_i on $\partial\mathbb{B}_d$. It follows that the essential norm $\|M_f\|_e$ of an element $f \in \mathcal{A}_d$ is given by

$$\|M_f\|_e = \sup_{z\in\mathbb{B}_d} |f(z)| \quad , \quad f \in \mathcal{A}_d. \tag{5.1.5}$$

Another consequence of the above theorem is

$$\mathcal{T}_d = \overline{\operatorname{span} \mathcal{A}_d\mathcal{A}_d^*}^{\|\cdot\|}. \tag{5.1.6}$$

It is worth noting that for $d = \infty$ equation (5.1.3) fails, because S is not essentially normal in that case. There is a naturally defined ideal $\mathcal{I} \triangleleft \mathcal{T}_d$ that contains $\mathcal{K}$ (and coincides with $\mathcal{K}$ when $d < \infty$) such that $\mathcal{T}_d/\mathcal{I}$ is commutative. This ideal $\mathcal{I}$ is given by

$$\mathcal{I} = \{A \in \mathcal{T}_d : \lim_{n\to 0} \|AP_{E^n}\| = 0\},$$

where P_{E^n} is the orthogonal projection $\mathcal{F}_+(E) \to E^n$. The counterpart of (5.1.3) still fails, instead one has

$$\mathcal{T}_\infty/\mathcal{I} = C(\overline{\mathbb{B}_\infty}).$$

See [120, Example 3.6] for details.

5.2. **Continuous multipliers versus $\mathcal{A}_d$.** It follows from (3.3.1) and (3.3.2) that $\mathcal{A}_d \subseteq C(\overline{\mathbb{B}_d}) \cap \mathcal{M}_d$. When $d = 1$ this containment is an equality, but for $d > 1$ the reverse containment does not hold.

Indeed, in [64] it is proved that there is a sequence of continuous multipliers $\{\psi_k\}$ such that $\lim_{k\to\infty} \|\psi_k\|_\infty = 0$ while $\inf_k \|M_{\psi_k}\|_e \geq 1/2$. It follows that (5.1.5) does not hold for the multipliers ψ_k. Since $\psi_k \in C(\overline{\mathbb{B}_d}) \cap \mathcal{M}_d$, it follows that

$$\mathcal{A}_d \subsetneq C(\overline{\mathbb{B}_d}) \cap \mathcal{M}_d. \tag{5.2.1}$$

5.3. **Nullstellensatz and approximation in homogeneous ideals.**

Definition 5.3.1. *Let $\mathcal{B} \subseteq \mathcal{O}(\mathbb{B}_d)$ be an algebra. An ideal $J \triangleleft \mathcal{B}$ is said to be a* homogeneous ideal *if for every $f \in J$ with homogeneous decomposition (3.5.1) and every $n \in \mathbb{N}$, it holds that $f_n \in J$.*

Definition 5.3.2. *Let $\mathcal{B} \subseteq \mathcal{O}(\mathbb{B}_d)$ be an algebra and $J \triangleleft \mathcal{B}$ an ideal. The* radical of *J is the ideal*

$$\sqrt{J} = \{f \in \mathcal{B} : \exists N.\, f^N \in J\}.$$

An ideal J is said to be a radical ideal *if $\sqrt{J} = J$.*

If $\mathcal{B} \subseteq \mathcal{O}(\mathbb{B}_d)$ is an algebra and $X \subseteq \mathbb{B}_d$ is a set, denote

$$I_{\mathcal{B}}(X) = \{f \in \mathcal{B} : f\big|_X \equiv 0\}.$$

For $J \subseteq \mathcal{B}$ denote

$$V(J) = \{z \in \mathbb{B}_d : \forall f \in J.\, f(z) = 0\}.$$

Theorem 5.3.3 ([50], Theorem 6.12; [104], Theorem 2.1.30)**.** *Let $\mathcal{B}$ be either $\mathcal{A}_d$ equipped with the norm topology, or $\mathcal{M}_d$ equipped with the weak-$*$ topology, and let $J \triangleleft \mathcal{B}$ be a closed homogeneous ideal. Then*

$$\sqrt{J} = I_{\mathcal{B}}(V(J)). \tag{5.3.4}$$

The above may be considered as a Nullstellensatz for homogeneous ideals in the algebra $\mathcal{B}$. Besides its intrinsic interest, it also immediately implies the following approximation-theoretic result.

Theorem 5.3.5 ([50], Corollary 6.13; [104], Corollary 2.1.31)**.** *Let $\mathcal{B}$ be either $\mathcal{A}_d$ equipped with the norm topology, or $\mathcal{M}_d$ equipped with the weak-$*$ topology, and let I be a radical homogeneous ideal in $\mathbb{C}[z]$. If $f \in \mathcal{B}$ vanishes on $V(I)$, then $f \in \overline{I}$.*

In other words, if a function $f \in \mathcal{A}_d$ vanishes on a homogeneous variety $V \subset \mathbb{B}_d$, then it can be approximated in norm (and, consequently, uniformly) by polynomials that vanish on V.

Remark 5.3.6. The results for $\mathcal{B} = \mathcal{A}_d$ were obtained in [50], while the extension to $\mathcal{B} = \mathcal{M}_d$ is from [104]. For brevity, this Section describes the results in the setting of either $\mathcal{A}_d$ or $\mathcal{M}_d$; but — as the proof depends only on the fact that $\mathcal{A}_d$ and $\mathcal{M}_d$ are algebras of multipliers on a Hilbert function space with circular symmetry — similar results hold in a more general setting, in particular in the setting of the ball algebra $A(\mathbb{B}_d)$ or $H^\infty(\mathbb{B}_d)$ (see the [50, 104] for further details). In the setting of non-homogeneous ideals, however, not much is known.

6. Model theory

The importance of the d-shift stems from the fact that it is a universal model for d-contractions, in fact, *the unique* universal model for d-contractions. The results of 6.1 and 6.2 have become well known thanks to their appearance in [20], though these results and the techniques that give them have been known before, at least in some form or other (see, e.g., [25, 26, 59, 94, 96, 98, 117]), and have been extended and generalized afterwards (see, e.g., [7, 11, 93, 100, 101]).

6.1. **Universality of the d-shift among pure row contractions.**

Recall the notation from Section 2.

Lemma 6.1.1. *Let T be a pure d-contraction on a Hilbert space H. Then there exists an isometry $W : H \to H^2_d \otimes \mathcal{D}_T$ such that for every multi-index α and all $g \in \mathcal{D}_T = \overline{\Delta_T H}$,*

$$W^*(z^\alpha \otimes g) = T^\alpha \Delta g. \tag{6.1.2}$$

Proof. Fix a Hilbert space E with orthonormal basis $\{e_1, \ldots, e_d\}$. In this proof, H^2_d and $\mathcal{F}_+(E)$ will be identified. Define an operator $W : H \to \mathcal{F}(E) \otimes \mathcal{D}_T$ by

$$Wh = \sum_{n=0}^{\infty} \sum_{i_1,\ldots,i_n=1}^{d} e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes \Delta T^*_{i_n} \cdots T^*_{i_1} h.$$

By purity, one has

$$\begin{aligned}
\|Wh\|^2 &= \sum_{n=0}^{\infty} \sum_{i_1,\ldots,i_n=1}^{d} \langle T_{i_1} \cdots T_{i_n} \Delta^2 T^*_{i_n} \cdots T^*_{i_1} h, h\rangle \\
&= \lim_{N\to\infty} \sum_{n=0}^{N} \langle (\Theta^n_T(I) - \Theta^{n+1}_T(I))h, h\rangle \\
&= \langle h, h\rangle - \lim_{N\to\infty} \langle \Theta^{N+1}_T(I)h, h\rangle = \|h\|^2.
\end{aligned}$$

From commutativity of T is follows that W maps H into $\mathcal{F}_+(E) \otimes \mathcal{D}_T$. Finally, letting $g \in \mathcal{D}_T$ and $h \in H$, it holds that

$$\begin{aligned}\langle W^*(e^\alpha \otimes g), h\rangle &= \sum_{n=0}^\infty \sum_{i_1,\ldots,i_n=1}^d \langle e^\alpha \otimes g, e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes \Delta T_{i_n}^* \cdots T_{i_1}^* h\rangle \\ &= \frac{|\alpha|!}{\alpha!}\|e^\alpha\|^2 \langle T^\alpha \Delta g, h\rangle \\ &= \langle T^\alpha \Delta g, h\rangle.\end{aligned}$$

Identifying z^α with e^α gives (6.1.2). ∎

If A is tuple of operators on G, a subspace $K \subseteq G$ is said to *co-invariant for A* if K is invariant for A^*, (equivalently, if $AK^\perp \subseteq K^\perp$).

Theorem 6.1.3. *Let T be a pure d-contraction on H. Then there exists a subspace $K \subset H_d^2 \otimes \mathcal{D}_T$ that is invariant for S^*, such that T is unitarily equivalent to the compression of $S \otimes I_{\mathcal{D}_T}$ to K. To be precise, there is an isometry $W : H \to H_d^2 \otimes \mathcal{D}_T$ such that $W(H) = K$ and*

$$T^* = W^*\big(S^* \otimes I_{\mathcal{D}_T}\big)\big|_K W. \tag{6.1.4}$$

Proof. Let W be as in Lemma 6.1.1 and denote $K = W(H)$. From (6.1.2) one finds $W^*(S \otimes I_{\mathcal{D}_T}) = TW^*$, thus $(S \otimes I_{\mathcal{D}_T})^* W = WT^*$. From this the invariance of K under $S^* \otimes I_{\mathcal{D}_T}$ as well as (6.1.4) follow. ∎

In particular, if one identifies H with K via W, then for every polynomial $p \in \mathbb{C}[z]$

$$p(T) = P_K(p(S) \otimes I)P_K. \tag{6.1.5}$$

6.2. **Drury's inequality.** The following facts are well known (see [113] or the chapter on commutative dilation theory by C. Ambrozie and V. Müller in this Handbook):

(1) (von Neumann's inequality [122]) For every contraction T and every polynomial p,
$$\|p(T)\| \leq \sup_{|z|\leq 1} |p(z)|.$$
(2) (Ando's inequality [10]) For every pair of commuting contractions S, T and every bivariate polynomial p,
$$\|p(S,T)\| \leq \sup_{|y|,|z|\leq 1} |p(y,z)|.$$
(3) (Varopoulos's example [118]) There exists a triple of commuting contractions R, S, T and a polynomial in three variables p such that
$$\|p(R,S,T)\| > \sup_{|x|,|y|,|z|\leq 1} |p(x,y,z)|.$$

Thus, the naive generalization of von Neumann's inequality to the multivariate setting,

$$\|p(T)\| \overset{?}{\leq} \|p\|_{\infty,\mathbb{D}^k} \tag{6.2.1}$$

for every k-tuple of commuting contractions, fails. The failure of von Neumann's inequality (6.2.1) in the multivariate setting, and the search for a suitable replacement that does work for several commuting operators, have been and are still the subject of great interest. A candidate for a replacement of von Neumann's inequality was obtained by Drury in 1978 [59].

Theorem 6.2.2. *Let T be a d-contraction. Then for every matrix valued polynomial $p \in \mathbb{C}[z_1, \ldots, z_d] \otimes M_k(\mathbb{C})$,*

$$\|p(T)\| \leq \|p(S)\|. \tag{6.2.3}$$

Proof. It is enough to prove this inequality for rT instead of T, for all $r \in (0,1)$. But as rT is pure, the inequality $\|p(rT)\| \leq \|p(S)\|$ is a direct consequence Theorem 6.1.3 (or equality (6.1.5)). ∎

When $d = 1$ then the above theorem reduces to von Neumann's inequality. When $d = 2$ then the above theorem fundamentally differs from Ando's inequality: one cannot replace the right hand side by multiple of the sup norm of p on the ball (cf. Theorem 3.7.1).

6.3. **Universality of the d-shift among d-contractions.** The model theory for d-contractions reached final form in [20, Theorem 8.5], and is presented in Theorem 6.3.3 below. For a precise formulation additional terminology is required.

Definition 6.3.1. *Let A be a tuple of operators on a Hilbert space G and K a subspace of G which is co-invariant for A. K is said to be* full *if*

$$G = [C^*(1, A)K].$$

Definition 6.3.2. *A* spherical unitary *is a d-tuple Z of commuting normal operators such that $\sum_i Z_i Z_i^* = 1$.*

Fix $d \in \{1, 2, \ldots, \infty\}$. Given $n \in \{0, 1, 2, \ldots, \infty\}$, one denotes by $n \cdot S$ the direct sum of S with itself n times acting on $n \cdot H^2_d$. Given a spherical unitary $Z = (Z_1, \ldots, Z_d)$ on a Hilbert space H_Z, one writes $n \cdot S \oplus Z$ for the d-contraction

$$(\underbrace{S_1 \oplus \cdots \oplus S_1}_{n \text{ times}} \oplus Z_1, \ldots, \underbrace{S_d \oplus \cdots \oplus S_d}_{n \text{ times}} \oplus Z_d)$$

on $\underbrace{H^2_d \oplus \cdots \oplus H^2_d}_{n \text{ times}} \oplus H_Z$. The case where $n = 0$ or Z represents the nil operator is also alowed.

Theorem 6.3.3. *Let $d < \infty$ and let T be a d-contraction on a separable Hilbert space. Then there is an $n \in \{0, 1, 2, \ldots, \infty\}$, a spherical unitary Z on H_Z, and subspace $K \subseteq n \cdot H^2 \oplus H_Z$ that is co-invariant and full for $n \cdot S \oplus Z$, such that T is unitarily equivalent to the compression of $n \cdot S \oplus Z$ to K.*

The triple (n, Z, K) is determined uniquely, up to unitary equivalence, by the unitary equivalence class of T. Moreover, Z is the nil operator if and only if T is pure, and $n = \mathrm{rank}(T)$.

Proof. The main ingredient of the proof is a combination of Arveson's extension theorem [16] and Stinespring's dilation theorem [110]. This method has appeared first in [17], and has been reused many times to obtain many dilation theorems. It runs as follows.

Suppose that T acts on H. By Theorem 6.2.2, the map $S_i \mapsto T_i$ extends to a unital completely contractive homomorphism $\Psi : \mathcal{A}_d \to B(H)$. By Arveson's extension theorem [16, Theorem 1.2.9], Ψ extends to a unital completely positive map $\Psi : \mathcal{T}_d \to B(H)$. By Stinespring's theorem [110], there is a Hilbert space G, an isometry $V : H \to G$, and a $*$-representation $\pi : \mathcal{T}_d \to B(G)$ such that

$$\Psi(X) = V^* \pi(X) V \ , \ X \in \mathcal{T}_d,$$

and such that $G = [\pi(\mathcal{T}_d)VH]$. The space $K = VH$ is full and co-invariant for $\pi(S)$, and V implements a unitary equivalence between T and a compression of $\pi(S)$.

Using Theorem 5.1.2, basic representation theory (see [19]) shows that π breaks up as a direct sum $\pi = \pi_a \oplus \pi_s$, where π_a is a multiple of the identity representation and π_s annihilates the compacts. It follows that $\pi_a(S) = n \cdot S$, that $Z := \pi_s(S)$ is a spherical unitary, and that $\pi(S) = n \cdot S \oplus Z$ dilates VTV^*. That shows that a model as stated in the first part of the proof exists. The remaining details are omitted. ∎

Remark 6.3.4. The above theorem and proof are also valid in the case $d = \infty$, with the important change that Z is not longer a spherical unitary, but merely a commuting tuple satisfying $\sum Z_i Z_i^* = 1$. In particular, Z is not necessarily normal, hence in this case the model reveals far less than in the $d < \infty$ case.

Theorem 6.3.3 implies the following subnormality result due originally to A. Athavale [25].

Corollary 6.3.5. *Let T be a commuting d-tuple ($d < \infty$) on a Hilbert space such that $T_1^* T_1 + \ldots + T_d^* T_d = 1$. Then T is subnormal.*

6.4. **Uniqueness of the d-shift.** The d-shift serves as a universal model for pure row contractions (Theorems 6.1.3 and 6.2.2). For $d > 1$, and in contrast to the case $d = 1$, the d-shift turns out to be the *unique* model for pure row contractions in the following sense.

Theorem 6.4.1 ([20], Lemma 7.14; see also [105])**.** *Suppose $d \geq 2$, let T be a d-contraction acting on H, and let $K \subseteq H$ be a subspace such that the compressed tuple $P_K T P_K$ is unitarily equivalent to the d-shift. Then K reduces T.*

For additional uniqueness and maximality properties of the d-shift, see [20, Section 7].

6.5. **The noncommutative setting.** The methods used above to show that S is a universal model for d-contractions work in a greater generality, to provide various universal models for tuples of operators satisfying certain constraints.

The key to these results is to examine what happens to the proof of Lemma 6.1.1 when a row contraction T satisfies more, or less, assumptions other than the assumption of being a commuting tuple. When T satisfies no assumptions besides that it be a row contraction, then the range of W appearing in the proof of the lemma might be larger then $\mathcal{F}_+(E)$. Thus the commutative d-shift S has to be replaced by the noncommutative d-shift L on $\mathcal{F}(E)$.

A tuple $V = (V_1 \ldots, V_d)$ on a Hilbert space G is said to be a *row isometry* if $V_i^* V_j = \delta_{ij} I_G$ for all i, j, which means that all the V_is are isometries with mutually orthogonal ranges. A row isometry is said to be of *Cuntz type* if $\sum V_i V_i^* = I_G$ (the convergence being understood as strong operator convergence in the case $d = \infty$). Applying the same reasoning one obtains the following theorem of J. Bunce [38], A. Frazho [68] and G. Popescu [96], which is a natural generalization of the Sz.-Nagy isometric dilation theorem [111].

Theorem 6.5.1. *Let $d \in \{1, 2, \dots, \infty\}$ and let T be a row contraction on a separable Hilbert space. Let L be the noncommutative shift acting on $\mathcal{F}(E)$, where $\dim E = d$. Then there is an $n \in \{0, 1, 2, \dots, \infty\}$, a row isometry V of Cuntz type acting on H_V, and a subspace $K \subseteq n \cdot \mathcal{F}(E) \oplus H_Z$ that is co-invariant and full for $n \cdot L \oplus V$, such that T is unitarily equivalent to the compression of $n \cdot L \oplus V$ to K.*

The triple (n, V, K) is determined uniquely, up to unitary equivalence, by the unitary equivalence class of T. Moreover, V is the nil operator if and only if T is pure, and $n = \mathrm{rank}(T)$.

6.6. **Constrained dilations.** The universality of the commutative and noncommutative d-shifts (Theorems 6.3.3 and 6.5.1) can be interpreted in the following way.

Fix d and let E be a d-dimensional Hilbert spaces with fixed orthonormal basis $\{e_1, \dots, e_d\}$, giving rise to the noncommutative d-shift $L = (L_1, \dots, L_d)$. Let $\mathbb{C}\langle z \rangle = \mathbb{C}\langle z_1, \dots, z_d \rangle$ denote the free algebra in d variables. Let $\mathfrak{C}$ be the commutator ideal in $\mathbb{C}\langle z \rangle$, that is, the ideal generated by $fg - gf$, where $f, g \in \mathbb{C}\langle z \rangle$. Note that $\mathbb{C}\langle z \rangle / \mathfrak{C} = \mathbb{C}[z]$. Now consider the closed subspace $[\mathfrak{C}]$ in $\mathcal{F}(E)$ (here $\mathbb{C}\langle z \rangle$ is identified with a dense subspace of $\mathcal{F}(E)$ in the natural way). Then $[\mathfrak{C}]$ is an invariant subspace for L, and $\mathcal{F}_+(E) = [\mathfrak{C}]^\perp$. Recall also that $S = P_{\mathcal{F}_+(E)} L P_{\mathcal{F}_+(E)}$.

The noncommutative d-shift L is a universal for row contractions, and the commutative d-shift S is universal for *commuting* row contractions. Now, a row contraction T is commuting if and only if it satisfies the relations in $\mathfrak{C}$, that is, $p(T) = 0$ for every $p \in \mathfrak{C}$. Thus the above discussion can be summarized in the following way: *the universal model for row contractions which satisfy the relations in $\mathfrak{C}$ is obtained by compressing L to $\mathcal{F}_+(E) = [\mathfrak{C}]^\perp$.*

Popescu discovered that the same holds when $\mathfrak{C}$ is replaced by an arbitrary ideal $J \triangleleft \mathbb{C}\langle z \rangle$: using more or less the same methods as above one obtains a universal model for row contractions satisfying the relations in J by compressing the noncommutative d-shift L to the co-invariant subspace $\mathcal{F}_J = [J]^\perp$. See [100] for details; similar results for special classes of ideals appear in [36, 109].

6.7. **Constrained dilations in the commutative case.** The results of [100] discussed in the previous paragraph can be compressed to the commutative case, yielding the following model theory for d-contraction satisfying polynomial relations.

For $J \triangleleft \mathbb{C}[z_1, \dots, z_d]$ an ideal in the algebra of d-variable (commutative) polynomials, let $[J]$ be its closure in H^2_d, and denote $\mathcal{F}_J = [J]^\perp$ and $S^J = P_{\mathcal{F}_J} S P_{\mathcal{F}_J}$. The tuple S^J gives $\mathcal{F}_J$ the structure of a Hilbert module, and it can be identified naturally with the quotient of H^2_d by the submodule $[J]$.

A row contraction V is said to be *of Cuntz type* if $\sum V_i V_i^* = 1$.

Theorem 6.7.1. *Fix d, and let $J \triangleleft \mathbb{C}[z_1, \dots, z_d]$ be an ideal. Let T be a d-contraction such that $p(T) = 0$ for every $p \in J$. Then there is a cardinal n, a row contractions V of Cuntz type on H_V satisfying $p(V) = 0$ for all $p \in J$, and subspace $K \subseteq n \cdot \mathcal{F}_J \oplus H_Z$ that is co-invariant and full for $n \cdot S^J \oplus V$, such that T is unitarily equivalent to the compression of $n \cdot S^J \oplus V$ to K.*

Moreover, V is the nil operator if and only if T is pure, and $n = \mathrm{rank}(T)$.

Remark 6.7.2. Under some additional conditions (for example, if J is a homogeneous ideal) the triple (n, V, K) is determined uniquely, up to unitary equivalence, by the unitary equivalence class of T.

Remark 6.7.3. For non-pure d-contractions the above model may not be very effective, since there is not much information on what V looks like. It can be shown, however, that if S^J is essentially normal (equivalently, if $\mathcal{F}_J$ is an essentially normal Hilbert module) then V is a normal tuple with spectrum in $\overline{V(J)} \cap \partial\mathbb{B}_d$.

6.8. **Other commutative models.** See the the chapter on commutative dilation theory by C. Ambrozie and V. Müller in this Handbook for a systematic construction of alternative models, given either by weighted shifts or by multiplication operators on spaces of analytic functions, which include the d-shift as a special case.

6.9. **Noncommutative domains.** In a different direction of generalization, G. Popescu obtained universal models for tuples satisfying a variety of different norm constraints, which include the row contractive condition as a special case [101]. For example, under some assumptions on the coefficients a_α, Popescu obtains a model for all tuples T which satisfy

$$\sum_\alpha a_\alpha T^\alpha T^{\alpha *} \leq I.$$

6.10. **Commutant lifting.** The classical Sz.-Nagy and Foias model theory [113] finds some of its most profound applications via the commutant lifting theorem [112] (see also [69]). It is natural therefore to expect a commutant lifting theorem in the setting of the model of 6.3. The following theorem is due to Ball, Trent and Vinnikov [33] (see also [8]).

Theorem 6.10.1 ([33], Theorem 5.1)**.** *Let K_1 and K_2 be Hilbert spaces. For $i = 1,2$, suppose that $M_i \subseteq H^2_d \otimes K_i$ is co-invariant for $S \otimes I_{K_i}$. Suppose that $X \in B(M_1, M_2)$ satisfies*

$$X^*(S \otimes I_{K_2})^*\big|_{M_2} = (S \otimes I_{K_1})^* X^*.$$

Then there exists $\Phi \in \mathcal{M}_d(K_1, K_2)$ such that

1. $M_\Phi^*\big|_{M_2} = X^*$,
2. $\|M_\Phi\| = \|X\|$.

Theorem 6.10.1 provides a commutant lifting result for the model of 6.3 only in the case where Z is the nil operator. The following theorem of K. Davidson and T. Le handles the non-pure case. If T is a d-contraction and $\tilde{T} = n \cdot S \oplus Z$ is the dilation given by Theorem 6.3.3 on $\tilde{H} = n \cdot H^2_d \oplus H_Z$, then one may consider H as a subspace of $\tilde{H}$ and T as the co-restriction of $\tilde{T}$ to H.

Theorem 6.10.2 ([46], Theorem 1.1)**.** *Suppose that $T = (T_1, \ldots, T_d)$ is a d-contraction on a Hilbert space H, and that X is an operator on H that commutes with $T_1, \ldots, T_n$. Let $\tilde{T} = (\tilde{T}_1, \ldots, \tilde{T}_d)$ on $\tilde{H}$ denote the dilation of T on provided by Theorem 6.3.3. Then there is an operator Y on $\tilde{H}$ that commutes with each $\tilde{T}_i$ for $i = 1, \ldots, d$, such that*

1. $Y^*\big|_H = X^*$.
2. $\|Y\| = \|X\|$.

Remark 6.10.3. There is also a commutant lifting theorem in the setting of 6.5 (see [96, Theorem 3.2]), and this commutant lifting theorem can be "compressed" down to co-invariant subspaces of L, giving rise to a commutant lifting theorem (for pure row contractions) in the constrained setting of 6.6. In particular one can obtain Theorem 6.10.1 above as a bi-product of the noncommutative theory in this way (see [46, Section 3] or [100, Theorem 5.1]).

7. Interpolation theory and function theory on subvarieties

7.1. Zero sets and varieties.

Definition 7.1.1. *Let $\mathcal{F}$ be a space of functions on a set X. Then a set $Y \subseteq X$ is said to be a* zero set for $\mathcal{F}$ *if there is an $f \in \mathcal{F}$ such that $Y = \{x \in X : f(x) = 0\}$. Y is said to be a* weak zero set *if it is the intersection of zero sets.*

As $\mathcal{M}_d \subseteq H^2_d$, every zero set of $\mathcal{M}_d$ is a zero set of H^2_d. In the converse direction all that is known is the following.

Theorem 7.1.2. *If $V \subseteq \mathbb{B}_d$ is a zero set for H^2_d, then it is a weak zero set for $\mathcal{M}_d$.*

Proof. See [3, Theorem 9.27], where this result is proved for any complete Pick Hilbert function space and its multiplier algebra. ∎

Definition 7.1.3. *Say that V is a* variety *in $\mathbb{B}_d$ if it is is a weak zero set of $\mathcal{M}_d$, that is, if it is defined as*

$$V = V(F) := \{\lambda \in \mathbb{B}_d : f(\lambda) = 0 \textit{ for all } f \in F\},$$

for some $F \subseteq \mathcal{M}_d$.

Remark 7.1.4. By Theorem 7.1.2, replacing H^2_d by $\mathcal{M}_d$ would lead to an equivalent definition.

Remark 7.1.5. This is not the usual definition of *analytic variety*, as only subsets $F \subseteq \mathcal{M}_d$ are allowed. Considering the familiar case $d = 1$ shows that the above definition is more restrictive than the usual one: any discrete set in $\mathbb{D}$ is an analytic variety, but only sequences satisfying the *Blaschke condition* can be zero sets of functions in $H^\infty(\mathbb{D}) = \mathcal{M}_1$ [70, Section II.2]).

It is immediate that if J is the WOT-closed ideal generated by F, then $V(F) = V(J)$. Given $X \subseteq \mathbb{B}_d$, denote by J_X the WOT-closed ideal

$$J_X = \{f \in \mathcal{M}_d : f(x) = 0 \text{ for all } x \in X\}.$$

Then $J_X = J_{V(J_X)}$.

For $X \subseteq \mathbb{B}_d$, denote by $\mathcal{H}_X = \overline{\text{span}}\{k_x : x \in X\}$.

Lemma 7.1.6 ([50], Lemma 5.5)**.** *If J is a radical homogeneous ideal in $\mathbb{C}[z]$, then*

$$\mathcal{H}_{V(J)} = \mathcal{F}_J := H^2_d \ominus J.$$

Lemma 7.1.7 ([51], Section 2)**.** *If $V \subseteq \mathbb{B}_d$ is a variety and X is a set, then $V = V(J_V)$ and $\mathcal{H}_X = \mathcal{H}_{V(J_X)}$.*

$V(J_X)$ is the smallest variety containing X, thus the final assertion of the above lemma can be rephrased to say that the space $\mathcal{H}_X$ does not change when one replaces X by its "Zariski closure".

7.2. **The complete Pick property.**

Definition 7.2.1. *Let $\mathcal{H}$ be a Hilbert function space on X, and let $K^{\mathcal{H}}$ be its kernel. Then $\mathcal{H}$ said to have the* complete Pick property *if the following two conditions are equivalent:*

(1) *For all $m, n \in \mathbb{N}$, all n points $x_1, \ldots, x_n \in X$ and all matrices $W_1, \ldots, W_n \in M_m(\mathbb{C})$, there is a contractive operator valued multiplier $\Phi \in \mathrm{Mult}(\mathcal{H}) \otimes M_m(\mathbb{C})$ such that $\Phi(x_i) = W_i$ for all $i = 1, \ldots, n$,*

(2) *The following $mn \times mn$ matrix is positive semi-definite:*

$$\left[(I - W_j W_i^*) K^{\mathcal{H}}(x_j, x_i)\right]_{i,j=1}^n \geq 0. \tag{7.2.2}$$

If $\mathcal{H}$ has the complete Pick property then it is said to be a *complete Pick space*, the kernel $K^{\mathcal{H}}$ is said to be a *complete Pick kernel*, and the multiplier algebra $\mathrm{Mult}(\mathcal{H})$ is said to be a *complete Pick algebra*. Some researchers use the term *complete Nevanlinna-Pick kernel* instead of complete Pick kernel, etc. The terminology comes from the fact that, if $m = 1$, $\mathcal{H}$ is the Hardy space on the disc $H^2(\mathbb{D})$ and $K^{\mathcal{H}}$ is the Szegő kernel $s(z, w) = \frac{1}{1 - z\overline{w}}$, then (7.2.2) is the necessary and sufficient condition given by Pick's classical interpolation theorem [70, Theorem I.2.2].

The reader is referred to [3] for background and complete treatment of interpolation problems of this sort.

Remark 7.2.3. One may also consider the *operator valued Pick property*, where the matrices $W_1, \ldots, W_n \in M_m(\mathbb{C})$ in the above definition are replaced with an n-tuple of operators on some Hilbert space K, and the required Φ is a $B(K)$ valued function on X multiplying $\mathcal{H} \otimes K$ into itself. However, it can be shown that the operator valued Pick property is equivalent to the complete Pick property.

In any Hilbert function space (7.2.2) is a necessary condition for the existence of a contractive multiplier Φ that satisfies $\Phi(x_i) = W_i$ for all $i = 1, \ldots, n$ [3, Theorem 5.8]. Complete Pick spaces are the spaces in which (7.2.2) is also a sufficient condition.

Theorem 7.2.4. *The Drury-Arveson space H^2_d has the complete Pick property.*

Proof. This theorem has several proofs.

A Hilbert function space theoretic proof was given by J. Agler and J. McCarthy [2] (following works of McCullough [89] and Quiggin [103]). In fact [2] characterizes all complete Pick kernels, showing that an irreducible kernel $K^{\mathcal{H}}$ is a complete Pick kernel if and only if for any finite set $x_1, \ldots, x_n$, the matrix

$$\left[\frac{1}{K^{\mathcal{H}}(x_j, x_i)}\right]_{i,j=1}^n$$

has exactly one positive eigenvalue. The kernel (3.2.1) is easily seen to satisfy this property.

A proof based on the commutant lifting theorem 6.10.1 was given by Ball, Trent and Vinnikov [33, p. 118] (see also [15] for a proof via noncommutative commutant lifting). The proof, based on a deep idea which goes back to [107], runs as follows.

Let $x_1, \ldots, x_n \in \mathbb{B}_d$ and $W_1, \ldots, W_n \in M_m(\mathbb{C})$ be as in Definition 7.2.1. Put $H = \mathbb{C}^m$, and define

$$N_1 = \overline{\mathrm{span}}\{k_{x_i} \otimes h : i = 1, \ldots, n;\ h \in H\}$$

and

$$N_2 = \overline{\text{span}}\{k_{x_i} \otimes W_i^* h : i = 1, \dots, n;\ h \in H\}.$$

By (3.8.2), N_1 and N_2 are co-invariant. Now define $X : N_2 \to N_1$ to be the adjoint of the operator $X^* : N_1 \to N_2$ defined by

$$X^*(k_{x_i} \otimes h) = k_{x_i} \otimes W_i^* h \quad , \quad i = 1, \dots, n;\ h \in H.$$

It is clear that $X^*(S \otimes I)^*\big|_{N_1} = (S \otimes I)^* X^*$, and the condition (7.2.2) implies that $\|X^*\| \leq 1$. By Theorem 6.10.1 there exists a contractive multiplier $\Phi \in \mathcal{M}_d(H)$ satisfying $M_\Phi^*\big|_{N_1} = X^*$. Since

$$M_\Phi^* k_{x_i} \otimes h = k_{x_i} \otimes \Phi(x_i)^* h$$

for all $h \in H$, it follows that $\Phi(x_i) = W_i$.

An alternative proof is provided in [33, p. 108] (see also [63]) using what is sometimes called “the lurking isometry” argument. The main idea is that (7.2.2) is used to construct directly a unitary as in (3.8.6) which realizes the interpolating multiplier by formula (3.8.7).

Finally, there is also a proof that passes through the noncommutative setting via a distance formula, found independently by K. Davidson and D. Pitts [47] and by A. Arias and G. Popescu [15]. The roots of this proof can also be traced back to [107]. Here are a few details of the proof, compressed to the commutative setting.

Suppose that (7.2.2) holds, and for simplicity assume that $W_1, \dots, W_n$ are all in $\mathbb{C}$. It is easy to see that there is *some* function $f \in \mathcal{M}_d$ that satisfies $f(x_i) = W_i$ for $i = 1, \dots, n$. The norm of f could be anything, but it can be modified by adding a function vanishing on $\{x_1, \dots, x_n\}$. Let J be the ideal

$$J = \{g \in \mathcal{M}_d : g(x_i) = 0,\ i = 1, \dots, n\}.$$

If h is another multiplier satisfying $h(x_i) = W_i$ for $i = 1, \dots, n$, then there is some $g \in J$ such that $h = f + g$. Thus, there is a multiplier $h \in \mathcal{M}_d$ satisfying $\|h\| \leq 1$ and $h(x_i) = W_i$ for $i = 1, \dots, n$ if and only if $\inf_{g \in J} \|f + g\| = \text{dist}(f, J) \leq 1$. By the Arias-Popescu/Davidson-Pitts distance formula alluded to above ([15, Proposition 1.3] and [47, Theorem 2.1]),

$$\text{dist}(f, J) = \|P_N M_f P_N\|, \tag{7.2.5}$$

where $N = [J]^\perp = \text{span}\{k_{x_i} : i = 1, \dots, n\}$. A computation now shows that $\|P_N M_f P_N\| \leq 1$ is equivalent to (7.2.2). ∎

Remark 7.2.6. The second and fourth proofs described above (using commutant lifting or the distance formula) generalize easily to give additional interpolation theorems for the algebra $\mathcal{M}_d$, such as Carathéodory interpolation (see [15, 47]). The third proof (the “lurking isometry” argument) can be used to obtain interpolation results in other algebras of functions (for example $H^\infty(\mathbb{D}^2)$), and further results as well (see [1, 31]). The first proof is based on the characterization of complete Pick kernels, from which it follows that the kernel (3.2.1) of the space H^2_d plays a universal role; this is discussed in the next paragraph.

7.3. **The universal kernel.** For $d \in \{1, 2, \dots, \infty\}$, the notation k^d will be used below to denote the kernel (3.2.1) of H^2_d, to emphasize the dependence on d.

Definition 7.3.1. *Let $\mathcal{H}$ be an Hilbert function space on a set X with kernel $K^{\mathcal{H}}$. The kernel $K^{\mathcal{H}}$ is said to be* irreducible *if*

(1) *For every $x \neq y$ in X, $K_x^{\mathcal{H}} = K^{\mathcal{H}}(\cdot, x)$ and $K_y^{\mathcal{H}} = K^{\mathcal{H}}(\cdot, y)$ are linearly independent.*
(2) *For all $x, y \in X$, $K^{\mathcal{H}}(x, y) \neq 0$.*

It is a fact that every (complete) Pick kernel can be broken up into irreducible pieces [3, Lemma 7.2].

Definition 7.3.2. *If $\mathcal{H}$ is a Hilbert function space on X with kernel $K^{\mathcal{H}}$ and $\mu : X \to \mathbb{C}$ is a non-vanishing function, then one denotes by $\mu\mathcal{H}$ the Hilbert function space $\{\mu f : f \in \mathcal{H}\}$.*

Remark 7.3.3. The kernel of $\mu\mathcal{H}$ is given by

$$K^{\mu\mathcal{H}}(x, y) = \mu(x)\overline{\mu(y)}K^{\mathcal{H}}(x, y).$$

It follows from this and from Theorem 3.8.2 that $\mathcal{H}$ and $\mu\mathcal{H}$ have identical multiplier algebras, meaning that the set of multipliers is the same and that the multiplier norm is also the same.

Agler and McCarthy showed that H^2_d is a universal complete Pick space in the sense of the following theorem.

Theorem 7.3.4 ([2], Theorem 4.2)**.** *Let $\mathcal{H}$ be a Hilbert function space with an irreducible kernel $K^{\mathcal{H}}$. Then $K^{\mathcal{H}}$ is a complete Pick kernel if and only if there is a cardinal number $d \leq \aleph_0$, an injective function $f : X \to \mathbb{B}_d$ and a non-vanishing function $\delta : X \to \mathbb{C}$ such that*

$$K^{\mathcal{H}}(x, y) = \delta(x)\overline{\delta(y)}k^d(f(x), f(y)) = \frac{\delta(x)\overline{\delta(y)}}{1 - \langle f(x), f(y)\rangle}. \tag{7.3.5}$$

Moreover, if this happens, then the map $K_x^{\mathcal{H}} \mapsto \delta(x)k^d_{f(x)}$ is an isometry from $\mathcal{H}$ onto a subspace of $\delta \circ f^{-1}H^2_d$.

A consequence of this result is that every for complete Pick space $\mathcal{H}$, there is cardinal number $1 \leq d \leq \infty$ and a variety $V \subseteq \mathbb{B}_d$ such that $\mathrm{Mult}(\mathcal{H})$ is completely isometrically isomorphic to the restriction algebra $\mathcal{M}_V = \{f\big|_V : f \in \mathcal{M}_d\}$ (see Theorem 8.4.1 below).

7.4. **Generalized interpolation problems.** For further results on interpolation in H^2_d see [27] and the reference therein; for interpolation in a broader framework including Drury-Arveson space see [32].

8. Submodules, quotient modules and quotient algebras

8.1. **Submodules and quotients.** Let K be Hilbert space. A subspace $L \subseteq H^2_d \otimes K$ that is invariant under $S \otimes I_K$ is a Hilbert module over $\mathbb{C}[z]$ in its own right, and is referred to as a *submodule* of $H^2_d \otimes K$. Algebraically, this determines a quotient module $H^2_d \otimes K/L$. The quotient module can be normed using the quotient norm, making it a Hilbert module.

Put $N = L^\perp$. As N is co-invariant for $S \otimes I_K$, it is also a Hilbert module determined by the action of $T = P_N S P_N$. The Hilbert modules $H^2_d \otimes K/L$ and N are unitarily equivalent.

A natural problem is to determine all submodules and all quotients of $H^2_d \otimes K$. This is a fundamental problem, since, by Theorem 6.1.3, every pure contractive Hilbert module is a quotient of $H^2_d \otimes K$ for some K. The case $K = \mathbb{C}$ is the best understood.

8.2. **Invariant subspaces of H^2_d and ideals.** In [48, Theorem 2.1] it was shown that there is a bijective correspondence between two sided WOT-closed ideals in $\mathcal{L}_d$ and subspaces of $\mathcal{F}_d$ which are invariant under L and also under the right shift. The bijective correspondence is the map sending an ideal J to its range space $[J\mathcal{F}_d] = [J \cdot 1]$. The following two theorems concerning ideals and invariant subspaces in $\mathcal{M}_d$ follow from this bijective correspondence together with 4.4 and 4.9 (see [51, Section 2] for some details).

Denote by $\mathrm{Lat}(\mathcal{M}_d)$ and $\mathrm{Id}(\mathcal{M}_d)$ the lattices of the closed invariant subspaces of $\mathcal{M}_d$ and the WOT-closed ideals in $\mathcal{M}_d$, respectively.

Theorem 8.2.1. *Define a map $\alpha : \mathrm{Id}(\mathcal{M}_d) \to \mathrm{Lat}(\mathcal{M}_d)$ by $\alpha(J) = [J \cdot 1]$. Then α is a complete lattice isomorphism whose inverse β is given by*

$$\beta(K) = \{f \in \mathcal{M}_d : f \cdot 1 \in K\}.$$

Theorem 8.2.2. *If J is* WOT*-closed ideal in $\mathcal{M}_d$ with $\alpha(J)^\perp = N$, then $\mathcal{M}_d/J$ is completely isometrically isomorphic and weak-$*$ homeomorphic to $P_N\mathcal{M}_dP_N$.*

8.3. **Quotients of H^2_d and quotients of $\mathcal{M}_d$ associated to varieties.** Let $V \subseteq \mathbb{B}_d$ be a variety (see Section 7.1). The space $\mathcal{H}_V$ can be considered as a Hilbert function space on V, and its multiplier algebra $\mathrm{Mult}(\mathcal{H}_V)$ is an algebra of functions on V. Denote $\mathcal{M}_V = \{g : V \to \mathbb{C} : \exists f \in \mathcal{M}_d. \, f\big|_V = g\}$. Using Theorems 7.2.4 and 8.2.2 the following theorem is deduced.

Theorem 8.3.1. *Let $V \subseteq \mathbb{B}_d$ be a variety. Then $\mathcal{H}_V = [J_V \cdot 1]^\perp$, and*

$$\mathrm{Mult}(\mathcal{H}_V) = \mathcal{M}_V \cong \mathcal{M}_d/J_V \cong P_{\mathcal{H}_V}\mathcal{M}_dP_{\mathcal{H}_V}$$

where $\cong$ denotes completely isometric and WOT*-continuous isomorphisms, given by*

$$f\big|_V \longleftrightarrow f + J_V \longleftrightarrow P_{\mathcal{H}_V}M_fP_{\mathcal{H}_V}.$$

8.4. **The universal complete Pick algebra.** Theorems 7.3.4 and 8.3.1 imply the following result.

Theorem 8.4.1. *Let H be a separable, irreducible complete Pick Hilbert function space on a set X. Then there is a cardinal $d \in \{1, 2, \ldots, \aleph_0\}$ and a variety $V \subseteq \mathbb{B}_d$ such that $\mathrm{Mult}(H)$ is complete isometrically isomorphic to $\mathcal{M}_V$. V can be chosen to be the smallest variety containing $f(X)$, where f is the as in Theorem 7.3.4.*

8.5. **Maximal ideal spaces.** Being commutative Banach algebras, the algebras $\mathcal{M}_V$ are determined to a very a large extent by their maximal ideal space $\mathfrak{M}(\mathcal{M}_V)$, that is, the space of complex homomorphism from $\mathcal{M}_V$ to $\mathbb{C}$. Elements of $\mathfrak{M}(\mathcal{M}_V)$ are also referred to as *characters*. The full maximal ideal space is too big to be tractable — it is the space of WOT-continuous characters that is amenable to analysis (see Section 11).

Theorem 8.5.1 ([48], Theorem 3.3; [51], Proposition 3.2)**.** *Let $V \subseteq \mathbb{B}_d$ be a variety. There is a continuous projection $\pi : \mathfrak{M}(\mathcal{M}_V) \to \overline{\mathbb{B}_d}$ given by*

$$\pi(\rho) = (\rho(S_1), \ldots, \rho(S_d)) \ , \ \rho \in \mathfrak{M}(\mathcal{M}_V).$$

For each $\lambda \in V$ there is a unique character $\rho_\lambda \in \pi^{-1}(\lambda)$ given by

$$\rho_\lambda(f) = f(\lambda) = \langle M_fk_\lambda, k_\lambda\rangle/\|k_\lambda\|^2 \ , \ f \in \mathcal{M}_V.$$

ρ_λ is WOT*-continuous and every* WOT*-continuous character arises this way. Moreover, $\pi(\mathfrak{M}(\mathcal{M}_V)) \cap \mathbb{B}_d = V$, and $\pi\big|_{\pi^{-1}(V)}$ is a homeomorphism.*

In the case $\mathcal{M}_V = \mathcal{M}_d$ (i.e., the case $V = \mathbb{B}_d$), π is onto $\overline{\mathbb{B}_d}$, and for every $\lambda \in \partial\mathbb{B}_d$ the fiber $\pi^{-1}(\lambda)$ is canonically homeomorphic to the fiber over 1 in $\mathfrak{M}(H^\infty(\mathbb{D}))$.

8.6. **Beurling type theorems.** In [90] S. McCullough and T. Trent obtained the following generalization of the classical Beurling-Lax-Halmos theorem [35, 77, 86].

Theorem 8.6.1 ([90])**.** *Let L be a subspace of $H^2_d \otimes K$. The following are equivalent.*

1. *L is invariant under $S \otimes I_K$.*
2. *L is invariant under $\mathcal{M}_d \otimes I_K$.*
3. *There is an auxiliary Hilbert space K_* and $\Phi \in \mathcal{M}_d(K_*, K)$ such that $M_\Phi M_\Phi^*$ is the projection onto L and $L = M_\Phi(H_2 \otimes K_*)$.*

Actually, a version of this theorem holds in any complete Pick space [90], thus in particular it holds for quotients of H^2_d of the type $\mathcal{H}_V$ considered in Section 8.3. In [4, 5, 37] finite dimensional invariant subspaces of $S^* \otimes I_K$ were studied, and further information was obtained.

Since $M_\Phi M_\Phi^*$ is a projection, M_Φ is a partial isometry. A multiplier Φ for which M_Φ is a partial isometry is called a *inner function.* When $d = \dim K = 1$, it can be shown that (unless L is trivial) K_* can be chosen to be one dimensional and Φ can be chosen so that M_Φ is an isometry. In this case Φ is a scalar valued function on the disc which has absolute value 1 a.e. on the circle, i.e. an *inner function* in the classical sense, and one recovers Beurling's theorem [35] (see [70, Chapter II]).

Theorem 8.6.1 was obtained by Arveson in the case where $\dim K = 1$ [21, Section 2]. In this case $\Phi \in \mathcal{M}_d(K_*, \mathbb{C})$, and this means that there is a sequence $\{\phi_n\}_{n=0}^{\dim K_*}$ such that $P_L = \sum M_{\phi_n} M_{\phi_n}^*$ and $L = \sum M_{\phi_n} H^2_d$ (just put $\phi_n = \Phi_n(1 \otimes e_n)$ where $\{e_n\}$ is an orthonormal basis for K_*). Now

$$\sum |\phi_n(z)|^2 \|k_z\|^2 = \langle \sum M_{\phi_n} M_{\phi_n}^* k_z, k_z \rangle \leq \|k_z\|^2,$$

so $\sup_{\|z\|<1} \sum |\phi_n(z)|^2 \leq 1$. In particular, for every n, $\phi_n \in H^\infty(\mathbb{B}_d)$, and therefore the radial limit $\tilde{\phi_n}(w) = \lim_{r \nearrow 1} \phi_n(rw)$ exists for a.e. $w \in \partial\mathbb{B}_d$ (in fact the limit exists through much larger regions of convergence, see [106, Theorem 5.6.4]). Arveson raised the problem of whether or not $\sum_n |\tilde{\phi}_n(w)|^2 = 1$ for a.e. $w \in \partial\mathbb{B}_d$. This problem was solved by D. Greene, S. Richter and C. Sundberg [72].

For every $\lambda \in \mathbb{B}_d$, let $E_\lambda : H^2_d \otimes K \to K$ denote the point evaluation functional $E_\lambda f \otimes k = f(\lambda)k$.

Theorem 8.6.2 ([72])**.** *Let K be a separable Hilbert space, let L be an invariant subspace of $H^2_d \otimes K$, and let K_* and Φ be as in Theorem 8.6.1. If $d < \infty$, then for a.e. $w \in \partial\mathbb{B}_d$, the radial limit $\tilde{\Phi}(w) := \lim_{r \nearrow 1} \Phi(rw)$ exists and is a partial isometry with*

$$\operatorname{rank} \Phi(w) = \sup_{\lambda \in \mathbb{B}_d} \dim(E_\lambda L). \tag{8.6.3}$$

In particular, if $\dim K = 1$, then for a.e. $w \in \partial\mathbb{B}_d$

$$\sum_n |\tilde{\phi}_n(w)|^2 = 1. \tag{8.6.4}$$

8.7. **Rigidity phenomena.** Recall that Beurling's theorem says that every submodule of $H^2(\mathbb{D})$ has the form $\phi H^2(\mathbb{D})$, where ϕ is an inner function. Theorems 8.6.1 and 8.6.2 shows that a very similar result holds for submodules of H^2_d, thereby providing a strong analogy between the submodule theories of $H^2(\mathbb{D})$ and H^2_d. On the other hand, there are some big differences as well. For example, a consequence of Beurling's theorem is that every two submodules of $H^2(\mathbb{D})$ are unitarily equivalent; the following theorem of K. Guo, J. Hu and X. Xu shows that for $d \geq 2$, the situation with submodules of H^2_d is almost the opposite.

Theorem 8.7.1 ([73], Section 5; [41], Section 6). *Let M and N be submodules of H^2_d, $d \geq 2$. Consider the following conditions.*

(1) *M or N is the closure of a polynomial ideal,*
(2) *$M \subseteq N$.*

Under the assumption that one of the above conditions holds, if M is unitarily equivalent to N, then $M = N$.

9. The curvature invariant of a contractive Hilbert module

9.1. **The curvature invariant.** In [21] Arveson introduced a numerical invariant for contractive Hilbert modules of finite rank, the *curvature invariant*.

Let $d < \infty$, and fix a contractive Hilbert module of finite rank d. Recall that this means that there is a d contraction T on H such that $\operatorname{rank} H := \operatorname{rank}(T) = \dim \Delta H < \infty$, where $\Delta = \Delta_T = \sqrt{I - \sum T_i T_i^*}$. For $z \in \mathbb{B}_d$, define the operator valued functions

$$T(z) = \overline{z}_1 T_1 + \ldots \overline{z}_d T_d,$$

and

$$F(z) = \Delta(1 - T(z)^*)^{-1}(1 - T(z))^{-1}\Delta.$$

For z, $F(z)$ is an operator on the finite dimensional space ΔH, hence has a trace.

Lemma 9.1.1 ([21], Theorem A). *For almost every $w \in \partial\mathbb{B}_d$ the limit*

$$\kappa_0(w) = \lim_{r \nearrow 1}(1 - r^2)\operatorname{trace} F(rw) \tag{9.1.2}$$

exists and satisfies $0 \leq \kappa_0(w) \leq \operatorname{rank}(H)$.

Definition 9.1.3. *The* curvature invariant *of H is defined to be*

$$\kappa(H) = \int_{\partial\mathbb{B}_d} \kappa_0(w) d\sigma(w), \tag{9.1.4}$$

where σ is normalized area measure on the sphere.

One also writes $\kappa(T)$ for the curvature of H. From (9.1.2), $\kappa(H)$ is a real number between 0 and $\operatorname{rank}(H)$.

Theorem 9.1.5 ([21], Theorem 2.1). *Suppose that H is a pure contractive Hilbert module of finite rank. Then $\kappa(H) = \operatorname{rank}(H)$ if and only if H is unitarliy equivalent to the free Hilbert module $H^2_d \otimes \Delta H$ of rank* $\operatorname{rank}(H)$.

The curvature invariant is evidently invariant under unitary equivalence. The above theorem shows that the curvature contains non-trivial operator theoretic information. Arveson used the curvature invariant to prove the $\dim K = 1$ case of Theorem 8.6.2 for the case where the submodule L contained a polynomial [21, Theorem E].

9.2. **The Euler characteristic.** The analytically defined curvature invariant is closely tied to an algebraic invariant called the *Euler characteristic*.

If H is a finite rank contractive Hilbert module, then the linear space

$$M_H = \{p \cdot \xi | \, p \in \mathbb{C}[z], \, \xi \in \Delta H\}$$

is a finitely generated Hilbert module over the ring $\mathbb{C}[z]$. By Hilbert's syzygy theorem [60, Corollary 19.8] there is a finite free resolution

$$0 \to F_n \to \ldots \to F_2 \to F_1 \to M_H \to 0 \tag{9.2.1}$$

where each F_k is the (algebraic) module direct sum of β_k copies of $\mathbb{C}[z]$.

Definition 9.2.2. *The* Euler characteristic *of H is defined by*

$$\chi(H) = \sum_{k=1}^{n} (-1)^{k+1} \beta_k.$$

Remark 9.2.3. One can show that $\chi(H)$ does not depend on the choice of free resolution (9.2.1).

Theorem 9.2.4. $0 \leq \kappa(H) \leq \chi(H) \leq \mathrm{rank}(H)$.

9.3. **Graded modules and Arveson's "Gauss-Bonnet" theorem.** A Hilbert module H is said to be *graded* if there exists a strongly continuous unitary representation Γ of the circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ on H such that

$$\Gamma(\lambda) T_k \Gamma(\lambda)^{-1} = \lambda T_k \;\; k = 1, \ldots, d, \;\; \lambda \in \mathbb{T}.$$

Denoting $H_n = \{h \in H : \Gamma(\lambda) h = \lambda^n h\}$, one obtains the decomposition

$$H = \ldots H_{-1} \oplus H_0 \oplus H_1 \oplus H_2 \oplus \ldots, \tag{9.3.1}$$

and every operator T_k is of degree one in the sense that $T_k H_n \subseteq H_{n+1}$. The existence of a representation of $\mathbb{T}$ should be thought of as a kind of minimal symmetry that H possesses.

The Hilbert module $H^2_d \otimes K$ is a graded Hilbert module, and the decomposition (9.3.1) is the natural one induced by the degree of polynomials (there are no negatively indexed summands in the grading of $H^2_d \otimes K$). If $I \triangleleft \mathbb{C}[z]$ is a homogeneous ideal, then its closure in H^2_d is also a graded contractive Hilbert module.

Theorem 9.3.2 ([21], Theorem B)**.** *Let H be a contractive, pure, finite rank and graded Hilbert module. Then*

$$\kappa(H) = \chi(H). \tag{9.3.3}$$

In particular, the curvature is an integer.

In [67, Theorem 18] the above theorem was generalized to quotients of $H^2_d \otimes \mathbb{C}^r$ by polynomially generated submodules.

9.4. **Integrality of the curvature invariant.** Theorem 9.3.2 naturally raised the question whether the curvature invariant is always an integer. Using Theorem 8.6.2, Greene, Richter and Sundberg proved that this is so.

Recall that if H is a pure, contractive Hilbert module then, by Theorem 6.1.3, H can be identified with the quotient of $H^2_d \otimes K$ by a submodule L, where $\dim K = \mathrm{rank}(H)$.

Theorem 9.4.1 ([72]). *Let H be a pure, contractive Hilbert module of finite rank. Then*

$$\kappa(H) = \operatorname{rank}(H) - \sup_{\lambda \in \mathbb{B}_d} \dim(E_\lambda L).$$

In particular, $\kappa(H)$ is an integer.

9.5. **The curvature as index.** The following theorem of J. Gleason, S. Richter and C. Sundberg exhibits the curvature invariant as the index of a Fredholm tuple (for more details on spectral theory and Fredholm theory of commuting d-tuples, see the chapter on multi-parameter spectral theory by V. Müller in this Handbook).

Theorem 9.5.1 ([71], Theorem 4.5). *Let T be a pure d-contraction of finite rank. Denote the essential spectrum of T by $\sigma_e(T)$. Then $\sigma_e(T) \cap \mathbb{B}_d$ is contained in an analytic variety, and for all $\lambda \in \mathbb{B}_d \setminus \sigma_e(T)$ the tuple $T - \lambda = (T_1 - \lambda_1, \ldots, T_d - \lambda_d)$ is Fredholm, and*

$$\kappa(T) = (-1)^d \operatorname{ind}(T - \lambda).$$

This theorem implies that the curvature invariant is stable under compact perturbations:

Corollary 9.5.2. *Let T and T' be two pure d-contractions of finite rank. If $T_i - T_i'$ is compact for all i, then $\kappa(T) = \kappa(T')$.*

9.6. **Generalizations.** The curvature invariant has also been defined for row contractions which are not necessarily commutative [85, 99], and this notion has been generalized further for completely positive maps [92]. An extension to the setting where row-contractivity is replaced by a more general condition $p(T, T^*) \geq 0$ (for some polynomial p) is considered in [61].

10. Essential normality and the conjectures of Arveson and Douglas

In [21] it was shown that the curvature invariant (as well as the Euler characteristic) is stable under finite rank perturbations, but left open whether it is invariant under compact perturbations. This problem was taken up in [22] for graded Hilbert modules. By exhibiting the curvature invariant of H as a the index of a certain operator — the *Dirac operator* of the d-contraction T associated with H — it was shown that if H is essentially normal then $\kappa(T) = \kappa(T')$ whenever T' is unitarily equivalent to T modulo compacts (the curvature invariant was eventually shown to be equal to the index of a Fredholm tuple — hence invariant under compact perturbations — by Gleason, Richter and Sundberg; see 9.5 above. The conjectures in this section remained a subject of growing interest for other reasons, see [53]).

Based on these considerations Arveson raised the question whether every pure graded contractive Hilbert module of finite rank is essentially normal [22, Problem 2]. In fact, following the examination of several classes of examples, Arveson conjectured that every pure graded contractive Hilbert module of finite rank is p-essentially normal for all $p > d$. By Theorem 6.1.3 this can be reformulated as follows.

Conjecture 10.0.1. *Let K be a finite dimensional Hilbert space, and let $L \subseteq H^2_d \otimes K$ be a graded submodule. Then $(H^2_d \otimes K)/L$ is p-essentially normal for all $p > d$.*

This conjecture attracted a lot of attention [23, 24, 52, 53, 56, 57, 58, 62, 66, 74, 75, 76, 80, 81, 108], where the conjecture was proved in particular classes of submodules, but it is still far from being solved. In all cases where the conjecture was verified, the following stronger conjecture due to Douglas was also shown to hold.

Conjecture 10.0.2. *Let K be a finite dimensional Hilbert space, and let $L \subseteq H^2_d \otimes K$ be a graded submodule. Then $(H^2_d \otimes K)/L$ is p-essentially normal for all $p > \dim(L)$.*

Here $\dim(L)$ is defined as follows. Let $H = H_0 \oplus H_1 \oplus \ldots$ be the grading of $(H^2_d \otimes K)/L$. It is known that there is a polynomial $p_L(x)$ such that $p_L(n) = \dim H_n$ for sufficiently large n. Then $\dim L$ is defined to be $\deg p_L(x) + 1$.

10.1. **K-homology.** Let $I \triangleleft \mathbb{C}[z]$ be an ideal of infinite co-dimension. Denote $S^I = P_{I^\perp} S P_{I^\perp}$. Thus, S^I is the d-contraction acting on the quotient Hilbert module $H^2_d/[I]$. Define $\mathcal{T}_I = C^*(S^I, 1)$, and let $\mathcal{K}$ denote the compact operators on $H^2_d \ominus I$.

Lemma 10.1.1. *$\mathcal{K} \subseteq \mathcal{T}_I$.*

If H^2_d/I is essentially normal, then by the Lemma one has the following exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T}_I \longrightarrow C(X) \longrightarrow 0. \tag{10.1.2}$$

It can be shown (see, e.g., [74, Section 5]) that if I is homogeneous then $X = V(I) \cap \partial\mathbb{B}_d$, where $V(I) = \{z \in \mathbb{C}^d : p(z) = 0 \text{ for all } p \in I\}$. Thus one obtains an element in the odd K-homology group of the space $V(I) \cap \partial\mathbb{B}_d$. R. Douglas raises in [53, Section 3] the problem of determining which element of $K_1(V(I) \cap \partial\mathbb{B}_d)$ this extension gives rise to, and conjectures that it is a certain specific element, the so-called *fundamental class* of $V(I) \cap \partial\mathbb{B}_d$. K. Guo and K. Wang have found some evidence for this conjecture, see [74, 75].

10.2. **Some positive results.** In this section some positive results in the direction of Conjecture 10.0.2 are listed. For simplicity, only the case $L = [I]$, where I is a homogeneous ideal in $\mathbb{C}[z]$, is treated. There is not much loss in this; [24, Corollary 8.4] reduces the problem to the case where the submodule $L \subseteq H^2_d \otimes K$ is generated by terms of degree 1, and [108, Section 5] reduces the problem further to the case where $\dim K = 1$ and $L = [I]$ is the closure of a homogeneous ideal I that is generated by scalar valued polynomials of degree 2 (the second reduction involves the loss in the range of p's for which p-essential normality holds).

Theorem 10.2.1. *Let I be a homogeneous ideal in $\mathbb{C}[z]$, and let $L = [I]$ be its closure in H^2_d. If I satisfies any one of the following assumptions, then H^2_d/L is p-essentially normal for all $p > \dim I$.*

1. *I is generated by monomials.*
2. *I is principal.*
3. *$\dim I \leq 1$.*
4. *$d \leq 3$.*
5. *I is the radical ideal corresponding to a union of subspaces.*

Proof. The first item is proved in [23, 52] and the last one is proved in [81]; the rest are proved in [74]. Several different approaches and generalizations appear in the other references cited in the paragraph preceding Conjecture 10.0.2. ∎

10.3. **A non-graded counter example.** Conjecture 10.0.2 is stated for quotients of $H^2_d \otimes K$ by a graded submodule L. There is reason to believe that the conclusion is true also for the case where L is generated by K-valued polynomials, indeed some positive results have been obtained for quasi-homogeneous submodules [56, 75, 76] or in the case of principal submodules [57, 66, 76]. However, the conjecture cannot be stretched further to arbitrary submodules. In [71, p. 72] an example is given of a submodule $L \subset H^2_2$ such that L (and therefore also the quotient H^2_2/L) is not essentially normal. Thus, in general, a pure d-contraction of finite rank need not be essentially normal.

11. The isomorphism problem for complete Pick algebras

Let $V \subseteq \mathcal{M}_d$ be a variety as in Section 8.3. A natural problem is to study how the structures of V and $\mathcal{M}_V$ are related, and to try to classify the algebras $\mathcal{M}_V$ in terms of the varieties. Theorem 8.4.1 gives this problem additional motivation.

11.1. **Isometric and completely isometric isomorphism.** Let $\mathrm{Aut}(\mathbb{B}_d)$ denote the group of automorphisms of the ball, that is, the biholomorphisms of $\mathbb{B}_d$ onto itself.

Theorem 11.1.1 ([48] Section 4;[50], Theorem 9.2, [102], Theorems 3.5 and 3.10)**.** *For every $\phi \in Aut(\mathbb{B}_d)$ there exists a unitary $U : H^2_d \to H^2_d$ given by*

$$Uh(z) = (1 - |\phi^{-1}(0)|^2)^{1/2} k_{\phi^{-1}(0)}(z) h(\phi(z)). \tag{11.1.2}$$

Conjugation with U is an automorphism Φ of $\mathcal{M}_d$ and implements composition with ϕ,

$$\Phi(f) = UfU^* = f \circ \phi.$$

The following theorem due to K. Davidson, C. Ramsey and O. Shalit completely solves the classification problem of the algebras $\mathcal{M}_V$ up to completely isometric isomorphism.

Theorem 11.1.3 ([51], Theorems 4.4 and 5.10. See also [14])**.** *Every $\phi \in \mathrm{Aut}(\mathbb{B}_d)$ defines a completely isometric isomorphism $\Phi : \mathcal{M}_V \to \mathcal{M}_W$ by $\Phi(f) = f \circ \phi$, and every completely isometric isomorphism arises this way and is unitarily implemented by a unitary of the form (11.1.2). In particular, the algebras $\mathcal{M}_V$ and $\mathcal{M}_W$ are completely isometrically isomorphic if and only if they are unitarily equivalent, which happens if and only if there exists $\phi \in \mathrm{Aut}(\mathbb{B}_d)$ such that $\phi(W) = V$. When $d < \infty$ then $\mathcal{M}_V$ and $\mathcal{M}_W$ are isometrically isomorphic if and only if they are completely isometrically isomorphic.*

11.2. **Algebraic isomorphism.** A more delicate question is when two algebras $\mathcal{M}_V$ and $\mathcal{M}_W$ are algebraically isomorphic (since these algebras are semi-simple, this is equivalent to existence of a bounded isomorphism).

Theorem 11.2.1 ([51], Theorem 5.6; [45])**.** *Suppose that V, W are both subvarieties of $\mathbb{B}_d$, $d < \infty$, which are comprised of a finite union of irreducible components and a sequence of points. Let $\Phi : \mathcal{M}_V \to \mathcal{M}_W$ be an isomorphism. Then there exist holomorphic maps $\phi, \psi : \mathbb{B}_d \to \mathbb{C}^d$ such that*

1. *$\phi(W) = V$ and $\psi(V) = W$,*
2. *$\phi \circ \psi\big|_V = \mathbf{id}\big|_V$ and $\psi \circ \phi|_W = \mathbf{id}\big|_W$,*
3. *$\Phi(f) = f \circ \phi$ for all $f \in \mathcal{M}_V$ and $\Phi^{-1}(f) = f \circ \psi$ for all $f \in \mathcal{M}_W$.*

(4) *The restrictions of* ψ *to* V *and of* ϕ *to* W *are bi-Lipschitz maps with respect to the pseudohyperbolic metric.*

The following corollary follows from the above theorem and Theorem 11.1.1.

Corollary 11.2.2. *Every algebraic automorphism of* $\mathcal{M}_d$ *is given by composition with an automorphism of the ball, hence is completely isometric and unitarily implemented.*

Two varieties V, W for which there are maps ϕ, ψ as in Theorem 11.2.1 are said to be *biholomorphic*, and the maps ψ and ϕ are said to be *biholomorphisms* from V to W or vice-versa. In light of the above result, it is natural to ask: *given a biholomorphism* $\phi : W \to V$*, does it induce an algebraic isomorphism* $\mathcal{M}_V \to \mathcal{M}_W$*?* If $f \in \mathcal{M}_V$ and $\phi \in W \to V$ is holomorphic then evidently $f \circ \phi \in H^\infty(W)$; the crux of the matter is whether or not it is a multiplier. The answer is negative in general [45, 51]. The first positive result in this direction was obtained by D. Alpay, M. Putinar and V. Vinnikov.

Theorem 11.2.3 ([6], Proposition 2.1)**.** *Let* $d < \infty$*, and let* $\phi : \overline{\mathbb{D}} \to \mathbb{B}_d$ *be a proper injective* C^2 *function that is a biholomorphism of* $\mathbb{D}$ *onto* $V = \phi(\mathbb{D})$*. Then the map*

$$\Phi : \mathcal{M}_V \to H^\infty(\mathbb{D}) \ , \ \Phi(f) = f \circ \phi,$$

is a bounded isomorphism. In particular, $\mathcal{M}_V = H^\infty(V)$*.*

Combining this theorem with 8.3.1 one obtains the following variant of a hard-analytic extension theorem of Henkin [79].

Corollary 11.2.4 ([6], Theorem 2.2)**.** *Let* V *be as in Theorem 11.2.3. Then there is a constant* C *such that for any bounded analytic function* f *on* V *there is a multiplier* $F \in \mathcal{M}_d$ *(in particular,* $F \in H^\infty(\mathbb{B}_d)$*) such that* $f = F\big|_V$ *and*

$$\|F\|_\infty \le \|F\|_{\mathcal{M}_d} \le C\|f\|_\infty.$$

Theorem 11.2.3 and Corollary 11.2.4 were extended to the case where $\mathbb{D}$ is replaced by a planar domain by N. Arcozzi, R. Rochberg and E. Sawyer [13, Section 2.3.6] or a finite Riemann surface by M. Kerr, J. McCarthy and O. Shalit [83, Section 4], and in these extensions ϕ was allowed to be a finitely ramified holomap. In the three papers mentioned an additional assumption about V meeting the boundary of $\mathbb{B}_d$ transversally were imposed, but this assumption was later shown to be satisfied automatically [45]. The case of a bihlomorphic embedding of a disc in $\mathbb{B}_\infty$ was studied in [45, 51], and in particular it was shown that a continuum of non-isomorphic algebras can arise this way. In [45] it was also shown that the conclusion of Theorem 11.2.3 fails if the assumption is weakened slightly so that $\phi(-1) = \phi(1)$.

11.3. **Homogeneous varieties.**

Definition 11.3.1. *A variety* $V \subseteq \mathbb{B}_d$ *is said to be* homogeneous *if for all* $v \in V$ *and all* $\lambda \in \mathbb{D}$ *it holds that* $\lambda v \in V$*.*

A variety is homogeneous if and only if it is the zero set of a homogeneous ideal. There are some satisfactory results for the isomorphism problem in the case where V and W are homogeneous varieties. The following theorem was obtained by Davidson, Ramsey and Shalit in [51] under some technical assumptions, which were removed by M. Hartz in [78].

Theorem 11.3.2 ([50], Theorems 8.5 and 11.7; [78], Theorem 5.9)**.** *Let V and W be two homogeneous varieties in $\mathbb{B}_d$, with $d < \infty$. Then $\mathcal{M}_V$ and $\mathcal{M}_W$ are isomorphic if and only if there exist linear maps $A, B : \mathbb{C}^d \to \mathbb{C}^d$ such that $A(W) = V$, $B(V) = W$, $AB\big|_V = \mathbf{id}\big|_V$ and $BA\big|_W = \mathbf{id}\big|_W$.*

11.4. **The isomorphism problem for norm closed algebras of multipliers.** The algebras $\mathcal{A}_V := \mathcal{A}_d\big|_V = \{f\big|_V : f \in \mathcal{A}_d\}$ and $\mathcal{A}_d/I$ (where I is a closed ideal in $\mathcal{A}_d$) have also been considered, but in this setting less is known. The case of homogeneous varieties is completely settled by results of [50] and [78]. Some partial results are contained in [45, 51, 83].

Theorem 11.4.1. *Let V and W be two homogeneous varieties in $\mathbb{B}_d$. $\mathcal{A}_V$ and $\mathcal{A}_W$ are completely isometrically isomorphic if and only if there is a unitary U such that $U(W) = V$. If $d < \infty$, then $\mathcal{M}_V$ and $\mathcal{M}_W$ are isomorphic if and only if there exist linear maps $A, B : \mathbb{C}^d \to \mathbb{C}^d$ such that $A(W) = V$, $B(V) = W$, $AB\big|_V = \mathbf{id}\big|_V$ and $BA\big|_W = \mathbf{id}\big|_W$.*

12. Some harmonic analysis in H^2_d

The $d = 1$ instance of $\mathcal{M}_d$, which is simply the algebra $H^\infty(\mathbb{D})$ of bounded analytic functions on the disc, has been the arena of a long-standing, beautiful and fruitful interaction between function theory and functional analysis [70]. Among the most profound results in this setting are Carleson's interpolation and corona theorems [39, 40], and a technical tool which Carleson introduced — now called *Carleson measures* — has been of lasting significance. This section surveys some recent results in the case $1 < d < \infty$ regarding these three topics: interpolating sequences, Carleson measures, and the corona theorem. For a recent survey with emphasis on the harmonic analysis side of H^2_d see [12].

12.1. **Carleson measures for H^2_d.** Recall the Besov-Sobolev spaces $B^\sigma_p(\mathbb{B}_d)$ from Section 3.10.

Definition 12.1.1. *A positive measure μ on $\mathbb{B}_d$ is said to be a* Carleson measure for $B^\sigma_p(\mathbb{B}_d)$ *if there exists a constant C such that for all $f \in B^\sigma_p(\mathbb{B}_d)$,*

$$\|f\|_{L^p(\mu)} \leq C\|f\|_{B^\sigma_p(\mathbb{B}_d)} . \tag{12.1.2}$$

The space of all Carleson measures on $B^\sigma_p(\mathbb{B}_d)$ is denoted $CM(B^\sigma_p(\mathbb{B}_d))$. The infimum of C's appearing in the right hand side of (12.1.2) is the the Carleson measure norm *of μ, denoted $\|\mu\|_{CM(B^\sigma_p(\mathbb{B}_d))}$.*

An understanding of Carleson measures has turned out to be a key element in the analysis of the spaces $B^\sigma_p(\mathbb{B}_d)$. The focus of this survey is $H^2_d = B^{1/2}_2(\mathbb{B}_d)$, but in the literature one often finds a treatment for an entire range of p's or σ's. A characterization of the Carleson measures of $B^\sigma_p(\mathbb{B}_d)$ for ranges of p and σ that include $p = 2, \sigma = 1/2$ was obtained in [13],[114] and [121]. The reader is referred to these papers for additional details.

Remark 12.1.3. Consider the scale of spaces $B^\sigma_2(\mathbb{B}_d)$. It is interesting that the value $\sigma = 1/2$ seems to play a critical role in some approaches, while in others it does not. For example, the characterization of Carleson measures given in [13, Theorem 23] holds for $0 \leq \sigma < 1/2$, the case $\sigma = 1/2$ is handled differently. On the other hand, the methods of E. Tchoundja [114] work for the range $\sigma \in (0, 1/2]$, but not for $\sigma > 1/2$. However, using different techniques, A. Volberg and B. Wick give in [121, Theorem 2] a characterization of Carleson measures for $B_2^\sigma(\mathbb{B}_d)$ for all $\sigma > 0$.

12.2. **Characterization of multipliers.** The strict containment (3.7.2) and the incomparability of the multiplier norm and the sup norm lead to the problem of characterizing multipliers in function theoretic terms. One of the applications of Carleson measures is such a characterization. A geometric characterization of Carleson measures such as the one given in [13, Theorem 34] then enables, in principle, to determine in intrinsic terms whether a function is multiplier.

Theorem 12.2.1 (Theorem 2, [13]; Theorem 3.7, [95])**.** *Let $d < \infty$, let f be a bounded analytic function on $\mathbb{B}_d$, and fix $m > (d-1)/2$. Then $f \in \mathcal{M}_d$ if and only if the measure*

$$d\mu_{f,k} = \sum_{|\alpha|=m} \left| \frac{\partial^\alpha f}{\partial z^\alpha}(z) \right|^2 (1-|z|^2)^{2m-d} d\lambda(z)$$

is a Carleson measure for H^2_d. In this case one has the following equivalence of norms

$$\|f\|_{\mathcal{M}_d} \sim \|f\|_\infty + \|\mu_{f,m}\|_{CM(H^2_d)}. \tag{12.2.2}$$

The equivalence of norms (12.2.2) together with Theorem 6.2.2 (Drury's von Neumann inequality) gives a version of von Neumann's inequality for d-contractions that avoids mention of the d-shift, but is valid only up to equivalence of norms.

Corollary 12.2.3. *Let T be a d-contraction ($d < \infty$), and fix $m > (d-1)/2$. Then there exists a constant C such that for every polynomial $p \in \mathbb{C}[z]$,*

$$\|p(T)\| \leq C \left(\sup_{z \in \mathbb{B}_d} |p(z)| + \|\mu_{p,m}\|_{CM(H^2_d)} \right).$$

For an explicit description of the right hand side see [13, Theorem 4]. A function theoretic version of von Neumann's inequality for d-contractions resulting from the above corollary was also noted by Chen [42, Corollary 3].

12.3. **Interpolating sequences.**

Definition 12.3.1. *Let $Z = \{z_n\}_{n=1}^\infty$ be a sequence of points in $\mathbb{B}_d$. Z is said to be an* interpolating sequence for $\mathcal{M}_d$ *if the map*

$$\mathcal{M}_d \ni f \mapsto (f(z_n))_{n=1}^\infty \in \ell^\infty$$

maps $\mathcal{M}_d$ onto ℓ^∞.

There is also a notion of interpolating sequence for H^2_d, but since H^2_d contains unbounded functions, the definition has to be modified.

Definition 12.3.2. *Let $Z = \{z_n\}_{n=1}^\infty$ be a sequence of points in $\mathbb{B}_d$. Define a sequence $\{w_n\}_{n=1}^\infty$ of weights by $w_n = (1-\|z_n\|)^{1/2}$. Z is said to be an* interpolating sequence for H^2_d *if the map*

$$H^2_d \ni h \mapsto (w_n h(z_n))_{n=1}^\infty$$

maps H^2_d into and onto ℓ^2.

Remark 12.3.3. There exists a similar notion of interpolating sequence for an arbitrary Hilbert function space $\mathcal{H}$ with kernel $K^{\mathcal{H}}$, where the weights are given by $w_n = \|K^{\mathcal{H}}_{z_n}\|^{-1}$.

Theorem 12.3.4. *Let $Z = \{z_n\}_{n=1}^\infty$ be a sequence of points in $\mathbb{B}_d$ $(1 \leq d \leq \infty)$. Then Z is an interpolating sequence for $\mathcal{M}_d$ if and only if Z is an interpolating sequence for H^2_d.*

Proof. The theorem, due to D. Marshall and C. Sundberg, holds for arbitrary Hilbert function spaces with the Pick property. See [3, Theorem 9.19] or [88, Corollary 7] for a proof. ∎

The thrust of the above theorem is that it allows to approach the problem of understanding interpolating sequences for the algebra $\mathcal{M}_d$ by understanding the interpolating sequences for the (presumably more tractable) Hilbert space H^2_d. A characterization of interpolation sequences in $B^\sigma_p(\mathbb{B}_d)$ and $\mathrm{Mult}(B^\sigma_p(\mathbb{B}_d))$ for $\sigma \in [0, 1/2)$ was found by Arcozzi, Rochberg and Sawyer [13, Section 2.3.2]. The case $\sigma = 1/2$ (i.e., Drury-Arveson space) is an open problem [12].

12.4. **The corona theorem for multipliers of H^2_d.** Lennart Carleson's corona theorem [40] for $H^\infty(\mathbb{D})$ is the following.

Theorem 12.4.1 (Carleson's corona theorem, [40]). *Let $\delta > 0$, and suppose that $f_1, \ldots, f_N \in H^\infty(\mathbb{D})$ satisfy*

$$\sum_{i=1}^N |f_i(z)|^2 \geq \delta \ , \quad \text{for all } z \in \mathbb{D}.$$

Then there exist $g_1, \ldots, g_N \in H^\infty(\mathbb{D})$ such that

$$\sum_{i=1}^N g_i f_i = 1.$$

An equivalent way of phrasing this theorem is that the point evaluation functionals

$$H^\infty(\mathbb{D}) \ni f \mapsto f(\lambda)$$

are weak-$*$ dense in the maximal ideal space of $H^\infty(\mathbb{D})$, in other words $\mathbb{D}$ is dense in $\mathfrak{M}(H^\infty(\mathbb{D}))$ — hence the metaphor *corona*. In fact, Carleson proved a stronger result, which included bounds on the norm of $g_1, \ldots, g_N$ in terms of δ and the norms $f_1, \ldots, f_N$.

Over the years a lot of effort was put into proving an analogue of this celebrated theorem in several variables, and some results were obtained [9, 84, 87, 115, 116, 119]; see also the recent survey [54]. However, the most natural several variables analogues of Theorem 12.4.1, which are precisely the same statement in the theorem but with the disc $\mathbb{D}$ replaced by either the unit ball $\mathbb{B}_d$ or the polydisc $\mathbb{D}^d$, remain to this day out of reach.

The growing role that the Drury-Arveson space played in multivariable operator theory suggests that the "correct" multivariable analogue of $H^\infty(\mathbb{D})$ is not $H^\infty(\mathbb{B}_d)$ or $H^\infty(\mathbb{D}^d)$, but $\mathcal{M}_d$. Indeed, using a mixture of novel harmonic analytic techniques with available operator theoretic machinery, Costea, Sawyer and Wick [43] proved a corona theorem for $\mathcal{M}_d$. Their main technical result is the following result that they call the *baby corona theorem*.

Theorem 12.4.2 (Baby corona theorem. Theorem 2, [43])**.** *Fix $\delta > 0$ and $d < \infty$. Let $f_1, \dots f_N \in \mathcal{M}_d$ satisfy*

$$\sum_{n=1}^{N} |f_n(z)|^2 \geq \delta \ , \quad \text{for all } z \in \mathbb{B}_d \, . \tag{12.4.3}$$

Then for all $h \in H^2_d$, there exist $g_1, \dots, g_N \in H^2_d$ such that

$$\sum_{n=1}^{N} f_n g_n = h. \tag{12.4.4}$$

Moreover, there is a constant $C = C(d, \delta)$ such that whenever $f_1, \dots, f_N$ satisfy

$$\sum_{n=1}^{N} M^*_{f_n} M_{f_n} \leq I \tag{12.4.5}$$

then $g_1, \dots, g_N$ can be chosen to satisfy

$$\sum_{n=1}^{N} \|g_n\|^2 \leq C \|h\|^2. \tag{12.4.6}$$

Remark 12.4.7. Note that C does not depend on N. In fact, the theorem also holds for $N = \infty$, and also in a semi-infinite matricial setting. Moreover, the theorem holds with $B^\sigma_p(\mathbb{B}_d)$ replacing H^2_d and $\mathrm{Mult}(B^\sigma_p(\mathbb{B}_d))$ replacing $\mathcal{M}_d$ for all $1 < p < \infty$ and $\sigma \geq 0$ (see [43]).

To see why Theorem 12.4.2 is called the "baby" corona theorem note the following. A full (or "grown-up") corona theorem for $\mathcal{M}_d$ would be that given $f_1, \dots, f_N \in \mathcal{M}_d$ satisfying (12.4.3), there are $\tilde{g}_1, \dots, \tilde{g}_N$ in $\mathcal{M}_d$ for which $\sum f_n \tilde{g}_n = 1$ (implying that $\mathbb{B}_d$ is dense in $\mathfrak{M}(\mathcal{M}_d)$). In the baby corona theorem (Theorem 12.4.2) $g_1, \dots, g_N$ are only required to be in the (much larger) space H^2_d. Clearly the full corona theorem implies the baby theorem, because if $\tilde{g}_1, \dots, \tilde{g}_N$ are as in the full corona theorem, then given h the functions $g_n := \tilde{g}_n h \in H^2_d$ clearly satisfy (12.4.4).

Stated differently, the assertion of Theorem 12.4.2 is that, given (12.4.3), the row operator $T := [M_{f_1}\, M_{f_2}\, \cdots\, M_{f_N}] : H^2_d \otimes \mathbb{C}^N \to H^2_d$ is surjective, equivalently, it says that

$$\sum_{n=1}^{N} M_{f_n} M^*_{f_n} \geq \epsilon^2 I \tag{12.4.8}$$

for some $\epsilon > 0$. On the other hand, the full corona theorem asserts that under the same hypothesis the tuple $(M_{f_1}, \dots, M_{f_N})$ is an invertible tuple in the Banach algebra $\mathcal{M}_d$.

In [18, Section 6] Arveson showed, in the setting of $H^\infty(\mathbb{D})$, that (12.4.8) implies a full corona theorem. This was extended to several variables by Ball, Trent and Vinnikov, using their commutant lifting theorem (Theorem 6.10.1).

Theorem 12.4.9 (Toeplitz corona theorem. p. 119, [33])**.** *Suppose $f_1, \dots f_N \in \mathcal{M}_d$ satisfy (12.4.8). Then there are $g_1, \dots, g_N \in \mathcal{M}_d$ such that*

$$\sum_{n=1}^{N} f_n g_n = 1.$$

Moreover, $g_1, \dots, g_N$ can be chosen such that $\sum \|M_{g_n}\|^2 \leq \epsilon^{-2}$.

Remark 12.4.10. The converse is immediate.

Remark 12.4.11. Both the theorem and its converse hold for $d = \infty$. In fact, the theorem and its converse hold for any multiplier algebra of a complete Pick space.

As a consequence of Theorems 12.4.2 and 12.4.9, one has the full corona theorem for $\mathcal{M}_d$.

Theorem 12.4.12 (Corona theorem for $\mathcal{M}_d$. Theorem 1, [43])**.** *Let $\delta > 0$, and suppose that $f_1, \ldots, f_N \in \mathcal{M}_d$ satisfy*

$$\sum_{i=1}^{N} |f_i(z)|^2 \geq \delta \ , \quad \text{for all } z \in \mathbb{B}_d.$$

Then there exist $g_1, \ldots, g_N \in \mathcal{M}_d$ such that

$$\sum_{i=1}^{N} g_i f_i = 1.$$

Remark 12.4.13. Since for $\sigma \in [0, 1/2]$ the space $B_2^\sigma(\mathbb{B}_d)$ is a complete Pick space, the above theorem also holds for the algebra $\mathrm{Mult}(B_2^\sigma(\mathbb{B}_d))$, $\sigma \in [0, 1/2]$ (see Remarks 12.4.7 and 12.4.11).

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