ON AMENABLE SEMIGROUPS OF RATIONAL FUNCTIONS

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ABSTRACT. We characterize left and right amenable semigroups of polynomials of one complex variable with respect to the composition operation. We also prove a number of results about amenable semigroups of arbitrary rational functions. In particular, we show that under quite general conditions a semigroup of rational functions is left amenable if and only if it is a subsemigroup of the centralizer of some rational function.

1. Introduction

The concept of amenable group was introduced by von Neumann in 1929 in the paper [40]. Defined initially in terms of invariant measures in relation with the Banach-Tarski paradox, nowadays the group amenability is known to be equivalent to variety of different conditions and to have connections to numerous branches of mathematics (see e.g. [39], [18] and the bibliography therein). The notion of amenability has been extended to semigroups by Day [6], who also introduced the term itself. Naturally, the absence of inverse elements in semigroups requires substantial changes in definitions and leads to new phenomenons. For example, a semigroup can be left amenable but not right amenable, amenable semigroups can contain non-amenable semigroups etc. (see e.g. [35]).

Let us recall that a semigroup $S$ is called left amenable if it admits a finitely additive probability measure $\mu$, defined on all the subsets of $S$, such that for every element $a \in S$ and subset $T \subseteq S$ the measure of the set

$$a^{-1}T = \{ s \in S \mid as \in T \}$$

is equal to the measure of $T$, that is,

$$\mu(a^{-1}T) = \mu(T).$$

Equivalently, $S$ is left amenable if there is a mean on $l_\infty(S)$, which is invariant under the natural left action of $S$ on the dual space $l_\infty(S)^*$ (see e.g. [35]). The right amenability is defined similarly. A semigroup is called amenable if there exists a mean on $l_\infty(S)$ which is invariant under the left and the right action of $S$ on $l_\infty(S)^*$. By the theorem of Day (see [5], [6]), this is equivalent to the condition that $S$ is left and right amenable.

In this paper, we investigate the amenability of semigroups of polynomials and more generally of rational functions of one complex variable with respect to the composition operation. To our best knowledge, for the first time this topic has been investigated only recently in the paper [3]. Among other things, it was shown in [3] that if $S$ is a semigroup of polynomials of degree at least two, containing no polynomials conjugate to $z^n$ or $\pm T_n$, then the condition that $S$ is amenable and

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any right invariant mean is a left invariant mean is equivalent to the condition that for any \( A, B \in S \) there exist integers \( l, k \geq 1 \) such that

\[
A^ol = B^ok.
\]

On the other hand, it was shown that a semigroup of polynomials \( S \), containing no polynomials conjugate to \( z^n \) or \( \pm T_n \), is right amenable if and only if there exists an \( S \)-invariant probability measure \( \mu \) on \( \mathbb{CP}^1 \) such that the measure of maximal entropy of any element of \( S \) of degree at least two coincides with \( \mu \).

In this paper, we generalize the above results in several directions. In particular, we obtain a comprehensive description of left amenable and amenable semigroups of polynomials, complementing the description of right amenable semigroups obtained in [3]. We also show that amenability conditions established in [3] follow from a weaker algebraic condition of reversibility. To formulate our results explicitly we introduce several definitions. Let us recall that a semigroup \( S \) is called left reversible if for any \( a, b \in S \) the right ideals \( aS \) and \( bS \) have a non-empty intersection, that is, if for any \( a, b \in S \) there exist \( x, y \in S \) such that

(1) \[ ax = by. \]

It is well-known and follows easily from the definition that any left amenable semigroup is left reversible.

For a rational function \( R \) of degree at least two, we define the group \( G(R) \) as the group of Möbius transformations \( \sigma \) such that

(2) \[ R \circ \sigma = \nu \circ R \]

for some Möbius transformations \( \nu \). It is easy to see that \( G(R) \) is indeed a group and that the map

\[
\gamma_R : \sigma \rightarrow \nu_\sigma
\]

is a homomorphism from \( G(R) \) to the group \( \text{Aut}(\mathbb{C}R^1) \). Moreover, the group \( G(R) \) is finite, unless there exist \( \alpha, \beta \in \text{Aut}(\mathbb{C}R^1) \) such that

\[
R = \alpha \circ z^n \circ \beta
\]

(see [29], [34]). For a subgroup \( \Gamma \) of \( G(R) \) such that \( \gamma_R(\Gamma) \subseteq \Gamma \), we define \( S_{\Gamma, R} \) as the semigroup of rational functions generated by \( \Gamma \) and \( R \). Notice that (2) implies that a rational function \( A \) belongs to \( S_{\Gamma, R} \) if and only if \( A \) has the form

\[
A = \sigma \circ R^{os},
\]

where \( s \geq 0 \) and \( \sigma \in \Gamma \).

For a rational function \( P \) of degree at least two, we define \( C(P) \) as the semigroup of rational functions commuting with \( P \). Further, we denote by \( Z \) the semigroup consisting of polynomials of the form \( az^n \), where \( a \in \mathbb{C}^* \) and \( n \geq 1 \), and by \( \mathcal{J} \) the semigroup consisting of polynomials of the form \( \pm T_n \), \( n \geq 1 \). We say that two semigroups of polynomials \( S_1 \) and \( S_2 \) are conjugate if there exists \( \alpha \in \text{Aut}(\mathbb{C}) \) such that

\[
\alpha \circ S_1 \circ \alpha^{-1} = S_2.
\]

In this notation, our main result concerning left amenable semigroups of polynomials is following.
Theorem 1.1. Let $S$ be a semigroup of polynomials not contained in $\text{Aut}(\mathbb{C})$ and not conjugate to a subsemigroup of $\mathbb{Z}$ or $\mathbb{T}$. Then the following conditions are equivalent:

1) The semigroup $S$ is left reversible.
2) The semigroup $S$ is left amenable.
3) The semigroup $S$ is amenable.
4) The semigroup $S$ is a subsemigroup of $S_{\Gamma,R}$ for some $R \in \mathbb{C}[z]$ of degree at least two and a subgroup $\Gamma$ of $G(R)$ such that $\gamma_R(\Gamma) = \Gamma$.
5) The semigroup $S$ is a subsemigroup of $C(P)$ for some $P \in \mathbb{C}[z]$ of degree at least two not conjugate to $z^n$ or $\pm T_n$.

Furthermore, if $S$ contains only polynomials of degree at least two, then any of the above conditions is equivalent to the condition that for any $A, B \in S$ there exist $k, l \geq 1$ such that $A^k = B^l$.

To formulate the analogue of Theorem 1.1 for right amenable semigroups of polynomials we introduce two other types of special semigroups. Let us recall that by the results of Freire, Lopes, Mañé ([11]) and Lyubich ([22]), for any rational function $P$ of degree $n \geq 2$ there exists a unique probability measure $\mu_P$ on $\mathbb{CP}^1$, which is invariant under $P$, has support equal to the Julia set $J_P$, and achieves maximal entropy $\log n$ among all $P$-invariant probability measures.

For a rational function $P$ of degree at least two, we denote by $E(P)$ the semigroup consisting of rational functions $Q$ of degree at least two such that $\mu_Q = \mu_P$, completed by $\mu_P$-invariant Möbius transformations. Finally, for a compact set $K \subset \mathbb{C}$ such that there exists at least one polynomial of degree greater than one for which $K$ is a completely invariant set, we denote by $I(K)$ the semigroup of all polynomials $A$ satisfying $A^{-1}\{K\} = K$. In this notation, the following statement holds.

Theorem 1.2. Let $S$ be a semigroup of polynomials not contained in $\text{Aut}(\mathbb{C})$ and not conjugate to a subsemigroup of $\mathbb{Z}$ or $\mathbb{T}$. Then the following conditions are equivalent:

1) The semigroup $S$ is right reversible.
2) The semigroup $S$ is right amenable.
3) The semigroup $S$ is a subsemigroup of $S_{\Gamma,R}$ for some $R \in \mathbb{C}[z]$ of degree at least two and a subgroup $\Gamma$ of $G(R)$ such that $\gamma_R(\Gamma) \subseteq \Gamma$.
4) The semigroup $S$ is a subsemigroup of $I(K)$ for some compact set $K \subset \mathbb{C}$, which is neither a union of concentric circles nor a segment.
5) The semigroup $S$ is a subsemigroup of $E(P)$ for some $P \in \mathbb{C}[z]$ of degree at least two not conjugate to $z^n$ or $\pm T_n$.
6) The semigroup $S$ contains no free subsemigroup of rank two.

Furthermore, if $S$ contains only polynomials of degree at least two, then any of the above conditions is equivalent to the condition that for any $A, B \in S$ there exist $k_1, l \geq 1$ and $k_2 \geq 0$ such that $A^{k_1} = A^{k_2} \circ B^l$.

It was conjectured in [3] that whenever a semigroup of rational functions of degree at least two $S$ contains no free subsemigroup of rank two, all elements of $S$ have the same measure of maximal entropy. Theorem 1.2 confirms this conjecture in the polynomial case. Moreover, Theorem 1.2 implies the following statement in spirit of von Neumann conjecture for amenable groups: if a semigroup of polynomials $S$ is not right amenable, then $S$ necessarily has a free subsemigroup of rank two.
In addition to the polynomial case, we study the amenability of semigroups of arbitrary rational functions, and prove a partial generalization of Theorem 1.1 to a wide class of such functions. Recall that a semigroup $S$ is called left (resp. right) cancellative if the equality $ab = ac$ (resp. $ba = ca$) for $a, b, c \in S$ implies the equality $b = c$. Any semigroup of rational function is obviously right cancellative but not necessary left cancellative.

Following [33], we say that a rational function $A$ of degree at least two is tame if the algebraic curve

$$A(x) - A(y) = 0$$

has no factors of genus zero or one distinct from the diagonal. By the Picard theorem, this condition is equivalent to the condition that the equality

$$A \circ f = A \circ g,$$

where $f$ and $g$ are functions meromorphic on $\mathbb{C}$, implies that $f \equiv g$. We say that a semigroup of rational functions $S$ is tame, if $S$ consists of tame rational functions. Clearly, any tame semigroup of rational functions $S$ is cancellative, so the tameness condition can be regarded as a strengthening of the cancellativity condition. Notice that tame rational functions form a subsemigroup of $\mathbb{C}(z)$.

In the above notation, our main result about left amenable semigroups of rational functions is following.

**Theorem 1.3.** Let $S$ be a tame semigroup of rational functions. Then the following conditions are equivalent.

1) The semigroup $S$ is left reversible.
2) The semigroup $S$ is left amenable.
3) The semigroup $S$ is amenable.
4) The semigroup $S$ is a subsemigroup of $\mathbb{C}(P)$ for some tame $P \in \mathbb{C}(z)$.
5) For any $A, B \in S$ there exist $k, l \geq 1$ such that $A^k = B^l$.
6) The semigroup $S$ contains no free subsemigroup of rank two.

Our approach to the study of left amenable semigroups of rational functions relies on using the reversibility condition. Specifically, applying condition 1 to powers of $a$ and $b$, we conclude that if a semigroup of rational functions $S$ is left amenable, then for any $A, B \in S$ of degree at least two the algebraic curves

$$(3) \quad A^m(x) - B(y) = 0, \quad n \geq 1,$$

and, more generally, the algebraic curves

$$(4) \quad A^m(x) - B(y) = 0, \quad n \geq 1, \quad m \geq 1,$$

have a factor of genus zero.

The problems of describing pairs of rational functions satisfying the above conditions arose recently in the context of arithmetic dynamics. Specifically, the problem of describing $A$ and $B$ such that all curves (3) have a factor of genus zero or one is a geometric counterpart of the following problem of the arithmetic nature posed in [3]: which rational functions $A$ defined over a number field $K$ have a $K$-orbit containing infinitely many points from the value set $B(\mathbb{P}^1(K))$? These problems were solved for arbitrary rational functions in [28] in terms of semiconjugacies and Galois coverings.
In turn, the problem of describing pairs of rational functions $A$ and $B$ such that all curves (4) have a factor of genus zero or one is a geometric counterpart of the problem of describing pairs of rational functions $A$ and $B$ having orbits with infinite intersection. For polynomials, these problems were solved in [13], [14], where it was shown that for non-special $A$ and $B$ any of these conditions is equivalent to the condition that $A$ and $B$ have a common iterate. Results of [13], [14] were generalized to tame rational functions in [33], and our proof of Theorem 1.3 is based on ideas and results of [33].

Notice that in the context of right amenability the analogues of the above problems about algebraic curves can be formulated in terms of intersections of subfields of rational functions as follows: given rational functions $A$ and $B$, under what conditions the fields

$$C(A^n) \cap C(B), \quad n \geq 1,$$

and, more generally, the fields

$$C(A^n) \cap C(B^m), \quad n \geq 1,$$

contain a non-constant rational function $f$? These problems however have a different flavor, and in this paper we mostly restrict ourselves to the polynomial case only, which can be analyzed using the Ritt theory [36] and other results specific to the polynomial case. Nevertheless, we prove the following result completely describing amenability of semigroups $S_{\Gamma,R}$ introduced above.

**Theorem 1.4.** Let $R$ be a rational function of degree $n \geq 2$ not conjugate to $z^{\pm n}$, and $\Gamma$ a subgroup of $G(R)$ such that $\gamma_R(\Gamma) \subseteq \Gamma$. Then any subsemigroup of $S_{\Gamma,R}$ is right amenable. On the other hand, $S_{\Gamma,R}$ is left amenable if and only if $\gamma_R(\Gamma) = \Gamma$. Moreover, in the last case any subsemigroup of $S_{\Gamma,R}$ is amenable.

The paper is organized as follows. In the second section, we first recall some basic definitions and results about abstract amenable semigroups. Then, we define $C_\infty(P)$ as the semigroup of all rational functions commuting with some iterate of a fixed rational function $P$ of degree at least two. Assuming that $P$ is not special, that is, that $P$ is neither a Lattès maps nor conjugate to $z^n$ or $\pm T_n$, and using results about commuting rational functions from the papers [37] and [31], we describe properties of semigroups $C_\infty(P)$, and, in particular, show their amenability.

In the third and the fourth sections, we study the semigroups $E(P)$ and $S_{\Gamma,R}$. In particular, we prove Theorem 1.4. Assuming that $R$ is not conjugate to $z^{\pm n}$, we also show that $S_{\Gamma,R} \subseteq E(R)$, and that $S_{\Gamma,R} \subseteq C_\infty(R)$, whenever $\gamma_R(\Gamma) = \Gamma$. In the fifth section, we study tame semigroups of rational functions and prove an extended version of Theorem 1.3. The proof is based on results of the paper [33] completed by the following stabilization result of independent interest: for a tame rational function $P$ the semigroup $C_\infty(P)$ coincides with the semigroup $C(P^{nk})$ for some $k \geq 1$.

Finally, in the sixth and the seventh sections, we consider the polynomial case. First, we recall main statements of the theory of functional decompositions of polynomials and prove some results about polynomial functional equations involving iterates. Then, we obtain a number of results about left and right amenable semigroups of polynomials and, in particular, prove Theorem 1.1 and Theorem 1.2.
2. AMENABILITY OF SEMIGROUPS $C_\infty(f)$ AND THEIR SUBSEMIGROUPS

2.1. Amenable semigroups. We start by recalling some definitions and results concerning abstract amenable semigroups. Mostly, we will discuss the “left” case, leaving the formulation of the corresponding definitions and results in the “right” case to the reader. Nevertheless, the results used only in the “right” case will be given accordingly. Notice that switching between left and right amenability in proofs reduces to switching between a semigroup $S$ with a binary operation $f(x, y)$ and a semigroup $S'$ with the same set of elements and a binary operation $f'(x, y) = f(y, x)$. We emphasize however that the left or the right amenability does not imply in general the opposite type of amenability.

We recall that a semigroup $S$ is called left amenable if it admits a finitely additive probability measure $\mu$, defined on all the subsets of $S$, which is left invariant in the following sense. For all $T \subseteq S$ and $a \in S$ the equality

$$\mu(a^{-1}T) = \mu(T)$$

holds, where the set $a^{-1}T$ is defined by the formula

$$a^{-1}T = \{ s \in S \mid as \in T \}.$$

Equivalently, $S$ is left amenable if there is a mean on $l_\infty(S)$ which is invariant under the natural left action of $S$ on the dual space $l_\infty(S)^*$ (see e.g. [35]). The right amenability is defined similarly. A semigroup is called amenable if there exists a mean on $l_\infty(S)^*$, which is invariant under the left and the right action of $S$ on $l_\infty(S)^*$. By the theorem of Day (see [5], [6]), this is equivalent to the condition that $S$ is left and right amenable.

The following statement describes some families of amenable and not amenable semigroups, which appear below (see [6], [35]).

**Theorem 2.1.** Every abelian semigroup is amenable. Every finite group is amenable. On the other hand, the free semigroup of rank two is not left or right amenable.

Recall that a semigroup $S$ is called left cancellative if the equality

$$ab = ac$$

for $a, b, c \in S$ implies the equality $b = c$. A semigroup $S$ is said to satisfy the left Følner condition if for every finite subset $H$ of $S$ and every $\varepsilon > 0$ there is a finite subset $F$ of $S$ with

$$|sF \setminus F| \leq \varepsilon|F|$$

for all $s \in H$. If for every finite subset $H$ of $S$ and every $\varepsilon > 0$ there is a finite subset $F$ of $S$ with

$$|F \setminus sF| \leq \varepsilon|F|,$$

then $S$ is said to satisfy the strong left Følner condition. It is known that the strong Følner condition implies the left amenability ([2]), while the left amenability implies the Følner condition ([12], [23]). In case if $S$ is left cancellative, the sets $F$ and $sF$ have the same cardinality, implying that

$$|sF \setminus F| = |F \setminus sF|.$$

Thus, the following criterion holds.
Theorem 2.2. A left cancellative semigroup is left amenable if and only if it satisfies the left Følner condition. □

In addition to Theorem 2.2 we will use the following criterion (see [6], p. 516).

Lemma 2.3. If $\Sigma_n$ is a set of left amenable subsemigroups in a semigroup $\Sigma$ such that for every $m, n$ there exists $p$ such that $\Sigma_n, \Sigma_m \subseteq \Sigma_p$ and $\Sigma = \bigcup_{i=1}^{\infty} \Sigma_i$, then $\Sigma$ is left amenable.

We recall that a semigroup $S$ is called left reversible if for any $a, b \in S$ the condition

$$aS \cap bS \neq \emptyset$$

holds, or, equivalently, if for any $a, b \in S$ there exist $x, y \in S$ such that

$$ax = by.$$  (5)

The following statement is obtained easily from the definitions (see [35], Proposition 1.23).

Proposition 2.4. Every left amenable semigroup is left reversible. □

In distinction with the group case, a subsemigroup of a left amenable semigroup or even of an amenable group is not necessarily left amenable. However, the following result holds (see [12], [7]).

Theorem 2.5. Let $S$ be a cancellative semigroup such that $S$ contains no free subsemigroup on two generators. If $S$ is left amenable, then every subsemigroup of $S$ is left amenable. □

For a semigroup $U$, we denote by $\text{End}(U)$ the set of endomorphisms of $U$. Suppose that $U$ and $T$ are semigroups with a homomorphism $\rho : T \to \text{End}(U)$. Denoting for $a \in T$ the endomorphism $\rho(a)$ of $U$ by $\rho_a$, we define the semidirect product of $U$ by $T$ as the semigroup $S = U \rtimes T$ of ordered pairs $(u, a)$, where $u \in U$ and $a \in T$, with the operation

$$(u, a)(v, b) = (u\rho_a(v), ab).$$

The following result was proved in [19].

Theorem 2.6. If $U$ and $T$ are right amenable semigroups with a homomorphism $\rho : T \to \text{End}(U)$, then $S = U \rtimes T$ is right amenable. □

Another criterion for the right amenability is following. Let $S$ be a right reversible semigroup, and let $\sim$ be the relation on $S$, which identifies $x$ and $y$ if there exists $s \in S$ for which

$$s \circ x = s \circ y.$$  (6)

In this notation, the following statement holds (see [35], Proposition 1.24 and Proposition 1.25).

Theorem 2.7. Let $S$ be a right reversible semigroup. The relation $\sim$ is a congruence on $S$ and the semigroup $S/\sim$ is right cancellative. Moreover, $S$ is right amenable if and only if $S/\sim$ is right amenable. □

Finally, we need the following simple statement.

Lemma 2.8. Let $S$ be a left cancellative semigroup, which contains no free subsemigroup of rank two. Then $S$ is left reversible.
Proof. By condition, for any \( a, b \in S \) the semigroup generated by \( a \) and \( b \) is not free. Therefore, there exist \( x_1, x_2, \ldots, x_k \in \{a, b\} \) and \( y_1, y_2, \ldots, y_l \in \{a, b\} \) such that
\[
 x_1 x_2 \cdots x_k = y_1 y_2 \cdots y_l,
\]
but the words in \( a \) and \( b \) in the parts of this equality are different. It follows from the left cancellativity that without loss of generality we may assume that \( x_1 \neq y_1 \). Thus, either \( x_1 = a \) and \( y_1 = b \), or \( x_1 = b \) and \( y_1 = a \). Moreover, we may assume that \( k \geq 2 \) and \( l \geq 2 \), since equality (7) implies the equality
\[
 x_1 x_2 \cdots x_k c = y_1 y_2 \cdots y_l c
\]
for any \( c \in S \). Thus, if, say, \( x_1 = a \) and \( y_1 = b \), the elements
\[
 x = x_2 \cdots x_k, \quad y = y_1 y_2 \cdots y_l
\]
of \( S \) satisfy (5). \( \square \)

2.2. Archimedean, power joined, and power twisted semigroups of rational functions. Let \( S \) be a semigroup of rational functions. We denote by \( \underline{S} \) and \( \overline{S} \) the subsets of \( S \) consisting of rational functions of degree one and of degree greater than one, correspondingly. It is easy to see that \( \underline{S} \) and \( \overline{S} \) in fact are subsemigroups of \( S \). When working with semigroups of rational functions it is convenient to use the following terminology. A rational function \( G \) is called a compositional left factor of a rational function \( F \) if \( F = G \circ H \) for some rational function \( H \). Compositional right factors are defined in a similar way.

We recall that an abstract semigroup \( S \) is called power joined if for any \( a, b \in S \) there exist \( k, l \geq 1 \) such that \( a^k = b^l \), and it is called left (resp. right) Archimedean if for any \( a, b \in S \) there exists \( n \geq 1 \) and \( x \) such that \( a^n = xb \) (resp. \( a^n = bx \)). Thus, a semigroup \( S \) of rational functions is power joined if any two elements \( A, B \) of \( S \) have a common iterate, that is, satisfy
\[
 A^{o_k} = B^{o_l}
\]
for some \( k, l \geq 1 \). On the other hand, \( S \) is left (resp. right) Archimedean if for any functions \( A, B \in S \) the function \( B \) is a compositional right (resp. left) factor of some iterate of \( A \). We say that a semigroup \( S \) of rational functions is power twisted if for any two elements \( A, B \) of \( S \) there exist \( k_1, l \geq 1 \) and \( k_2 \geq 0 \) such that
\[
 A^{o_{k_1}} = A^{o_{k_2}} \circ B^{o_l}.
\]
It is clear that any power joined semigroup is power twisted. Notice also that if \( S \) is power twisted and \( \underline{S} \) is not empty, then \( S = \underline{S} \). Indeed, if \( \deg A = 1 \), then (9) implies that \( \deg B = 1 \) for any \( B \in S \).

Lemma 2.9. Let \( S \) be a semigroup of rational functions of degree at least two. If \( S \) is power joined, then \( S \) is left and right Archimedean and left and right reversible. On the other hand, if \( S \) is power twisted, then \( S \) is left Archimedean and right reversible.

Proof. By condition, for any \( A, B \in S \) there exist \( k, l \geq 1 \) such that (8) holds. Moreover, since equality (8) implies the equality \( A^{o_{2k}} = B^{o_{2l}} \), without loss of generality we may assume that (8) holds for some \( k, l \geq 2 \), implying that the equalities
\[
 A^{o_k} = B \circ Y, \quad A \circ X = B \circ Y,
\]
and

$$A^k = Y \circ B, \quad X \circ A = Y \circ B$$

hold for the rational functions

(10) 

$$X = A^{\circ (k-1)}, \quad Y = B^{\circ (l-1)},$$

which belong to $S$ (the assumption $k, l \geq 2$ is necessary since for $l = 1$ and $k = 1$ these functions reduce to the function $z$, which does not necessarily belong to $S$).

To prove the second part of the lemma, we observe that if $k_2 = 0$ in (9), then (9) reduces to (8). Thus, without loss of generality we may assume that $k_2 \geq 1$ implying that $k_3 \geq 2$. Now the statement of the lemma follows from the equalities

$$A^{\circ k_1} = (A^{\circ k_2} \circ B^{\circ (l-1)}) \circ B, \quad A^{\circ (k_1-1)} \circ A = (A^{\circ k_2} \circ B^{\circ (l-1)}) \circ B. \quad \square$$

**Lemma 2.10.** Let $S$ be a semigroup of rational functions such that $\overline{S}$ is finite. If $S$ is power joined, then $S$ is left and right reversible. On the other hand, if $S$ is power twisted, then $S$ is right reversible.

**Proof.** If $S$ is power joined, then $\overline{S}$ is left and right reversible by Lemma 2.9. Thus, to prove the first part of the lemma we only must construct solutions of

(11) 

$$A \circ X = B \circ Y, \quad X \circ A = Y \circ B$$

in case if one of the functions $A$ and $B$ is of degree one. If both $A$ and $B$ have degree one, then the finiteness of $\overline{S}$ implies that $A^k = z$ and $B^l = z$ for some $k, l \in \mathbb{N}$. In particular, the function $z$ belongs to $S$. Moreover, functions (11) satisfy (11). Finally, if, say, the degree of $A \in S$ is one, while the degree of $B \in S$ is greater than one, then equalities (11) hold for

$$X = A^{\circ (k-1)} \circ B, \quad Y = z, \quad \text{and} \quad X = B \circ A^{\circ (k-1)}, \quad Y = z,$$

correspondingly, where $k$ satisfies $A^k = z$. The proof of the second part of the lemma is similar. \hfill \square

2.3. **Semigroups** $C_\infty(P)$. We recall that a rational function is called special if it is either a Lattès map or is conjugate to $z^n$ or $\pm T_n$, where $T_n$ stands for the Chebyshev polynomial. Let $P$ be a non-special rational function of degree at least two. We define $C(P)$ as the collection of rational functions, including rational functions of degree one, commuting with $P$. It is clear that $C(P)$ is a semigroup with respect to the operation of functional composition $\circ$. The subsemigroup $C(P)$ will be denoted by $\text{Aut}(P)$. Clearly, $\text{Aut}(P)$ is a group. Notice that since elements of $\text{Aut}(P)$ permute fixed points of $P^k, k \geq 1$, and any M"{o}bius transformation is defined by its values at any three points, the group $\text{Aut}(P)$ is finite. In particular, for any $A \in \text{Aut}(P)$ the equality

(12) 

$$A^k = z$$

holds for some $k \geq 1$. It is clear that the rational function $z$ and the cyclic semigroup $\langle P \rangle$ generated by $P$ belong to $C(P)$. However, a subsemigroup $S$ of $C(P)$ not necessarily contains $z$. Nevertheless, since $\text{Aut}(P)$ is finite, if $S$ contains a function of degree one $A$ then $z \in S$ by (12).

The following fact is proved easily by a direct calculation (see [31], Lemma 2.1).

**Lemma 2.11.** If $A$ and $U$ are rational functions such that $A \circ U \in C(P)$ and $U \in C(P)$, then $A \in C(P).$ \hfill $\square$
Along with $C(P)$ and $\text{Aut}(P)$ we define sets $C_\infty(P)$ and $\text{Aut}_\infty(P)$ by the formulas

$$C_\infty(P) = \bigcup_{i=1}^{\infty} C(P^{\circ k}), \quad \text{Aut}_\infty(P) = \bigcup_{k=1}^{\infty} \text{Aut}(P^{\circ k}).$$

Since

$$C(P^{\circ k}), C(P^{\circ l}) \subseteq C(P^{\circ \text{LCM}(k,l)})$$

and

$$\text{Aut}(P^{\circ k}), \text{Aut}(P^{\circ l}) \subseteq \text{Aut}(P^{\circ \text{LCM}(k,l)}),$$

the set $C_\infty(P)$ is a semigroup, and $\text{Aut}_\infty(P)$ is a group.

We recall that by the Ritt theorem ([37]) commuting rational functions of degree at least two either both special or have an iterate in common (see [37], [10], [31] for more details). This result implies the following characterizations of semigroups $C_\infty(P)$ and power joined semigroups of rational functions.

**Lemma 2.12.** For any non-special rational function $P$ of degree at least two the semigroup $C_\infty(P)$ coincides with the set of rational functions sharing an iterate with $P$.

**Proof.** If there exist $k, l \in \mathbb{N}$ such that $A^{\circ k} = P^{\circ l}$, then $A$ commutes with $P^{\circ l}$. On the other hand, if $A$ commutes with some iterate of $P$, then the Ritt theorem implies that $A$ and $P$ share an iterate. □

**Theorem 2.13.** Let $S$ be a semigroup of rational functions of degree at least two containing no special functions. Then $S$ is power joined if and only if $S$ is a subsemigroup of the semigroup $C_\infty(P)$ for some non-special $P \in \mathbb{C}(z)$ of degree at least two.

**Proof.** Assume that $S$ is power joined, and let $P$ be an arbitrary element of $S$. Then for any $A \in S$ the equality $A^{\circ k} = P^{\circ l}$ holds for some $k, l \in \mathbb{N}$, implying that $A$ commutes with $P^{\circ l}$. Therefore,

$$S \subseteq \bigcup_{i=1}^{\infty} C(P^{\circ k}) = C_\infty(P).$$

In the other direction, assume that $S \subseteq C_\infty(P)$ for some non-special $P \in \mathbb{C}(z)$ of degree at least two. Then [13] implies that for any $A, B \in S$ there exist $l \in \mathbb{N}$ such that both $A$ and $B$ commute with $P^{\circ l}$. It follows now from the Ritt theorem that there exist $k_1, k_2, r_1, r_2 \in \mathbb{N}$ such that the equalities

$$A^{\circ k_1} = P^{\circ l r_1}, \quad B^{\circ k_2} = P^{\circ l r_2}$$

hold, implying that

$$A^{\circ k_1 r_2} = B^{\circ k_2 r_1}. \quad \square$$

**Theorem 2.14.** Let $S$ be a subsemigroup of the semigroup $C_\infty(P)$ for some non-special rational function $P$ of degree at least two. Then $S$ is cancellative, and left and right reversible.
Proof. Suppose that
\[ F \circ X = F \circ Y \]
for some \( F, X, Y \in C_\infty(P) \). Clearly, if \( \deg F = 1 \), then \( X = Y \), so assume that \( \deg F > 1 \). Let \( k, l \in \mathbb{N} \) be numbers such that \( F^{\circ k} = P^{\circ l} \), and \( s \in \mathbb{N} \) a number such that both \( X, Y \) commute with \( P^{\circ s} \). Then \( X, Y \) commutes with \( F^{\circ ks} = P^{\circ ls} \). Since equality (14) implies the equality
\[ F^{\circ ks} \circ X = F^{\circ ks} \circ Y, \]
this yields that \( X \circ F^{\circ ks} = Y \circ F^{\circ ks} \).
Therefore, \( X = Y \) and hence the semigroup \( C_\infty(P) \) is cancellative, implying that any its subsemigroup is also cancellative.

Finally, since \( C_\infty(P) \) is power joined by Theorem 2.13 and Aut\((P) = C_\infty(P) \) is finite, the left and the right reversibility of \( C_\infty(P) \) follows from Lemma 2.10. □

2.4. Amenability of semigroups \( C_\infty(P) \). Let \( P \) be a non-special rational function of degree at least two. Following [31], we define an equivalence relation \( \sim_P \) on the semigroup \( C(P) \), setting \( Q_1 \sim_P Q_2 \) if
\[ Q_1 \circ P^{\circ l_1} = Q_2 \circ P^{\circ l_2} \]
for some \( l_1 \geq 0, l_2 \geq 0 \).

The following lemma is an easy corollary of the right cancellativity of the semigroup of rational functions (see [31], Lemma 3.1).

Lemma 2.15. Let \( A \) be an equivalence class of \( \sim_P \). For any \( n \geq 1 \) the class \( A \) contains at most one rational function of degree \( n \). Furthermore, if \( A_0 \in A \) is a function of minimum possible degree, then any \( A \in A \) has the form \( A = A_0 \circ P^{\circ l} \), \( l \geq 1 \). □

The following result was proved in [31].

Theorem 2.16. Let \( P \) be a non-special rational function of degree at least two. Then the relation \( \sim_P \) is a congruence on the semigroup \( C(P) \), and the quotient semigroup is a finite group. □

Theorem 2.16 combined with the Følner criterion for amenability permits to prove that any subsemigroup of \( C_\infty(P) \) is amenable. This result is only a light generalization of the result of [3] stating that any power joined subsemigroup of rational functions is amenable. Moreover, as in the paper [3], our proof relies on Theorem 2.16. However, our reduction to Theorem 2.16 is different.

Theorem 2.17. Let \( S \) be a subsemigroup of the semigroup \( C_\infty(P) \) for some non-special \( P \in \mathbb{C}(z) \) of degree at least two. Then \( S \) is amenable.

Proof. By Theorem 2.14 the semigroup \( C_\infty(P) \) is cancellative. Furthermore, \( C_\infty(P) \) cannot contain a free subsemigroup on two generators. Indeed, if \( A, B \in S \) are of degree greater than one, then \( A \) and \( B \) have a common iterate and hence \( \langle A, B \rangle \) is not free. On the other hand, if say \( A \) is of degree one, then \( \langle A, B \rangle \) is not free since (12) implies that \( A^{\circ(k+1)} = A \). Therefore, by Theorem 2.5 to prove the
theorem we only must show that $C_\infty(P)$ is amenable. Moreover, it follows from (13) by Lemma 2.3 that it is enough to prove the amenability of $C(P)$.

It follows from Lemma 2.15 and Theorem [2.16] that there exists a finite subset $X_1, X_2, \ldots, X_n$ of elements of $S$ such that

$$C(P) = \bigsqcup_{i=1}^{n} M_i,$$

where

$$M_i = \{X \in S \mid X = X_i \circ P^j, \ j \geq 0\}.$$

For $N \in \mathbb{N}$ and $i, 1 \leq i \leq n$, we set

$$M_{i,N} = \{X \in S \mid X = X_i \circ P^j, 0 \leq j \leq N\}$$

and

$$F_N = \bigsqcup_{i=1}^{n} M_{i,N}.$$

Let us show that for every finite subset $H$ of $C(P)$ and every $\varepsilon > 0$ the set $F_N$ with $N$ big enough satisfies the Følner condition

$$|F_N \setminus X \circ F_N| \leq \varepsilon|F_N|$$

for all $X \in H$.

By Theorem [2.16], for every $j, i, 1 \leq j, i \leq n$ there exist $m(j, i) \in \mathbb{N}$ and $k(j, i)$, $1 \leq k(j, i) \leq n$, such that

$$X_j \circ X_i = X_{k(j, i)} \circ P^{m(i,j)}.$$

Moreover, for fixed $j$ the map $i \to k(j, i)$ is a bijection of the set $\{1, 2, \ldots, n\}$. Set

$$L_1 = \max_{1 \leq j, i \leq n} m(j, i).$$

Since $H$ is a subset of $C(P)$, any element $X$ of $H$ can be represented in the form

$$X = X_j \circ P^l$$

for some $j, 1 \leq j \leq n$, and $l \geq 0$, and we define $L_2$ as the maximum number $l$ in such a representation (since $H$ is finite, such a number exists). Clearly,

$$|F_N| = (N + 1)n,$$

and it follows from (16) and (17) that for every $X \in H$ the inequality

$$|F_N \setminus X \circ F_N| \leq n(L_1 + L_2)$$

holds. Therefore, (15) holds for $N$ big enough and hence $C(P)$ is left amenable by Theorem [2.2]. Since the set $M_i, 1 \leq i \leq n$, coincides with the set

$$M'_i = \{X \in S \mid X = P^j \circ X_i, \ j \geq 0\},$$

a symmetric argument shows that $C(P)$ is right amenable. \hfill \Box
3. Semigroups $E(P)$

Let us recall that for a rational function $P$ of degree at least two we denote by $\mu_P$ the measure of maximal entropy for $P$, and by $E(P)$ the set of rational functions $Q$ of degree at least two such that $\mu_Q = \mu_P$, completed by $\mu_P$-invariant Möbius transformations.

**Lemma 3.1.** Let $P$ be a non-special rational function of degree at least two. Then the set $E(P)$ is a semigroup.

**Proof.** Let $A$ and $B$ be elements of $E(P)$ of degree $n$ and $m$ correspondingly. Assume first that $n, m \geq 2$. We recall that the measure $\mu_P$ is characterized by the balancedness property that

$$\mu_P(P(S)) = \mu_P(S) \deg P$$

for any Borel set $S$ on which $P$ is injective. Thus, we only must show that if $\mu_P$ is the balanced measure for $A$ and $B$, then $\mu_P$ is the balanced measure for $A \circ B$. Let $S$ be a Borel set on which $A \circ B$ is injective. Then $B$ is injective on $S$ and $A$ is injective on $B(S)$, implying that

$$\mu_P((A \circ B)(S)) = \mu_P(A(B(S))) = n \mu_P(B(S)) = nm \mu_P(S).$$

Thus, $\mu_P$ is the balanced measure for $A \circ B$.

Similarly, if $A \in E(P)$ is a function of degree $n \geq 2$, and $\sigma$ is a $\mu_P$-invariant Möbius transformation, then for any Borel set $S$ on which $A \circ \sigma$ is injective we have

$$\mu_P((A \circ \sigma)(S)) = \mu_P(A(\sigma(S))) = n \mu_P(\sigma(S)) = n \mu_P(S)$$

and

$$\mu_P((\sigma \circ A)(S)) = \mu_P(\sigma(A(S))) = \mu_P(A(S)) = n \mu_P(S),$$

implying that $\mu_P$ is the balanced measure for $A \circ \sigma$ and $\sigma \circ A$.

Finally, it is clear that if $\sigma_1$ and $\sigma_2$ are $\mu_P$-invariant Möbius transformation, then $\sigma_1 \circ \sigma_2$ is also such a transformation. $\square$

Algebraic conditions for non-special rational functions $A$ and $B$ to share the measure of maximal entropy were obtained in [20], [21]. In particular, it was shown in [21] that $A$ and $B$ have the same measure of maximal entropy if and only if there exist iterates $\hat{A}$ of $A$ and $\hat{B}$ of $B$ and positive integers $M$ and $N$ such that

$$\hat{A}^{-1} \circ \hat{A} \circ \hat{A}^N = (\hat{B}^{-1} \circ \hat{B}) \circ \hat{B}^M$$

for some choice of local branches of $\hat{A}^{-1}$ and $\hat{B}^{-1}$.

In particular, this criterion implies that $A$ and $B$ share the measure of maximal entropy whenever there exist $k_1, k_2, l \geq 1$ such that the equality (18) holds. Indeed, if some iterates $\hat{A}$ and $\hat{B}$ satisfy

$$\hat{A} \circ \hat{A}^N = \hat{A} \circ \hat{B}^M,$$

then (18) holds for some branch of $\hat{A}^{-1}$ and the branch of $\hat{B}^{-1}$ with $\hat{B}^{-1} \circ \hat{B} = z$. On the other hand, it is easy to see that equality (19) implies that for any $s \geq 1$ the equality

$$A^{(sk_1-(s-1)k_2)} = A^{sk_2} \circ B^{osl}$$

holds. For $s = k_2$ this gives

$$A^{sk_2} \circ A^{sk_2(k_1-k_2)} = A^{sk_2} \circ B^{osl},$$
implying that\textup{ (19) } holds for
\[\hat{A} = A^{\circ k_2}, \quad N = k_1 - k_2, \quad \hat{B} = B, \quad M = s l.\]
Moreover, condition\textup{ (18) } can be restated in terms of functional equations of type\textup{ (19) } as follows: \(A\) and \(B\) have the same measure of maximal entropy if and only if there exist iterates \(\hat{A}\) and \(\hat{B}\) such that
\[\hat{A} = \hat{A} \circ \hat{B}, \quad \hat{B} = \hat{B} \circ \hat{A}\]
(see\textup{ [42] } for more detail, and also\textup{ [32] } for some results about solutions of\textup{ (20) }).

Rational functions sharing an iterate share the measure of maximal entropy. Moreover, conditions\textup{ (18), (20) } can be regarded as generalizations of the condition that \(A\) and \(B\) share an iterate. Correspondingly, Theorem\textup{ 2.13 } can be generalized as follows.

**Theorem 3.2.** Let \(S\) be a semigroup of rational functions of degree at least two containing no special functions. Then \(S\) is power twisted if and only if \(S\) is a subsemigroup of the semigroup \(E(P)\) for some non-special \(P \in \mathbb{C}(z)\) of degree at least two.

**Proof.** Since condition\textup{ (19) } implies that \(A\) and \(B\) share the measure of maximal entropy, all elements of a power twisted semigroup \(S\) share the measure of maximal entropy with any element \(P \in S\).

On the other hand, since the equality \(\mu_A = \mu_B\) implies equalities\textup{ (20) }, any subsemigroup of \(E(P)\) is power twisted. \(\square\)

4. **Semidirect products**

Let us recall that for a rational function \(R\) of degree at least two, the group \(G(R)\) is defined as the group of M"{o}bius transformations \(\sigma\) such that
\[R \circ \sigma = \nu \circ R\]
for some M"{o}bius transformations \(\nu\). It is easy to see that \(G(R)\) is indeed a group and that the map
\[\gamma_R : \sigma \mapsto \nu_{\sigma}\]
is a homomorphism from \(G(R)\) to the group \(\text{Aut(} \mathbb{C} \mathbb{P}^1\text{)}\). Notice that the group \(\text{Aut(}R\text{)}\) is a subgroup of \(G(R)\). We say that a rational function \(R\) of degree \(n \geq 2\) is a quasi-power if there exist \(\alpha, \beta \in \text{Aut(} \mathbb{C} \mathbb{P}^1\text{)}\) such that
\[R = \alpha \circ z^n \circ \beta.\]

The following statement was proved in\textup{ [29] } (see also\textup{ [34] } for more results about \(G(R)\) and related groups).

**Theorem 4.1.** Let \(R\) be a rational function of degree at least two, which is not a quasi-power. Then the group \(G(R)\) is finite. \(\square\)

Notice that any finite group \(\Delta\) from the well-known list \(A_4, S_4, A_5, C_n, D_{2n}\) of finite subgroups of \(\text{Aut(} \mathbb{C} \mathbb{P}^1\text{)}\) can be realized as the group \(G(R)\) for some rational function \(R\). Indeed, it is enough to set \(R = Q \circ \beta\), where \(\beta\) is the invariant function for the group \(\Delta\), and \(Q\) is a sufficiently general rational function. Less obviously, for any finite subgroup \(\Delta\) of \(\text{Aut(} \mathbb{C} \mathbb{P}^1\text{)}\) there exist rational functions \(R\) such that \(\Delta = \text{Aut(}R\text{)}\) (see\textup{ [10] }). Notice that in the first case the kernel of homomorphism\textup{ (22) } coincides with the whole \(\Delta\), while in the second this kernel is trivial.
Assume that $\Gamma$ a subgroup of $G(R)$ such that $\gamma_R(\Gamma) \subseteq \Gamma$. Then (22) is an endomorphism of $G(R)$. Furthermore, $\gamma_R$ defines in an obvious way a homomorphism

$$\rho_R : \langle R \rangle \to \text{End}(\Gamma).$$

We denote by $S_{\Gamma,R}$ the semigroup generated by $\Gamma$ and $R$. It is clear that a rational function $A$ belongs to $S_{\Gamma,R}$ if and only if

$$A = \sigma \circ R^s$$

for some $s \geq 0$ and $\sigma \in \Gamma$. Moreover, in the notation of Section 2.1, we have:

$$S_{\Gamma,R} = \Gamma \times \rho_R \langle R \rangle.$$

**Lemma 4.2.** Let $R$ be a rational function of degree $n \geq 2$ not conjugate to $z^{\pm n}$, and $\Gamma$ a subgroup of $G(R)$ such that $\gamma_R(\Gamma) = \Gamma$. Then $S_{\Gamma,R}$ is power joined and $S_{\Gamma,R}$ is left and right reversible.

**Proof.** First of all, we observe that $\Gamma$ is finite. Indeed, since the group $G(z^n)$ consists of the Möbius transformations $cz^{\pm 1}, \ c \in \mathbb{C} \setminus \{0\}$ (see [29], Lemma 4.1), it is easy to see that the condition $\gamma_R(\Gamma) \subseteq \Gamma$ holds for a quasi-power $R$ only if $R$ is conjugate to $z^{\pm n}$. Thus, $G(R)$ is finite by Theorem 4.1, implying that $\Gamma$ is finite.

Since the group $G(R)$ is finite, the condition $\gamma_R(\Gamma) = \Gamma$ implies that the restriction $\gamma_R : \Gamma \to \Gamma$ is an automorphism. Therefore, there exists $l \geq 1$ such that the iterate $\gamma_R^l$ is the identical automorphism, implying that any element of $\Gamma$ commutes with the iterate $R^l$. In turn, this implies that for any $A \in S_{\Gamma,R}$ the iterate $A^l$ commute with $R^l$, since (23) implies that

$$A^l = \delta \circ R^l$$

for some $\delta \in \Gamma$. Thus, $A$ and $R$ share an iterate by the Ritt theorem, implying by Lemma 2.12 and Theorem 2.13 that $S_{\Gamma,R}$ is power joined. Finally, since $\Gamma = S_{\Gamma,R}$ is finite, Lemma 2.10 implies that $S_{\Gamma,R}$ is left and right reversible. □

In case if instead of the condition $\gamma_R(\Gamma) = \Gamma$ the weaker condition $\gamma_R(\Gamma) \subseteq \Gamma$ holds, the following statement is true.

**Lemma 4.3.** Let $R$ be a rational function of degree $n \geq 2$ not conjugate to $z^{\pm n}$, and $\Gamma$ a subgroup of $G(R)$ such that $\gamma_R(\Gamma) \subseteq \Gamma$. Then $S_{\Gamma,R}$ is power twisted and $S_{\Gamma,R}$ is right reversible.

**Proof.** To prove that $S_{\Gamma,R}$ is power twisted we must show that any $A, B \in S_{\Gamma,R}$ satisfy (24). It follows from (23) that considering instead of $A$ and $B$ their iterates without loss of generality we may assume that $\deg A = \deg B$ and

$$A = \sigma \circ B$$

for some $\sigma \in \Gamma$. Furthermore, it follows from (23) and (24) that for any $k \geq 1$ there exists $\sigma_k \in \Gamma$ such that

$$A^{\sigma_k} = \sigma_k \circ B^{\sigma_k}.$$

Therefore, since $\Gamma$ is finite, there exist $k_1, k_2 \geq 1$ such that $k_2 > k_1$ and

$$A^{\sigma_{k_1}} = \delta \circ B^{\sigma_{k_1}}, \quad A^{\sigma_{k_2}} = \delta \circ B^{\sigma_{k_2}}$$

for some $\delta \in \Gamma$, implying that (24) holds for $l = k_2 - k_1$. Finally, Lemma 2.10 implies that $S_{\Gamma,R}$ is right reversible. □
The following two results describe amenability properties of subsemigroups of $S_{\Gamma,R}$ according to whether the condition $\gamma_R(\Gamma) = \Gamma$ or the weaker condition $\gamma_R(\Gamma) \subseteq \Gamma$ holds.

**Theorem 4.4.** Let $R$ be a rational function of degree $n \geq 2$ not conjugate to $z^{\pm n}$, and $\Gamma$ a subgroup of $G(R)$ such that $\gamma_R(\Gamma) \subseteq \Gamma$. Then $S_{\Gamma,R}$ is left amenable if and only if $\gamma_R(\Gamma) = \Gamma$. Moreover, if $\gamma_R(\Gamma) = \Gamma$, then $S_{\Gamma,R} \subseteq C_{\infty}(R)$ and any subsemigroup of $S_{\Gamma,R}$ is amenable.

**Proof.** It is easy to see that if $\sigma_0 \in \Gamma$ does not belong to the subgroup $\gamma_R(\Gamma)$ of $\Gamma$, then there exist no $X,Y \in S_{\Gamma,R}$ such that the equality

$$R \circ X = (\sigma_0 \circ R) \circ Y$$

holds. Therefore, if $\gamma_R(\Gamma)$ is a proper subset of $\Gamma$, then $S_{\Gamma,R}$ is not left reversible and hence is not left amenable.

On the other hand, if $\gamma_R(\Gamma) = \Gamma$, then Lemma 4.2 and Theorem 2.13 imply that $S_{\Gamma,R} \subseteq C_{\infty}(R)$. Thus, any subsemigroup of the semigroup $S_{\Gamma,R}$ is amenable by Theorem 2.17. □

**Theorem 4.5.** Let $R$ be a rational function of degree $n \geq 2$ not conjugate to $z^{\pm n}$, and $\Gamma$ a subgroup of $G(R)$ such that $\gamma_R(\Gamma) \subseteq \Gamma$. Then $S_{\Gamma,R} \subseteq E(R)$ and any subsemigroup of $S_{\Gamma,R}$ is right amenable.

**Proof.** Set

$$\Gamma_k = \gamma_R^{o_k}(\Gamma).$$

Since

$$\Gamma \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \ldots,$$

there exists $k_0$ such that $\Gamma_k = \Gamma_{k_0}$ for all $k \geq k_0$. We set $\hat{\Gamma} = \Gamma_{k_0}$ and $\Gamma_0 = \ker \gamma_R^{o_{k_0}}$, so that $\hat{\Gamma} = \Gamma / \Gamma_0$. Since $\gamma_R : \hat{\Gamma} \to \hat{\Gamma}$ is an isomorphism, the above definitions imply that for $\alpha_1, \alpha_2 \in \Gamma$ the equality

$$\gamma_R^{o_k}(\alpha_1) = \gamma_R^{o_k}(\alpha_2)$$

holds for some $k \geq k_0$ if and only if elements $\alpha_1$ and $\alpha_2$ belong to the same coset of $\Gamma_0$ in $\Gamma$.

Let us consider now equivalence classes on $S_{\Gamma,R}$ corresponding to equivalence relation (3) and show that $S_{\Gamma,R} / \sim$ is isomorphic to the semigroup $S_{\hat{\Gamma},R}$. Since $S_{\hat{\Gamma},R}$ is right reversible by Lemma 4.3, it follows from the first part of Theorem 2.7 that it is enough to prove that the equality

$$(\beta \circ R^{o_s}) \circ (\alpha_1 \circ R^{o_{s_1}}) = (\beta \circ R^{o_s}) \circ (\alpha_2 \circ R^{o_{s_2}}),$$

where $\alpha_1, \alpha_2 \in \Gamma$ and $s_1, s_2 \geq 0$, holds for some $s \geq 0$ and $\beta \in \Gamma$ if and only if $s_1 = s_2$ and $\alpha_1, \alpha_2$ belong to the same coset of $\Gamma_0$ in $\Gamma$. To prove the “if” part, we observe that if $\alpha_2 = \delta \circ \alpha_1$, where $\delta \in \Gamma_0$, then (25) holds for $s = k_0$ and any $\beta$. On the other hand, if equality (25) holds, then obviously $s_1 = s_2$ and

$$(\beta \circ R^{o_s}) \circ (\alpha_1 \circ R^{o_{s_1}}) = (\beta \circ R^{o_s}) \circ (\alpha_2 \circ R^{o_{s_2}}).$$

In turn, (26) implies that for any $l \geq 0$ the equality

$$R^{o(s+l)} \circ (\alpha_1 \circ R^{o_{s_1}}) = R^{o(s+l)} \circ (\alpha_2 \circ R^{o_{s_2}})$$

holds. Thus, without loss of generality we may assume that $s \geq k_0$ in (26), implying that $\alpha_1$ and $\alpha_2$ belong to the same coset.
Since \( S_{\Gamma,R}/\sim \) is isomorphic to the semigroup \( S_{\hat{\Gamma},R} \), it follows from the second part of Theorem 2.7 that to prove that any subsemigroup of \( S_{\Gamma,R} \) is right amenable it is enough to prove that any subsemigroup of \( S_{\hat{\Gamma},R} \) is right amenable. In turn, the last statement is a corollary of Theorem 4.4. \( \square \)

Notice that analogues of Theorem 4.4 and Theorem 4.5 obviously remain true for \( R \) conjugate to \( z^{\pm n} \) if to require the finiteness of \( \Gamma \). For example, the semigroup generated by the polynomial \( z^2 \) and the Möbius transformation \( z \to -z \) is the simplest example of a right amenable semigroup which is not left amenable. On the other hand, the semigroup generated by \( z^3 \) and \( z \to e^{2\pi i/5} z \), say, is amenable.

5. Tame semigroups of rational functions

5.1. Tame rational functions. We recall that a rational function \( A \) of degree at least two is called tame if the algebraic curve

\[
A(x) - A(y) = 0
\]

has no factors of genus zero or one distinct from the diagonal. By the Picard theorem, the condition that the function \( A \) is tame is equivalent to the condition that the equality

\[
(27) \quad A \circ f = A \circ g,
\]

where \( f \) and \( g \) are functions meromorphic on \( \mathbb{C} \), implies that \( f \equiv g \). Notice that any rational function of degree two is not tame since the curve

\[
\frac{A(x) - A(y)}{x - y} = 0
\]

has degree one, implying that its genus is zero. Thus, a tame rational function has degree at least three. Notice that a generic rational function of degree at least four is tame. Specifically, a rational function of degree at least four is tame whenever it has only simple critical values (25).

We say that a semigroup of rational functions \( S \) is tame, if it contains tame rational functions only. Clearly, the tameness condition can be regarded as a strengthening of the cancellativity condition.

Lemma 5.1. Tame rational functions form a cancellative subsemigroup of \( \mathbb{C}(z) \).

Proof. Let \( A \) and \( B \) be tame rational functions. Assume that \( f \) and \( g \) are functions meromorphic on \( \mathbb{C} \) such that

\[
(B \circ A) \circ f = (B \circ A) \circ g.
\]

Since \( B \) is tame and \( A \circ f \) and \( A \circ g \) are meromorphic on \( \mathbb{C} \), the last equality implies equality (27), which in turn implies that \( f \equiv g \). Thus, tame rational functions form a subsemigroup of \( \mathbb{C}(z) \), and it is clear that this subsemigroup is cancellative. \( \square \)

Our approach to the amenability of tame semigroups of rational functions is based on the three results about tame rational functions from the paper [33] given below.

Let \( A^{od} = U \circ V \) be a decomposition of an iterate \( A^{od} \) of a rational function \( A \) into a composition of rational functions \( U \) and \( V \). We say that this decomposition is induced by the decomposition \( A^{od'} = U' \circ V' \), where \( d' < d \), if there exist \( k_1, k_2 \geq 0 \) such that

\[
U = A^{ok_1} \circ U', \quad V = V' \circ A^{ok_2}.
\]
The first statement we need is following (33).

**Theorem 5.2.** Let \( P \) be a tame rational function of degree \( n \geq 2 \). Then there exists an integer \( N \), depending on \( n \) only, such that any decomposition of \( P^d \) with \( d \geq N \) is induced by a decomposition of \( P^N \).

We recall that functional decompositions \( R = U \circ V \) of a rational function \( R \) into compositions of rational functions \( U \) and \( V \), considered up to the equivalence

\[
U \to U \circ \mu, \quad V \to \mu^{-1} \circ V, \quad \mu \in \text{Aut}(\mathbb{CP}^1),
\]

are in a one-to-one correspondence with imprimitivity systems of the monodromy group of \( R \). In particular, the number of such classes is finite. Consequently, Theorem 5.2 implies that for any tame rational function \( P \) there exist finitely many rational functions \( F_1, F_2, \ldots, F_s \) such that a rational function \( F \) is a right factor of an iterate of \( P \) if and only if \( F \) has the form

\[
F = \mu \circ F_i \circ P^l, \quad l \geq 0, \quad 1 \leq i \leq s, \quad \mu \in \text{Aut}(\mathbb{CP}^1).
\]

It is easy to see that if rational functions \( A \) and \( B \) have a common iterate, then each iterate of \( B \) is a compositional left and right factor of some iterate of \( A \). The following result provides a partial converse statement (33).

**Theorem 5.3.** Let \( A \) and \( B \) be tame rational functions. Then the following conditions are equivalent.

1) Each iterate of \( B \) is a compositional left factor of some iterate of \( A \).
2) Each iterate of \( B \) is a compositional right factor of some iterate of \( A \).
3) The functions \( A \) and \( B \) have a common iterate.

For rational functions \( A \) and \( B \), we define an algebraic curve \( C_{A,B} \) by the formula

\[
C_{A,B} : A(x) - B(y) = 0.
\]

The last result about tame rational functions we need below is following (see 33, Corollary 3.6).

**Theorem 5.4.** Let \( A \) and \( B \) be rational functions such that the curve \( C_{A^s, B} \) has an irreducible factor \( C \) of genus zero or one for some \( s \geq 1 \). Assume in addition that \( B \) is tame, \( \deg A \geq 2 \), and

\[
s > \log_2 \left[ 84(\deg B - 1)(\deg B)! \right].
\]

Then \( A^s = B \circ L \) for some rational function \( L \), and \( C \) is the graph \( L(x) - y = 0 \).

5.2. **Stabilization of semigroups** \( C(P^{os}) \). For a rational function \( P \) of degree at least two, the groups in the sequence \( G(P^{ok}) \), \( k \geq 1 \), in general are different. Nevertheless, the following statement holds (34).

**Theorem 5.5.** Let \( P \) be a rational function of degree \( n \geq 2 \). Then the sequence \( G(P^{ok}) \), \( k \geq 1 \), contains only finitely many non-isomorphic groups, and, unless \( P \) is a quasi-power, the orders of these groups are finite and uniformly bounded in terms of \( n \) only.

Among other things, Theorem 5.5 implies that, unless \( P \) is conjugate to \( z^{\pm n} \), the group \( \text{Aut}_\infty(P) \) is finite, so that

\[
\text{Aut}_\infty(P) = \text{Aut}(P^{os})
\]
for some $s \geq 1$ (see [34] for more detail). In this section, we prove the following generalization of equality (30) for tame rational functions.

**Theorem 5.6.** Let $P$ be a tame rational function. Then $C_\infty(P) = C(P^{os})$ for some $s \geq 1$.

**Proof.** Assume that $F \in C_\infty(P)$. By the Ritt theorem, $F$ is a compositional right factor of an iterate of $P$. On the other hand, by Theorem 5.2 there exist rational functions $F_1, F_2, \ldots, F_s$ such that any compositional right factor of an iterate of $P$ has the form $P^{os}$. Furthermore, by Lemma 2.11 the function $\alpha \circ F_i \circ P^{os}$ commutes with $P^{os}$, $s \geq 1$, if and only if $\alpha \circ F_i$ commutes with $P^{os}$.

Let us observe now that if $\alpha \circ F_i$ commutes with $P^{os}$, and $\alpha' \circ F_i$ commutes with $P^{os'}$ for some $\alpha, \alpha' \in \text{Aut}(\mathbb{C}\mathbb{P}^1)$ and $s, s' \geq 1$, then the both functions $\alpha \circ F_i$ and $\alpha' \circ F_i$ commute with $P^{osLCM(s,s')}$.

Therefore, since

$$\alpha \circ F_i = (\alpha' \circ \alpha^{-1}) \circ \alpha' \circ F_i,$$

Lemma 2.11 implies that $\alpha' \circ \alpha^{-1}$ also commutes with $P^{osLCM(s,s')}$. Thus, $\alpha' = \nu \circ \alpha$ for some $\nu \in \text{Aut}_\infty(P)$. Since the group $\text{Aut}_\infty(P)$ is finite, this yields that there exist finitely many rational functions $G_1, G_2, \ldots, G_r \in C_\infty(P)$ such that $F$ belongs to $C_\infty(P)$ if and only if $F$ has the form

$$G_i \circ P^{osl}, \quad 1 \leq i \leq r, \quad l \geq 0.$$

Finally, if $G_i$, $1 \leq i \leq r$, commutes with $P^{osk}$, $k_i \geq 1$, then $G_i$ also commutes with $P^N$, where $N = \text{LCM}(k_1, k_2, \ldots, k_r)$. Thus, $C_\infty(P) \subseteq C(P^{osN})$, implying that $C_\infty(P) = C(P^{osN})$. \hfill $\square$

5.3. **Amenable semigroups.** The following result is an extended version of Theorem 1.3 from the introduction (we recall that tame rational functions have degree at least two).

**Theorem 5.7.** Let $S$ be a tame semigroup of rational functions. Then the following conditions are equivalent.

1) The semigroup $S$ is left reversible.
2) The semigroup $S$ is left amenable.
3) The semigroup $S$ is amenable.
4) The semigroup $S$ is a subsemigroup of $C(P)$ for some tame $P \in \mathbb{C}(z)$.
5) The semigroup $S$ is power joined.
6) The semigroup $S$ is left or right Archimedean.
7) The semigroup $S$ contains no free subsemigroup of rank two.
8) For every $A, B \in S$ there exist $z_1, z_2 \in \mathbb{C}\mathbb{P}^1$ such that the forward orbits $O_A(z_1)$ and $O_B(z_2)$ have an infinite intersection.

**Proof.** Since a special rational function cannot be tame (see [33], Section 2), it follows from Theorem 2.13 that a tame power joined semigroup of rational functions is a subsemigroup of $C_\infty(F)$ for some tame rational function $F$. On the other hand, Theorem 5.6 implies that $C_\infty(F) = C(P)$, where $P = F^{os}$ for some $s \geq 1$. This proves the implication 5 $\Rightarrow$ 4. The implication 4 $\Rightarrow$ 3 holds by Theorem 2.17. The implication 3 $\Rightarrow$ 2 is clear. The implication 2 $\Rightarrow$ 1 holds by Proposition 2.7.

The implication 1 $\Rightarrow$ 5 follows from Theorem 5.4. Indeed, let $A$ and $B$ be arbitrary elements of $S$. It follows from the left reversibility, that for any $s \geq 1$ there exist $C_s, D_s \in S$ such that the equality $A^{os} \circ C_s = B \circ D_s$ holds. Since for
s big enough inequality (29) holds, it follows from Theorem 5.4 that the function $B$ is a compositional left factor of some iterate of $A$ (notice that this fact by itself does not imply that $S$ is right Archimedean since in the equality $B^{2n} = A \circ X$ the function $X$ may not belong to $S$). Moreover, using the same reasoning for iterates of $B$ we conclude that each iterate of $B$ is a compositional left factor of some iterate of $A$, implying that $A$ and $B$ have a common iterate by Theorem 5.3. This finishes the proof of the equivalences $1 \iff 2 \iff 3 \iff 4 \iff 5$.

The implication $5 \Rightarrow 6$ follows from Lemma 2.9. On the other hand, if $S$ is left (right) Archimedean, then for any $A, B \in S$ each iterate of $B$ is a compositional right (left) factor of some iterate of $A$, implying by Theorem 5.3 that $A$ and $B$ share an iterate. Thus, $5 \iff 6$. Further, it is clear that $5$ implies 8, while the inverse implication was proved in [33] using Theorem 5.3 and Theorem 5.4. For the reader convenience, we repeat the argument. By the Faltings theorem ([17]), if an irreducible algebraic curve $C$ defined over a finitely generated field $K$ of characteristic zero has infinitely many $K$-points, then $g(C) \leq 1$. On the other hand, it is easy to see that if $O_A(z_1) \cap O_B(z_2)$ is infinite, then for every pair $(i, j) \in \mathbb{N} \times \mathbb{N}$ the algebraic curve

$$A^{\alpha_i}(x) - B^{\alpha_j}(y) = 0$$

has infinitely many points $(x, y) \in O_A(z_1) \times O_B(z_2)$. Defining now $K$ as the field generated over $\mathbb{Q}$ by $z_1, z_2$, and the coefficients of $A, B$, and observing that the orbits $O_A(z_1)$ and $O_B(z_2)$ belong to $K$, we conclude that for every pair $(i, j) \in \mathbb{N} \times \mathbb{N}$ curve (31) has a factor of genus zero or one. It follows now from Theorem 5.3 that each iterate of $B$ is a compositional left factor of some iterate of $A$, implying that $A$ and $B$ have a common iterate by Theorem 5.3.

Since any tame semigroup of rational functions is left cancellative, the implication $7 \Rightarrow 1$ follows from Lemma 2.9. Finally, let us observe that if $S$ is a subsemigroup of $C(P)$, then $S$ cannot contain a free subsemigroup $S'$ of rank two, since such $S'$ would be a not left amenable subsemigroup of the semigroup $C(P)$, in contradiction with Theorem 2.17. Therefore, $4 \Rightarrow 7$. \hfill $\square$

6. Polynomial functional equations involving iterates

6.1. Polynomial decompositions. We say that a polynomial $A$ is special if it is conjugate to $z^n$ or to $\pm T_n$. Since a polynomial cannot be a Lattès map, this definition is consistent with the previous definition of special rational functions.

The following two lemmas follow easily from from the characterization of the polynomials $z^n$ and $T_n$ in terms of their ramification (see e.g. [14], Lemma 3.5 and Lemma 3.9).

**Lemma 6.1.** Any decomposition of $z^n$, $n \geq 2$, into a composition of polynomials has the form

$$z^n = (z^{n/d} \circ \mu) \circ (\mu^{-1} \circ z^d),$$

where $d | n$ and $\mu$ is a polynomial of degree one. On the other hand, any decomposition of $T_n$, $n \geq 2$, has the form

$$T_n = (T_{n/d} \circ \mu) \circ (\mu^{-1} \circ T_d),$$

where $d | n$ and $\mu$ is a polynomial of degree one. \hfill $\square$
For brevity, we will say that two polynomials $A$ and $B$ are \textit{linearly equivalent} if there exist polynomials of degree one $\sigma$ and $\nu$ such that the equality $A = \sigma \circ B \circ \nu$ holds.

\textbf{Lemma 6.2.} Let $A$ be a polynomial of degree $d \geq 2$ such that $A^l$, $l > 1$, is linearly equivalent to $z^d$. Then $A$ is conjugate to $z^d$. Similarly, if $A^l$, $l > 2$, is linearly equivalent to $T^l_d$, then $A$ is conjugate to $\pm T^l_d$. \hfill $\Box$

It is easy to see that if $A$ is a polynomial, and $A = U \circ V$ is a decomposition of $A$ into a composition of rational functions $U$ and $V$, then there exists a Möbius transformation $\mu$ such that $U \circ \mu$ and $\mu^{-1} \circ V$ are polynomials. Thus, considering decompositions of a polynomial $A$ into compositions of rational functions we can restrict ourselves by the consideration of decompositions into compositions of polynomials. Furthermore, if an iterate of a rational function $F$ is a polynomial, then either $F$ itself is a polynomial or $F$ is conjugate to $1/z^d$. This implies, in particular, the following statement.

\textbf{Lemma 6.3.} Let $P$ be a polynomial of degree at least two not conjugate to $z^n$. Then the semigroup $C(P)$ consists of polynomials.

\textit{Proof.} Indeed, if $Q \in C(P)$, then by the Ritt theorem equality $P^{ok} = Q^{ol}$ holds for some $l, k \geq 1$. Therefore, if $Q$ is not a polynomial, then $P^{ok}$ is conjugate to a power, implying that $P$ is conjugate to a power by Lemma 6.2 in contradiction with the assumption. On the other hand, if $Q$ has degree one, then the condition $P \circ Q = Q \circ P$ implies easily that $Q^{-1}\{\infty\} = \infty$. \hfill $\Box$

In distinction with the general case, polynomial solutions of the functional equation
\begin{equation}
A \circ C = B \circ D
\end{equation}
admit essentially a complete description.

Specifically, the following result follows easily from the fact that the monodromy group of a polynomial of degree $n$ contains a cycle of length $n$.

\textbf{Theorem 6.4 (15).} Let $A, C, B, D$ be polynomials such that $A \circ C = B \circ D$. Then there exist polynomials $U, V, A, C, B, D$, where
\[
\deg U = \gcd(\deg A, \deg B), \quad \deg V = \gcd(\deg C, \deg D),
\]
such that
\[
A = U \circ \tilde{A}, \quad B = U \circ \tilde{B}, \quad C = \tilde{C} \circ V, \quad D = \tilde{D} \circ V,
\]
and
\[
\tilde{A} \circ \tilde{C} = \tilde{B} \circ \tilde{D}.
\]

Notice that Theorem 6.4 implies that if $\deg B | \deg A$ in (32), then the equalities
\[
A = B \circ R, \quad D = R \circ C
\]
hold for some polynomial $R$.

Theorem 6.4 reduces describing solutions of (32) to describing solutions satisfying
\begin{equation}
\gcd(\deg A, \deg B) = 1, \quad \gcd(\deg C, \deg D) = 1.
\end{equation}

The following result called “the second Ritt theorem” (36) describes such solutions.
Theorem 6.5 (35). Let \( A, C, B, D \) be polynomials such that (32) and (33) hold. Then there exist polynomials \( \sigma_1, \sigma_2, \mu, \nu \) of degree one such that, up to a possible replacement of \( A \) by \( B \) and of \( C \) by \( D \), either
\[
A = \nu \circ z^n R(z) \circ \sigma_1^{-1}, \quad C = \sigma_1 \circ z^n \circ \mu
\]
\[
B = \nu \circ z^n \circ \sigma_2^{-1}, \quad D = \sigma_2 \circ z^n R(z^n) \circ \mu,
\]
where \( R \) is a polynomial, \( n \geq 1 \), \( s \geq 0 \), and \( \gcd(s, n) = 1 \), or
\[
A = \nu \circ T_m \circ \sigma_1^{-1}, \quad C = \sigma_1 \circ T_n \circ \mu,
\]
\[
B = \nu \circ T_n \circ \sigma_2^{-1}, \quad D = \sigma_2 \circ T_m \circ \mu,
\]
where \( T_n, T_m \) are the Chebyshev polynomials, \( n, m \geq 1 \), and \( \gcd(n, m) = 1 \).

Theorem 6.5 implies the following corollary.

Corollary 6.6. Let \( A, C, B, D \) be polynomials such that (32) and (33) hold and \( \deg A > \deg B \). Then \( B \) and \( C \) are linearly equivalent either to powers or to Chebyshev polynomials. \( \square \)

For a partial generalization of Corollary 6.6 to rational functions we refer the reader to the paper [27].

6.2. Decompositions of iterates. In order to describe amenable semigroups of polynomials we will use the polynomial versions of Theorem 5.2, Theorem 5.3, and Theorem 5.6 given below. These versions are more precise since they hold for all non-special polynomials, not only for tame ones.

The more precise version of Theorem 5.2 for polynomials is the following result (see [41], [26], [33]).

Theorem 6.7. Let \( A \) be a polynomial of degree \( n \geq 2 \) not conjugate to \( z^n \) or \( \pm T_n \).
Then there exists an integer \( N \), depending on \( n \) only, such that any decomposition of \( A^d \) with \( d \geq N \) is induced by a decomposition of \( A^N \). \( \square \)

The following analogue of Theorem 6.6 is obtained from Theorem 6.7 in the same way as Theorem 5.6 is obtained from Theorem 5.2.

Theorem 6.8. Let \( P \) be a non-special polynomial of degree at least two. Then \( C_\infty(P) = C(P^s) \) for some \( s \geq 1 \). \( \square \)

Theorem 6.7 implies the following useful criterion.

Theorem 6.9. Let \( B \) be a polynomial of degree at least two. Assume that there exists a sequence of polynomials \( F_j \), \( j \geq 1 \), such that:
1) Each \( F_j \), \( j \geq 1 \), is a compositional left factor of some iterate of \( B \).
2) Each \( F_j \), \( j \geq 1 \), is linearly equivalent to a special polynomial.
3) \( \deg F_j \to \infty \).

Then \( B \) is special. The same conclusion holds if to replace the first condition by the condition that each \( F_j \), \( j \geq 1 \), is a compositional right factor of some iterate of \( B \).

Proof. We consider the “left” case. The proof in the “right” case is similar. Assume that \( B \) is not special. Then Theorem 6.7 implies that there exist a left compositional factor \( C \) of some iterate of \( B \) and different \( j_1, j_2 \geq 1 \) such that
\[
F_{j_1} = B^{s_{j_1}} \circ C \circ \mu_1, \quad F_{j_2} = B^{s_{j_2}} \circ C \circ \mu_2
\]
for some \( l_1, l_2 \geq 1 \) and \( \mu_1, \mu_2 \in \text{Aut}(\mathbb{C}) \). Moreover, we can find \( j_1 \) and \( j_2 \) such that
\( l_1 - l_2 > 2 \). Equalities \( (34) \) yield that
\[
F_{j_1} = B^{\circ (l_1 - l_2)} \circ F_{j_2} \circ (\mu_2^{-1} \circ \mu_1),
\]
implying by Lemma \( 6.1 \) that \( B^{\circ (l_1 - l_2)} \) is linearly equivalent to a special polynomial. It follows now from Lemma \( 6.2 \) that \( B \) is special. The contradiction obtained proves the corollary. \( \square \)

Finally, the next two results are the “left” and the “right” polynomial analogues of Theorem \( 5.3 \). The first of them was established previously in the papers \( [13], [14] \) (see \( [13] \), Proposition 3.3 and \( [14] \), Proposition 4.1). The proof given below uses similar ideas but is somewhat shorter and is easily modified to fit the right case.

**Theorem 6.10.** Let \( A \) and \( B \) be non-special polynomials of degree at least two. Then each iterate of \( B \) is a compositional left factor of some iterate of \( A \) if and only if there exist \( k, l \geq 1 \) such that \( A^{\circ k} = B^{\circ l} \).

**Proof.** The “if” part is obvious. To prove the “only if” part, we observe first of all that without loss of generality we may assume that the group \( G(B) \) is finite. Indeed, considering instead of \( B \) its second iterate and using Lemma \( 6.2 \) we may assume that \( B \) is not a quasi-power, implying that \( G(B) \) is finite by Theorem \( 4.1 \). Furthermore, Lemma \( 6.2 \) implies that all the groups \( G(B^{\circ i}), i \geq 1 \), are finite.

By condition, for every \( i \geq 1 \) there exist \( s_i \geq 1 \) and \( R_i \in \mathbb{C}[z] \) such that
\[
A^{\circ s_i} = B^{\circ s_i} \circ R_i.
\]
Therefore, Theorem \( 6.7 \) implies that there exist an index \( i_0 \geq 1 \) and an increasing sequence of non-negative integers \( f_k, k \geq 1 \), such that
\[
B^{\circ f_k} = A^{\circ r_k} \circ B^{\circ i_0} \circ \mu_k, \quad k \geq 1,
\]
for some \( \mu_k \in \text{Aut}(\mathbb{C}\mathbb{P}^1) \) and \( r_k \geq 1 \). Furthermore, since \( (35) \) implies that for every \( k \geq 1 \) the function \( B^{\circ i_0} \circ \mu_k \) is a compositional right factor of an iterate of \( B \), we conclude that there exists an index \( k_0 \geq 1 \) and an increasing sequence of non-negative integers \( k_l, l \geq 1 \), such that
\[
B^{\circ i_0} \circ \mu_k = \delta_l \circ B^{\circ i_0} \circ \mu_{k_0}, \quad l \geq 1,
\]
for some \( \delta_l \in \text{Aut}(\mathbb{C}\mathbb{P}^1) \).

Clearly, the Möbius transformations \( \mu_{k_0}^{-1} \circ \mu_{k_0}^{-1} \), \( l \geq 1 \), belong to the group \( G(B^{\circ i_0}) \). Therefore, the finiteness of \( G(B^{\circ i_0}) \) yields that
\[
\mu_{k_1} \circ \mu_{k_0}^{-1} = \mu_{k_0} \circ \mu_{k_0}^{-1}
\]
for some \( l_2 > l_1 \), implying that \( \mu_{k_1} = \mu_{k_1} \). It follows now from \( (35) \) that
\[
B^{\circ f_{k_1} - f_{k_2}} = A^{\circ (r_{k_1} - r_{k_1})} \circ B^{\circ f_{k_2}},
\]
implying that
\[
B^{\circ f_{k_1} - f_{k_2}} = A^{\circ (r_{k_1} - r_{k_1})}.
\]
Since \( l_2 > l_1 \) and the sequences \( k_l, l \geq 1 \), and \( f_k, k \geq 1 \), are increasing, the inequality \( f_{k_2} > f_{k_1} \) holds, and therefore \( A \) and \( B \) have a common iterate. \( \square \)
**Theorem 6.11.** Let $A$ and $B$ be non-special polynomials of degree at least two. Then each iterate of $B$ is a compositional right factor of some iterate of $A$ if and only if there exist $k_1, k_2, l \geq 1$ such that $A^{o k_1} = A^{o k_2} \circ B^o$.

**Proof.** The proof is obtained by a modification of the proof of Theorem 6.10. Namely, assuming that for every $i \geq 1$ there exist $s_i \geq 1$ and $R_i \in \mathbb{C}[z]$ such that

$$A^{o s_i} = R_i \circ B^o,$$

we conclude that there exists $i_0 \geq 1$ and a sequence $f_k$, $k \geq 1$, such that

$$(36) \quad B^{o f_k} = \mu_k \circ B^{o \delta_k} \circ A^{o r_k}, \quad k \geq 1,$$

for some $\mu_k \in \text{Aut}(\mathbb{C}[z])$ and $r_k \geq 1$. Moreover, there exist an index $k_0 \geq 1$ and an increasing sequence $k_l, l \geq 1$, such that

$$\mu_{k_l} \circ B^{o \delta_l} = \mu_{k_0} \circ B^{o \delta_l} \circ B^{o \delta_l}, \quad l \geq 1,$$

for some $\delta_l \in \text{Aut}(\mathbb{C}[z])$. Finally, for some $l_2 > l_1$ the equality $\delta_{l_2} = \delta_{l_1}$ holds, implying that $\mu_{k_{l_2}} = \mu_{k_{l_1}}$. Now (36) implies that

$$B^{o f_{l_2}} = B^{o f_{l_1}} \circ A^{o (r_{l_2} - r_{l_1})}.$$ 

$\square$

### 6.3. Functional equations involving iterates

The following result was proved in [14] (see [14], Proposition 5.1 and Proposition 6.3, and also the paper [28] for a description of arbitrary rational functions $A$ and $B$ such that all algebraic curves have a factor of genus zero). Again, we give an independent proof which can be modified to fit the right case.

**Theorem 6.12.** Let $A$ and $B$ be polynomials of degree $n \geq 2$ and $m \geq 2$ respectively such that for any $i, j \geq 1$ there exist polynomial $C_{i,j}, D_{i,j}$ satisfying

$$A^{o i} \circ C_{i,j} = B^{o j} \circ D_{i,j}.$$  

Then, unless both $A$ and $B$ are special, there exist $k, l \geq 1$ such that $A^{o k} = B^{o l}$.

**Proof.** For a number $n \in \mathbb{N}$, we denote by $\mathcal{P}(n)$ the set of prime factors of $n$. Assume first that

$$\mathcal{P}(\deg B) \subseteq \mathcal{P}(\deg A).$$

In this case, for any $j \geq 1$ the number $\deg B^{o j}$ is a divisor of the number $\deg A^{o i}$ for $i$ big enough. Therefore, by Theorem 6.14. applied to equality (37), for any $j \geq 1$ the polynomial $B^{o j}$ is a compositional left factor of some iterate of $A$, implying by Theorem 6.10 that $A$ and $B$ share an iterate. By symmetry, the same conclusion holds if

$$\mathcal{P}(\deg A) \subseteq \mathcal{P}(\deg B).$$

Assume now that neither of conditions (38), (39) holds. In this case, there exist $p_1 \in \mathcal{P}(\deg A)$ such that $p_1 \not\in \mathcal{P}(\deg B)$, and $p_2 \in \mathcal{P}(\deg B)$ such that $p_2 \not\in \mathcal{P}(\deg A)$. Applying Theorem 6.14 to equality (37), we can find polynomials $U_{i,j}, V_{i,j}, A_{i,j}, C_{i,j}, B_{i,j}, D_{i,j}$, where

$$\deg U_{i,j} = \text{GCD}(\deg A^{o i}, \deg B^{o j}), \quad \deg V_{i,j} = \text{GCD}(\deg C_{i,j}, \deg D_{i,j}),$$

such that

$$A^{o i} = U_{i,j} \circ A_{i,j}, \quad B^{o j} = U_{i,j} \circ B_{i,j}, \quad C_{i,j} = C_{i,j} \circ V_{i,j}, \quad D = D_{i,j} \circ V_{i,j},$$

respectively.
and
\[(42) \quad \bar{A}_{i,j} \circ \bar{C}_{i,j} = \bar{B}_{i,j} \circ \bar{D}_{i,j}.\]

Moreover,
\[
\gcd(\deg \bar{A}_{i,j}, \deg \bar{B}_{i,j}) = 1, \quad \gcd(\deg \bar{C}_{i,j}, \deg \bar{D}_{i,j}) = 1,
\]
and
\[(43) \quad \deg \bar{A}_{i,j} \geq p_1^l, \quad \deg \bar{B}_{i,j} \geq p_2^l.\]

Since the second equality in (41) implies that for any \(i, j \geq 1\) the inequality
\[
\deg \bar{B}_{i,j} \leq \deg B^{o_j}
\]
holds, applying Corollary 6.10 to equality (12) for fixed \(j\) and \(i\) big enough and taking into account the first inequality in (43), we see that \(\bar{B}_{i,j}\) is linearly equivalent to a special polynomial. It follows now from the second inequality in (43) and the second equality in (11) that there exists a sequence of polynomials \(F_j, \ j \geq 1, \) where \(F_j = \bar{B}_{i,j}\) for some \(i = i(j),\) satisfying conditions of Theorem 6.9. Thus, \(B\) is special. Moreover, by symmetry, \(A\) is also special.

**Theorem 6.13.** Let \(A\) and \(B\) be non-special polynomials of degree \(n \geq 2\) and \(m \geq 2\) respectively such that for any \(i, j \geq 1\) there exist polynomial \(C_{i,j}, D_{i,j}\) satisfying
\[
C_{i,j} \circ A^{o_i} = D_{i,j} \circ B^{o_j}.
\]

Then, unless both \(A\) and \(B\) are special, there exist \(k_1, l_1 \geq 1, \) \(k_2 \geq 0, \) such that \(A^{o_{k_1}} = A^{o_{k_2}} \circ B^{o_l}, \) and \(l_1, k_1 \geq 1, \) \(l_2 \geq 0, \) such that \(B^{o_{l_1}} = B^{o_{l_2}} \circ A^{o_k}.
\]

**Proof.** Assume first that at least one of conditions (38), (39), say, (38), holds. Then, unless both \(A\) and \(B\) are special, there exist polynomials \(U_{i,j}, V_{i,j}, \bar{A}_{i,j}, \bar{C}_{i,j}, \bar{B}_{i,j}, \bar{D}_{i,j},\) where
\[(44) \quad \deg U_{i,j} = \text{GCD}(\deg C_{i,j}, \deg D_{i,j}), \quad \deg V_{i,j} = \text{GCD}(\deg A^{o_i}, \deg B^{o_j}),\]
such that
\[(45) \quad C_{i,j} = U_{i,j} \circ \bar{C}_{i,j}, \quad D_{i,j} = U_{i,j} \circ \bar{D}_{i,j}, \quad A^{o_i} = \bar{A}_{i,j} \circ V_{i,j}, \quad B^{o_j} = \bar{D}_{i,j} \circ V_{i,j}, \quad \bar{C}_{i,j} \circ \bar{A}_{i,j} = \bar{D}_{i,j} \circ \bar{B}_{i,j},\]
and inequalities (43) hold for some primes \(p_1, p_2.\) Now an obvious modification of the proof of Theorem 6.12 shows that \(A\) and \(B\) are special.

The next two results are analogues of Theorem 6.12 and Theorem 6.13 in case if at least one of polynomials \(A\) and \(B\) is special.
Theorem 6.14. Let $B$ be a polynomial of degree $m \geq 2$ such that for some $n \geq 2$ and any $i, j \geq 1$ there exist polynomials $C_{i,j}, D_{i,j}$ satisfying

$$z^n \circ C_{i,j} = B \circ j \circ D_{i,j}. \tag{46}$$

Then there exists $b \in \mathbb{C} \setminus \{0\}$ such that $B = bz^m$. On the other hand, if for any $i, j \geq 1$ there exist polynomials $C_{i,j}, D_{i,j}$ satisfying

$$\pm T_n \circ i \circ C_{i,j} = B \circ j \circ D_{i,j}, \tag{47}$$

then $B = \pm T_m$.

Proof. Since (46) is a particular case of (37), it follows from Theorem 6.12 that either there exists $\delta \in \text{Aut}(\mathbb{C})$ such that one of the equalities

$$B = \delta \circ z^m \circ \delta^{-1}, \tag{47}$$

$$B = \delta \circ T_m \circ \delta^{-1} \tag{48}$$

holds, or $z^n$ and $B$ share an iterate. Moreover, in the second case, considering instead of $B$ its second iterate and applying Lemma 6.2, we conclude that (47) still holds.

Assume first that $\text{GCD}(n, m) > 1$. Then, by (47), one can find $(i, j) \in \mathbb{N} \times \mathbb{N}$ such that the number

$$l = l(i, j) = \deg U_{i,j}$$

is arbitrarily large. Moreover, it follows from (41) and (47), (48) by Lemma 6.1 that there exist $\beta, \beta' \in \text{Aut}(\mathbb{C})$ such that either

$$U_{i,j} = z^l \circ \beta = \delta \circ z^l \circ \beta' \tag{49}$$

or

$$U_{i,j} = z^l \circ \beta = \delta \circ T_l \circ \beta', \tag{50}$$

according to whether condition (47) or (48) holds. Since we can find $i, j$ such that $l > 2$, and $z^l$ and $T_l$ are not linearly equivalent for $l > 2$, equality (48) is impossible, implying that equality (48) is also impossible. On the other hand, considering critical values of the polynomials in (49) we see that $\delta = cz, c \in \mathbb{C} \setminus \{0\}$, implying that

$$B = bz^m, \quad b \in \mathbb{C} \setminus \{0\}. \tag{51}$$

Assume now that $\text{GCD}(n, m) = 1$. In this case, applying Theorem 6.5 to equality (46) for fixed $j$ and $i$ big enough, and using again that $z^l$ and $T_l$ are not linearly equivalent for $l > 2$, we conclude easily that there exists $\nu \in \text{Aut}(\mathbb{C})$ such that

$$z^{\nu_{i}} = \nu \circ z^{\nu_{i}} \circ \beta, \quad B^{\nu_{j}} = \nu \circ z^{\nu_{j}} \circ \beta'$$

for some $\beta, \beta' \in \text{Aut}(\mathbb{C})$. The first equality implies that $\nu = cz, c \in \mathbb{C} \setminus \{0\}$. Now the second equality implies by Lemma 6.2 that (51) holds.

The proof of the second part of the theorem is obtained similarly and essentially reduces to the statement that the equality

$$T_l \circ \gamma = \delta \circ T_1, \tag{52}$$

where $\gamma, \delta \in \text{Aut}(\mathbb{C}P^1)$, implies the equality $\delta = \pm z$. In turn, this is a corollary of the well-known fact that $T_l, l > 2$, has only two finite critical values $\pm 1$. \qed
Theorem 6.15. Let $B$ be a polynomial of degree $m \geq 2$ such that for some $n \geq 2$ and any $i, j \geq 1$ there exist polynomial $C_{i,j}, D_{i,j}$ satisfying
\begin{equation}
C_{i,j} \circ z^{in} = D_{i,j} \circ B^{oj}.
\end{equation}
Then there exists $b \in \mathbb{C} \setminus \{0\}$ such that $B = bz^m$. On the other hand, if for any $i, j \geq 1$ there exist polynomial $C_{i,j}, D_{i,j}$ satisfying
\begin{equation}
C_{i,j} \circ (\pm T_n)^{oi} = D_{i,j} \circ B^{oj},
\end{equation}
then $B = \pm T_m$.

Proof. The proof is obtained by a modification of the proof of Theorem 6.14. Namely, if $\text{GCD}(n, m) > 1$, then equalities (53), (44), (45) imply that there exist $\beta, \beta' \in \text{Aut}(\mathbb{C})$ such that, according to whether condition (47) or (48) holds, either
\begin{equation}
V_{i,j} = \beta \circ z^{ol} = \beta' \circ z^{ol} \circ \delta^{-1}
\end{equation}
or
\begin{equation}
V_{i,j} = \beta \circ z^{ol} = \beta' \circ T_l \circ \delta^{-1},
\end{equation}
where
\begin{equation}
l = l(i, j) = \deg V_{i,j},
\end{equation}
and considering critical points of the polynomials in (52), we see that (51) holds. In case if $\text{GCD}(n, m) = 1$, we conclude that there exists $\mu \in \text{Aut}(\mathbb{C})$ such that
\begin{equation}
z^{oni} = \beta \circ z^{oni} \circ \mu, \quad B^{oj} = \beta' \circ z^{omj} \circ \mu
\end{equation}
for some $\beta, \beta' \in \text{Aut}(\mathbb{C})$, implying that (52) holds.

To prove the second part, we must prove that equality (52) implies $\gamma = \pm z$. In turn, this statement follows from the fact that the only points in the preimage $T_n^{-1}\{-1, 1\}, n > 2$, which are not critical are the points $\pm 1$. □

7. Amenable semigroups of polynomials

7.1. Semigroups $I(K)$ and $C(P)$. Let $K \subset \mathbb{C}$ be a compact set. We denote by $I(K)$ the set of polynomials $A$ such that $A^{-1}\{K\} = K$, and by $\Omega_K$ the subset of $I(K)$ consisting of polynomials of degree one. It is clear that whenever $I(K) \neq \emptyset$ the set $I(K)$ is a semigroup, and $\Omega_K$ is a group. In this section, we describe the structure of the semigroup $I(K)$ for a compact set $K$, which is neither a union of concentric circles nor a segment. In addition, we describe the structure of the semigroup $C(P)$ for a non-special polynomial $P$. Our approach is entirely based on the following result (see [24], Theorem 3, and also the related papers [8], [9]).

Theorem 7.1. Let $K \subset \mathbb{C}$ be a compact set, which is neither a union of concentric circles nor a segment, and $A_1, A_2$ polynomials such that
\begin{equation}
A_1^{-1}\{K\} = A_2^{-1}\{K\} = K.
\end{equation}
Then the group $\Omega_K$ is finite and there exist a polynomial $F$ such that $F^{-1}\{K\} = K$ and
\begin{equation}
A_1 = \mu_1 \circ F^{s_1}, \quad A_2 = \mu_2 \circ F^{s_2}
\end{equation}
for some $\mu_1, \mu_2 \in \Omega_K$ and $s_1, s_2 \geq 0$. □
Notice that Theorem 7.4 generalizes results of [1], [38] about polynomials sharing Julia sets.

**Corollary 7.2.** Let $K \subset \mathbb{C}$ be a compact set, which is neither a union of concentric circles nor a segment, such that $\overline{I(K)} \neq \emptyset$. Then for any $\mu \in \Omega_K$ and $A \in \mathcal{I}(K)$ there exists $\nu \in \Omega_K$ such that
\begin{equation}
A \circ \mu = \nu \circ A.
\end{equation}

**Proof.** It follows from Theorem 7.1 that there exist $s \geq 1$ and $F \in \mathcal{I}(K)$ such that
\begin{equation}
A = \mu_1 \circ F^{os}, \quad A \circ \mu = \mu_2 \circ F^{os}
\end{equation}
for some $\mu_1, \mu_2 \in \Omega_K$. Therefore, (55) holds for $\nu = \mu_2 \circ \mu_1^{-1}$. \hfill \Box

Notice that, in the notation of Section 4 Corollary 7.2 states that for any $A \in \mathcal{I}(K)$ the group $\Omega_K$ is a subgroup of $G(A)$ such that $\gamma_A(\Omega_K) \subseteq \Omega_K$.

**Corollary 7.3.** Let $K \subset \mathbb{C}$ be a compact set, which is neither a union of concentric circles nor a segment, such that $\overline{I(K)} \neq \emptyset$. Then there exists $R \in \mathcal{I}(K)$ such that a polynomial $A$ belongs to $I(K)$ if and only if $A$ has the form
\begin{equation}
A = \sigma \circ R^{os}
\end{equation}
for some $s \geq 0$ and $\sigma \in \Omega_K$.

**Proof.** Let $R$ be any polynomial of minimum possible degree which belongs to $\mathcal{I}(K)$. Then Theorem 7.1 implies that for every $A \in \mathcal{I}(K)$ there exists $F \in \Omega_K$ such that the equalities
\begin{equation}
A = \mu_1 \circ F^{os}, \quad R = \mu_2 \circ F
\end{equation}
hold for some $\mu_1, \mu_2 \in \Omega_K$ and $s \geq 1$, implying that
\begin{equation}
A = \mu_1 \circ (\mu_2^{-1} \circ R)^{os}.
\end{equation}
It follows now from Corollary 7.2 that (56) holds for any $A$ of degree at least two. On the other hand, if $\deg A = 1$, then (56) holds for $s = 0$. \hfill \Box

**Theorem 7.4.** Let $K \subset \mathbb{C}$ be a compact set, which is neither a union of concentric circles nor a segment, such that $\overline{I(K)} \neq \emptyset$. Then there exist $R \in \mathcal{I}(K)$ and a subgroup $\Gamma$ of $G(R)$ such that $\gamma_R(\Gamma) \subseteq \Gamma$ and $I(K) = S_{\Gamma,R}$.

**Proof.** The proof follows from Corollary 7.2 and Corollary 7.3 which imply that $I(K) = S_{\Omega_K,R}$. \hfill \Box

**Theorem 7.5.** Let $P$ be a non-special polynomial of degree at least two. Then there exist $R \in \overline{C(P)}$ and a subgroup $\Gamma$ of $G(R)$ such that $\gamma_R(\Gamma) = \Gamma$ and $C(P) = S_{\Gamma,R}$.

**Proof.** Let $R$ be any polynomial of minimum possible degree which belongs to $\overline{C(P)}$, and $K$ its filled-in Julia set. Since commuting polynomials have the same filled-in Julia sets, and the filled-in Julia set of a non-special polynomial is neither a union of concentric circles nor a segment (see e.g. [26], Lemma 4.7), arguing as in the proofs of Lemma 7.2 and Lemma 7.3 we conclude that for any $\mu \in \Sigma_K$ and $A \in \overline{C(P)}$ the equality (55) holds for some $\nu \in \Sigma_K$, and that any $A \in \overline{C(P)}$ can be represented in the form (56) for some $\sigma \in \Sigma_K$. Moreover, since $R \in \overline{C(P)}$, it is easy to see that a polynomial $A$, written in the form (56) belongs to $\overline{C(P)}$ if and only if $\sigma \in \overline{C(P)}$, implying that $C(P) = S_{\Sigma_K,R}.$
To finish the proof of the theorem we only must show that if 
\[ R \circ \mu = R \]
for some \( \mu \in C(P) \), then \( \mu = z \). For this purpose, we observe that \( P = \delta \circ R^{\circ s} \) for some \( \delta \in C(P) \) and \( s \geq 0 \). On the other hand, equality (57) implies the equality
\[ \delta \circ R^{\circ s} \circ \mu = \delta \circ R^{\circ s}. \]
Therefore, the condition \( \mu \in C(P) \) holds only if \( \mu = z \).

Notice that for a polynomial \( R \) of degree at least two the group \( G(R) \) is a finite cyclic group. Indeed, (21) implies that \( \nu \) and \( \mu \) are polynomials. Furthermore, conjugating we can assume that \( R \) has the form
\[ z^n + a_n z^{n-2} + \cdots + a_0, \]
where \( a_n = 1 \) and \( a_{n-1} = 0 \), and one can easily see that if (21) holds for a polynomial of the form (58) and polynomials
\[ \sigma = az + b, \quad \nu = cz + d, \]
then \( b = 0 \) and \( a \) is a root of unity.

7.2. Left amenable semigroups. The following result is an extended version of Theorem 1.1 for semigroups of polynomials of degree at least two containing no special polynomials.

**Theorem 7.6.** Let \( S \) be a semigroup of polynomials of degree at least two not containing special polynomials. Then the following conditions are equivalent.

1) The semigroup \( S \) is left reversible.
2) The semigroup \( S \) is left amenable.
3) The semigroup \( S \) is amenable.
4) The semigroup \( S \) is subsemigroup of \( \overline{S_{1,R}} \) for some \( R \in \mathbb{C}[z] \) of degree at least two and a subgroup \( \Gamma \) of \( G(R) \) such that \( \gamma_R(\Gamma) = \Gamma \).
5) The semigroup \( S \) is a subsemigroup of \( \overline{C(P)} \) for some \( P \in \mathbb{C}[z] \) of degree at least two not conjugate to \( z^n \) or \( \pm T_n \).
6) The semigroup \( S \) is power joined.
7) The semigroup \( S \) is right Archimedean.
8) For every \( A, B \in S \) there exist \( z_1, z_2 \in \mathbb{CP}^1 \) such that the forward orbits \( O_A(z_1) \) and \( O_B(z_2) \) have an infinite intersection.

**Proof.** It follows from Theorem 2.13 and Lemma 6.3 that if \( S \) is power joined, then \( S \) is a subsemigroup of \( \overline{C_\infty(P)} \) for some polynomial \( P \), and Theorem 6.8 implies that
\[ \overline{C_\infty(P)} = \overline{C(P^{\circ s})}. \]
for some \( s \geq 1 \). Thus, 6 \( \Rightarrow \) 5. The implications 5 \( \Rightarrow \) 4 follows from Theorem 7.5. The implication 4 \( \Rightarrow \) 3 follows from Theorem 4.4. The implications 3 \( \Rightarrow \) 2 and 2 \( \Rightarrow \) 1 are clear. The implication 1 \( \Rightarrow \) 6 follows from Theorem 6.12. Thus, the first six conditions of the theorem are equivalent.

Further, the implication 6 \( \Rightarrow \) 7 follows from Lemma 2.9. On the other hand, the implication 7 \( \Rightarrow \) 6 follows from Theorem 6.10. Therefore, 6 \( \Leftrightarrow \) 7. Finally, the implication 6 \( \Rightarrow \) 8 is clear, while the implication 8 \( \Rightarrow \) 6 is proved in the papers [13], [14]. Another proof of this implication is given in [33]. 

\[ \square \]
Notice that Theorem 7.6 is not true in its full generality without assumption that $S$ contains no special polynomials. Indeed, for example, for a Chebyshev polynomial $T_n$, $n \geq 2$, the semigroup $C(T_n)$ contains all Chebyshev polynomials $T_m$, $m \geq 2$, but not all pairs of Chebyshev polynomials share an iterate. Similarly, one can easily see that the semigroup generated by $T_6$ and $T_{12}$, say, is left and right Archimedean, but $T_6$ and $T_{12}$ have no common iterate. The assumption that $S$ contains only polynomials of degree at least two is also essential, since as it was already mentioned above any power joined semigroup $S$ with non-empty $S$ coincides with $S$.

Let us recall that we denote by $Z$ the semigroup consisting of polynomials of the form $az^n$, where $a \in \mathbb{C}^*$ and $n \geq 1$, and by $T$ the semigroup consisting of polynomials of the form $\pm T_n$, $n \geq 1$.

Notice that $Z \cong \mathbb{C}^* \times \mathbb{N}$, where the homomorphism $\rho : \mathbb{N} \to \text{End}(\mathbb{C}^*)$ is given by the formula $\rho_n(a) = a^n$, and that $T$ is isomorphic to the subsemigroup $\mathbb{Z}_2 \times \mathbb{N}$ of $\mathbb{C}^* \times \mathbb{N}$. We also recall that two semigroups of polynomials $S_1$ and $S_2$ are called conjugate if there exists $\alpha \in \text{Aut}(\mathbb{C})$ such that $\alpha \circ S_1 \circ \alpha^{-1} = S_2$.

The following result shows that if a left or right reversible semigroup of polynomials of degree at least two contains a special polynomial, then all its elements are special. More precisely, the following statement holds.

**Theorem 7.7.** Let $S$ be a semigroup of polynomials of degree at least two containing a special polynomial. Assume that $S$ is left or right reversible. Then $S$ is conjugate to a subsemigroup of $Z$ or $T$. In particular, $S$ is a subsemigroup of $Z$ or $T$ whenever $S$ is left or right amenable.

**Proof.** Assume that $S$ is left reversible. To prove the theorem in this case, it is enough to show if $S$ contains the polynomial $A = z^n$, then any other element $B$ of $S$ satisfies $B = A^n$, while if $S$ contains the polynomial $A = \pm T_n$, then $B = \pm T_m$. In turn, these statements are claimed by Theorem 6.14. Similarly, in case if $S$ right reversible, the theorem follows from Theorem 6.15. □

**Proof of Theorem 1.1.** The implications $5 \Rightarrow 4 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1$ are proved in the same way as in the proof of Theorem 7.6. Thus, to show that conditions 1-5 are equivalent we only must show that 1 implies 5. Since $S$ is not a subsemigroup of $\text{Aut}(\mathbb{C})$, the semigroup $S$ is not empty. Moreover, it follows from Theorem 7.7 that $S$ contains only non-special polynomials. Therefore, $S$ is a subsemigroup of $C(P)$ for some $P \in \mathbb{C}[z]$ of the required form by Theorem 7.6. If $S = \emptyset$, we are done. Moreover, in this case the rest of the theorem also follows from Theorem 7.6. On the other hand, if $S \neq \emptyset$, then for any element $\mu \in S$ and $A \in S$ the element $\mu \circ A$ belongs to $S$ and hence to $C(P)$. By Lemma 2.11 this implies that $\mu \in C(P)$. Thus, $S = S \cup \overline{S}$ is contained in $C(P)$. □
7.3. **Right amenable semigroups.** The following result is the analogue of Theorem 7.6 for right amenable semigroups.

**Theorem 7.8.** Let $S$ be a semigroup of polynomials of degree at least two not containing special polynomials. Then the following conditions are equivalent.

1) The semigroup $S$ is right reversible.
2) The semigroup $S$ is right amenable.
3) The semigroup $S$ is subsemigroup of $\overline{S_{\Gamma,R}}$ for some $R \in \mathbb{C}[z]$ of degree at least two and a subgroup $\Gamma$ of $G(R)$ such that $\gamma_R(\Gamma) \subseteq \Gamma$.
4) The semigroup $S$ is subsemigroup of $\overline{I(K)}$ for some compact set $K \subset \mathbb{C}$, which is neither a union of concentric circles nor a segment.
5) The semigroup $S$ is a subsemigroup of $\overline{E(P)}$ for some $P \in \mathbb{C}[z]$ of degree at least two not conjugate to $z^n$ or $\pm T_n$.
6) The semigroup $S$ is left Archimedean.
7) The semigroup $S$ is power twisted.
8) The semigroup $S$ contains no free subsemigroup of rank two.

**Proof.** The implication $7 \Rightarrow 6$ follows from Lemma 2.9 while the implication $6 \Rightarrow 7$ follows from Theorem 6.13. Thus, $6 \Leftrightarrow 7$. Further, it follows from Theorem 8.2 that if condition 7 holds, then $S$ is a subsemigroup of $E(P)$ for any fixed $P \in S$. Therefore, $7 \Rightarrow 5$.

Since the support of $\mu_P$ coincides with $J(P)$, polynomials sharing the measure of maximal entropy share the filled-in Julia sets. Thus, the implication $5 \Rightarrow 4$ follows from the above mentioned fact that the filled-in Julia set of a non-special polynomial cannot be a union of concentric circles or a segment. The implication $4 \Rightarrow 3$ follows from Theorem 7.4, the implication $3 \Rightarrow 2$ follows from Theorem 4.5, and the implication $2 \Rightarrow 1$ follows from Proposition 2.4. The implication $1 \Rightarrow 7$ follows from Theorem 6.13. Thus, conditions 1-7 are equivalent.

Finally, the implication $8 \Rightarrow 1$ follows from Lemma 2.8 since any semigroup of rational functions is right cancellative. On the other hand, the implication $3 \Rightarrow 8$ follows from Theorem 4.5. \qed

**Proof of Theorem 1.2.** The implications $5 \Rightarrow 4 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1$ are proved in the same way as in the proof of Theorem 7.8. Furthermore, to prove the implication $1 \Rightarrow 5$ it is enough to prove that if $S$ is contained in $E(P)$, then $S$ is also contained in $E(P)$. Assume that $\alpha \in S$. Then $P \circ \alpha \in S$ and, by the invariance of $\mu_P$, for any Borel set $S$ we have:

$$\mu_P((P \circ \sigma)^{-1}(S)) = \mu_P(S).$$

On the other hand,

$$\mu_P((P \circ \sigma)^{-1}(S)) = \mu_P(P^{-1}(\sigma(S))) = \mu_P(\sigma(S)).$$

Therefore,

$$\mu_P(\sigma(S)) = \mu_P(S).$$

Finally, the implications $6 \Rightarrow 1$ and $3 \Rightarrow 6$ are obtained in the same way as the implications $8 \Rightarrow 1$ and $3 \Rightarrow 8$ in Theorem 7.8. \qed

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