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ALGEBRAIC CURVES $A^{\circ l}(x) - U(y) = 0$ AND ARITHMETIC OF ORBITS OF RATIONAL FUNCTIONS

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Abstract. We give a description of pairs of complex rational functions A and U of degree at least two such that for every $d \geqslant 1$ the algebraic curve $A^{\circ d}(x) - U(y) = 0$ has a factor of genus zero or one. In particular, we show that if A is not a "generalized Lattès map", then this condition is satisfied if and only if there exists a rational function V such that $U \circ V = A^{\circ l}$ for some $l \geqslant 1$. We also prove a version of the dynamical Mordell–Lang conjecture, concerning intersections of orbits of points from $\mathbb{P}^1(K)$ under iterates of A with the value set $U(\mathbb{P}^1(K))$, where A and U are rational functions defined over a number field K.

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1. Introduction

In this paper we solve the following problem.

Problem 1.1. *Describe the pairs of complex rational functions A and U of degree at least two such that for every $d \geqslant 1$, the algebraic curve*

$$A^{\circ d}(x) - U(y) = 0 \tag{1}$$

has an irreducible factor of genus zero or one.

The motivation for this problem comes from the arithmetic dynamics. Specifically, in [3], the following problem was investigated: which rational functions A defined over a number field K have a K-orbit containing infinitely many distinct mth powers of elements from K? If such an orbit exists, then for every $d \geqslant 1$, the algebraic curve

$$A^{\circ d}(x) - y^m = 0 \tag{2}$$

has infinitely many K-points, implying by the Faltings theorem that it has a factor of genus zero or one. Thus, a geometric counterpart of the initial arithmetic problem is to describe rational functions A such that all curves (2) have a factor of genus zero or one. Now if instead of intersections of orbits of A with the set of mth powers we

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consider intersections with the value set $U(\mathbb{P}^1(K))$ of an arbitrary rational function U, we arrive at Problem 1.1.

The paper [3], based on painstaking calculations of the possible ramifications of rational functions A such that every curve (2) has a factor of genus zero or one, provides a very explicit description of such functions. In contrast, our approach is more geometric and provides an answer in the general case in terms of semiconjugacies and Galois coverings. Notice that Problem 1.1 is somewhat similar to the following problem considered in the paper [14]: for which rational functions U, there exists a sequence of rational functions F_d, $d \geqslant 1$, such that $\deg F_d \to \infty$, and for every $d \geqslant 1$, the curve

$$F_d(x) - U(y) = 0$$

is irreducible and of genus zero. It was shown in [14] that U satisfies this condition if and only if the Galois closure of the field extension $\mathbb{C}(z)/\mathbb{C}(U)$ has genus zero or one. Thus, this condition also holds for solutions of Problem 1.1 whenever curves (1) are irreducible. However, Problem 1.1 is distinct from the problem considered in [14] in two important respects. First, curves (1) can be reducible. Secondly, Problem 1.1 asks for a description of all *pairs* A, U such that curves (1) have an irreducible factor of genus zero or one, and not for a description of U for which *some* A with this property exists.

Let A and B be rational functions of degree at least two. Recall that the function B is called semiconjugate to the function A if there exists a non-constant rational function X such that the diagram

$$\begin{tikzcd} \mathbb{CP}^1 \arrow[r, "B"] \arrow[d, "X"] & \mathbb{CP}^1 \arrow[d, "X"] \\ \mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1 \end{tikzcd} \tag{3}$$

commutes. Semiconjugate rational functions appear naturally in complex and arithmetic dynamics (see for example the recent papers [4], [9], [13]). They are also closely related to Problem 1.1. Indeed, since the commutativity of diagram (3) implies that

$$A^{\circ d} \circ X = X \circ B^{\circ d}, \quad d \geqslant 1,$$

setting U equal to X, we see that for every $d \geqslant 1$, curve (1) has a component of genus zero parametrized by the rational functions X and $B^{\circ d}$.

More generally, if A, B and X satisfy (3), then curves (1) have a factor of genus zero for any rational function U, which is a "compositional left factor" of the function $A^{\circ l} \circ X$ for some $l \geqslant 0$, where by a compositional left factor of a holomorphic map $f\colon R_1 \to R_2$ between Riemann surfaces, we mean any holomorphic map $g\colon R' \to R_2$ such that $f = g \circ h$ for some holomorphic map $h\colon R_1 \to R'$. Indeed, it follows from (3) and

$$A^{\circ l} \circ X = U \circ V$$

that

$$A^{\circ l+k} \circ X = U \circ V \circ B^{\circ k}$$

for every $k \geqslant 0$, implying as above that the pair A, U is a solution of Problem 1.1. In particular, setting $B = A$ and $X = z$ in (3), we see that for every $d \geqslant 1$ curve (1) have a factor of genus zero whenever U is a compositional left factor of some iterate $A^{\circ l}$, $l \geqslant 1$.

Semiconjugate rational functions were studied at length in the recent series of papers [12], [16], [17], [18], [19] by using the theory of orbifolds on Riemann surfaces. Our approach to Problem 1.1 is based on the ideas and methods described in these papers. Roughly speaking, our main result states that, unless A belongs to a special family of functions, all corresponding solutions U of Problem 1.1 can be obtained as described above from some *fixed* semiconjugacy (3), where X is a *Galois covering* that depends only on A. Moreover, for "most" rational functions A, this Galois covering X_A is equal simply to the identity map. In other words, a rational function U is a solution of Problem 1.1 if and only if U is a compositional left factor of $A^{\circ l}$ for some $l \geqslant 1$.

To formulate our results explicitly we need several definitions. Recall that an *orbifold* $\mathcal{O}$ on $\mathbb{CP}^1$ is a ramification function $\nu\colon \mathbb{CP}^1 \to \mathbb{N}$ which takes the value $\nu(z) = 1$ except at a finite set of points. We assume that considered orbifolds are *good*, meaning that we forbid $\mathcal{O}$ to have exactly one point with $\nu(z) \neq 1$ or exactly two such points z_1, z_2 with $\nu(z_1) \neq \nu(z_2)$. Let f be a rational function and $\mathcal{O}_1$, $\mathcal{O}_2$ orbifolds with ramification functions ν_1 and ν_2. We say that $f\colon \mathcal{O}_1 \to \mathcal{O}_2$ is a *covering map* between orbifolds if for any $z \in \mathbb{CP}^1$ the equality

$$\nu_2(f(z)) = \nu_1(z) \deg_z f$$

holds. In case the weaker condition

$$\nu_2(f(z)) = \nu_1(z) \, \mathrm{GCD}(\deg_z f, \, \nu_2(f(z))$$

is satisfied, we say that $f\colon \mathcal{O}_1 \to \mathcal{O}_2$ is *a minimal holomorphic map* between orbifolds.

In the above terms, *a Lattès map* can be defined as a rational function A such that $A\colon \mathcal{O} \to \mathcal{O}$ is a covering self-map for some orbifold $\mathcal{O}$ (see [11] for a classical definition and for a proof of the equivalency between two definitions). Following [18], we say that a rational function A is a *generalized Lattès map* if there exists an orbifold $\mathcal{O}$ distinct from the non-ramified sphere such that $A\colon \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map. Thus, A is a Lattès map if there exists an orbifold $\mathcal{O}$ such that

$$\nu(A(z)) = \nu(z) \deg_z A, \quad z \in \mathbb{CP}^1,$$

and A is a generalized Lattès map if there exists an orbifold $\mathcal{O}$ such that

$$\nu(A(z)) = \nu(z) \, \mathrm{GCD}(\deg_z A, \, \nu(A(z))), \quad z \in \mathbb{CP}^1. \tag{4}$$

Similar to ordinary Lattès maps, generalized Lattès maps can be described in terms of semiconjugacies and group actions. In particular, the following statement is true (see [18]): a rational function A is a generalized Lattès map if and only if there exist a compact Riemann surface R of genus zero or one, a finite non-trivial group $\Gamma \subseteq \mathrm{Aut}(R)$, a group automorphism $\varphi\colon \Gamma \to \Gamma$, and a holomorphic map $B\colon R \to R$

such that the diagram

$$\begin{tikzcd} R \arrow[r, "B"] \arrow[d, "\pi"] & R \arrow[d, "\pi"] \\ \mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1, \end{tikzcd}$$

where $\pi\colon R \to R/\Gamma$ is the quotient map, commutes, and for any $\sigma \in \Gamma$ the equality

$$B \circ \sigma = \varphi(\sigma) \circ B$$

holds.

We say that a rational function is *special* if it is either a Lattès map or it is conjugate to $z^{\pm n}$ or $\pm T_n$, where T_n is the Chebyshev polynomial. For rational functions A and U, we denote by $g_d = g_d(A, U)$, $d \geqslant 1$, the minimal number g such that curve (1) has a component of genus g. In this notation our main result concerning Problem 1.1 is following.

Theorem 1.2. *Let A be a non-special rational function of degree at least two. Then there exist a rational Galois covering X_A and a rational function B such that the diagram*

$$\begin{tikzcd} \mathbb{CP}^1 \arrow[r, "B"] \arrow[d, "X_A"] & \mathbb{CP}^1 \arrow[d, "X_A"] \\ \mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1 \end{tikzcd} \tag{5}$$

commutes, and for a rational function U of degree at least two the sequence g_d, $d \geqslant 1$, is bounded if and only if U is a compositional left factor of $A^{\circ l} \circ X_A$ for some $l \geqslant 0$. Moreover, if A is not a generalized Lattès map, then g_d, $d \geqslant 1$, is bounded if and only if U is a compositional left factor of $A^{\circ l}$ for some $l \geqslant 1$.

Notice that our method provides an explicit description of the Galois covering X_A appearing in Theorem 1.2 via the "maximal" orbifold $\mathcal{O}$ for which (4) is satisfied. In particular, the function X_A is defined by the function A in a unique way up to a pre-composition with a Möbius transformation.

Theorem 1.2 can be illustrated as follows. A "random" rational function A is not a generalized Lattès map. Thus, a rational function U is a solution of Problem 1.1 if and only if U is a compositional left factor of $A^{\circ l}$ for some $l \geqslant 1$. A simple example of a generalized Lattès map is provided by any function of the form $A = z^r R^n(z)$, where R is a rational function, $n \geqslant 2$, $r \geqslant 1$, and $\mathrm{GCD}(r, n) = 1$. Indeed, one can easily check that (4) is satisfied for the orbifold $\mathcal{O}$ defined by the conditions

$$\nu(0) = n, \quad \nu(\infty) = n.$$

With a few exceptions, the rational function $A = z^r R^n(z)$ is not special, and diagram (5) from Theorem 1.2 has the form

$$\begin{tikzcd} \mathbb{CP}^1 \arrow[r, "z^r R(z^n)"] \arrow[d, "z^n"] & \mathbb{CP}^1 \arrow[d, "z^n"] \\ \mathbb{CP}^1 \arrow[r, "z^r R^n(z)"] & \mathbb{CP}^1. \end{tikzcd}$$

Thus, for such A a rational function U is a solution of Problem 1.1 if and only if there exists $l \geqslant 0$ such that U is a compositional left factor of the function $(z^r R^n(z))^{\circ l} \circ z^n$.

Assume now that considered rational functions A and U are defined over a number field K. As an application of our results, we prove a statement that, concerning intersections of orbits of points from $\mathbb{P}^1(K)$ under iterates of A with the value set $U(\mathbb{P}^1(K))$, can be considered to be a version of the dynamical Mordell–Lang conjecture. Recall that the dynamical Mordell–Lang conjecture states that if f is an endomorphism of a quasiprojective variety V over $\mathbb{C}$, then for any point $z_0 \in V$ and any subvariety $W \subset V$ the set of indices n such that the $X^{\circ n}(z_0) \in W$ is a finite union of arithmetic progressions (see [2] and the bibliography therein). In particular, this implies that if the f-orbit of z_0 has an infinite intersection with a proper subvariety W, then its Zariski closure is contained in a finite union of proper subvarieties, and therefore, it cannot coincide with whole V. Notice that singletons are considered as arithmetic progressions with the common difference equal zero, so any finite set is a union of arithmetic progressions.

A conjecture closely related to the dynamical Mordell–Lang conjecture was proposed in [9] (see also [1], [20]). It states that if f is a dominant endomorphism of a quasiprojective variety V defined over an algebraically closed field K of characteristic zero for which there exists no non-constant rational function g satisfying $g \circ f = g$, then there is a point $z_0 \in V(K)$ whose f-orbit is Zariski dense in V. This conjecture is complementary to the Mordell–Lang conjecture in the sense that the former states that there is a point with the dense orbit, while the dynamical Mordell–Lang conjecture asserts that the orbit of such a point intersects any subvariety W of V in at most finitely many points.

In this paper, we prove the following statement that is similar in spirit to the dynamical Mordell–Lang conjecture.

Theorem 1.3. *Let A and U be rational functions of degree at least two defined over a number field K, and x_0 a point from $\mathbb{P}^1(K)$. Then the set of indices n such that $A^{\circ n}(x_0) \in U(\mathbb{P}^1(K))$ is a finite union of arithmetic progressions. Moreover, if A is not a generalized Lattès map, then either the above set is finite, or $A^{\circ n}(x_0)$ belongs to $U(\mathbb{P}^1(K))$ for all but finitely many n.*

The first part of Theorem 1.3 confirms a conjecture proposed in the paper [3][1]. On the other hand, the second part asserts that if A is not a generalized Lattès map, then a stronger conclusion holds.

The paper is organized as follows. In the second section, we present relevant definitions and some results concerning orbifolds, fiber products, and generalized Lattès maps, mostly proved in the papers [12], [18]. In the third section, using the lower bounds obtained in [14] on the genus of algebraic curves of the form

$$F(x) - U(y) = 0,$$

[1]A proof of this conjecture was also announced by T. Hyde and M. Zieve in the "Workshop on Interactions between Model Theory and Arithmetic Dynamics" in July 2016 at the Fields Institute. To date, however, a complete proof of their results is still not available.

where F and U are rational functions, we solve Problem 1.1. In particular, we prove Theorem 1.2. We also solve Problem 1.1 for special A. In fact, we consider a more general version of Problem 1.1 in which U is allowed to be a holomorphic map

$$U\colon R \to \mathbb{CP}^1,$$

where R is a compact Riemann surface, and instead of curves (1) the fiber products of U and $A^{\circ d}$, $d \geqslant 1$, are considered.

Finally, in the fourth section we prove Theorem 1.3, combining the results of the third section with some results for semiconjugate maps between algebraic curves which are interesting in their own right. We also provide an example illustrating constructions and results of this paper.

2. Orbifolds and Generalized Lattès Maps

2.1. Riemann surface orbifolds. In this section, we recall basic definitions concerning orbifolds on Riemann surfaces (see [10, Appendix E]) and some results and constructions from the papers [12], [18]. We also prove some additional related results used later.

A *Riemann surface orbifold* is a pair $\mathcal{O} = (R, \nu)$ consisting of a Riemann surface R and a ramification function $\nu\colon R \to \mathbb{N}$ that takes the value $\nu(z) = 1$ except at isolated points. For an orbifold $\mathcal{O} = (R, \nu)$ the *Euler characteristic* of $\mathcal{O}$ is the number

$$\chi(\mathcal{O}) = \chi(R) + \sum_{z\in R}\left(\frac{1}{\nu(z)} - 1\right), \tag{6}$$

the set of *singular points* of $\mathcal{O}$ is the set

$$c(\mathcal{O}) = \{z_1, z_2, \dots, z_s, \dots\} = \{z \in R \mid \nu(z) > 1\},$$

and the *signature* of $\mathcal{O}$ is the set

$$\nu(\mathcal{O}) = \{\nu(z_1), \nu(z_2), \dots, \nu(z_s), \dots\}.$$

For orbifolds $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$, we write

$$\mathcal{O}_1 \preceq \mathcal{O}_2 \tag{7}$$

if $R_1 = R_2$, and for any $z \in R_1$, the condition $\nu_1(z) \mid \nu_2(z)$ holds. Clearly, (7) implies that

$$\chi(\mathcal{O}_1) \geqslant \chi(\mathcal{O}_2).$$

Let $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ be orbifolds and let $f\colon R_1 \to R_2$ be a holomorphic branched covering map. We say that $f\colon \mathcal{O}_1 \to \mathcal{O}_2$ is a *covering map between orbifolds* if for any $z \in R_1$, the equality

$$\nu_2(f(z)) = \nu_1(z)\deg_z f \tag{8}$$

holds, where $\deg_z f$ is the local degree of f at the point z. If for any $z \in R_1$, the weaker condition

$$\nu_2(f(z)) \mid \nu_1(z)\deg_z f \tag{9}$$

is satisfied instead of (8), we say that $f\colon \mathcal{O}_1 \to \mathcal{O}_2$ is a *holomorphic map between orbifolds*.

If $f\colon \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between orbifolds with compact R_1 and R_2, then the Riemann–Hurwitz formula implies that

$$\chi(\mathcal{O}_1) = d\chi(\mathcal{O}_2), \tag{10}$$

where $d = \deg f$. For holomorphic maps, the following statement is true (see [12, Proposition 3.2]).

Proposition 2.1. *Let $f\colon \mathcal{O}_1 \to \mathcal{O}_2$ be a holomorphic map between orbifolds with compact R_1 and R_2. Then*

$$\chi(\mathcal{O}_1) \leqslant \chi(\mathcal{O}_2)\deg f, \tag{11}$$

and the equality holds if and only if $f\colon \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between orbifolds. □

Let R_1, R_2 be Riemann surfaces and $f\colon R_1 \to R_2$ a holomorphic branched covering map. Assume that R_2 is provided with a ramification function ν_2. To define a ramification function ν_1 on R_1 so that f would be a holomorphic map between orbifolds $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$, we must satisfy condition (9), and it is easy to see that for any $z \in R_1$, a minimum possible value for $\nu_1(z)$ is defined by the equality

$$\nu_2(f(z)) = \nu_1(z)\,\mathrm{GCD}(\deg_z f, \nu_2(f(z)). \tag{12}$$

In case (12) is satisfied for any $z \in R_1$, we say that $f\colon \mathcal{O}_1 \to \mathcal{O}_2$ is a *minimal holomorphic map between orbifolds*. It follows from the definition that for any orbifold $\mathcal{O} = (R, \nu)$ and holomorphic branched covering map $f\colon R' \to R$, there exists a unique orbifold structure ν' on R', such that f becomes a minimal holomorphic map between orbifolds. We will denote the corresponding orbifold by $f^*\mathcal{O}$.

Below we will use the following property of the association from $\mathcal{O}$ to $f^*\mathcal{O}$ (see [12, Corollary 4.2]).

Proposition 2.2. *Let $f\colon R_1 \to R'$ and $g\colon R' \to R_2$ be holomorphic branched covering maps, and $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ orbifolds. Assume that $g \circ f\colon \mathcal{O}_1 \to \mathcal{O}_2$ is a minimal holomorphic map (resp. a covering map). Then $f\colon \mathcal{O}_1 \to g^*\mathcal{O}_2$ and $g\colon g^*\mathcal{O}_2 \to \mathcal{O}_2$ are minimal holomorphic maps (resp. covering maps).* □

A universal covering of an orbifold $\mathcal{O}$ is a covering map $\theta_{\mathcal{O}}\colon \tilde{\mathcal{O}} \to \mathcal{O}$ between orbifolds such that $\widetilde{R}$ is simply connected and $\widetilde{\mathcal{O}}$ is non-ramified, that is, $\widetilde{\nu}(z) \equiv 1$. If $\theta_{\mathcal{O}}$ is such a map, then there exists a group $\Gamma_{\mathcal{O}}$ of conformal automorphisms of $\widetilde{R}$ such that the equality

$$\theta_{\mathcal{O}}(z_1) = \theta_{\mathcal{O}}(z_2)$$

holds for $z_1, z_2 \in \widetilde{R}$ if and only if $z_1 = \sigma(z_2)$ for some $\sigma \in \Gamma_{\mathcal{O}}$. A universal covering exists and is unique up to a conformal isomorphism of $\widetilde{R}$ whenever $\mathcal{O}$ is *good*, that is, distinct from the Riemann sphere with one ramified point or with two ramified points z_1, z_2 such that $\nu(z_1) \neq \nu(z_2)$. Furthermore, $\widetilde{R}$ is the unit disk $\mathbb{D}$ if and only if $\chi(\mathcal{O}) < 0$, $\widetilde{R}$ is the complex plane $\mathbb{C}$ if and only if $\chi(\mathcal{O}) = 0$, and $\widetilde{R}$ is the Riemann sphere $\mathbb{CP}^1$ if and only if $\chi(\mathcal{O}) > 0$ (see for example [5, Section IV.9.12]).

Below we will always assume that considered orbifolds are good. Abusing notation, we will use the symbol $\widetilde{\mathcal{O}}$ both for the orbifold and for the Riemann surface $\widetilde{R}$.

Covering maps between orbifolds lift to isomorphisms between their universal coverings. More generally, the following proposition is true (see [12, Proposition 3.1]).

Proposition 2.3. *Let $f\colon \mathcal{O}_1 \to \mathcal{O}_2$ be a holomorphic map between orbifolds. Then for any choice of $\theta_{\mathcal{O}_1}$ and $\theta_{\mathcal{O}_2}$, there exist a holomorphic map $F\colon \widetilde{\mathcal{O}}_1 \to \widetilde{\mathcal{O}}_2$ and a homomorphism $\varphi\colon \Gamma_{\mathcal{O}_1} \to \Gamma_{\mathcal{O}_2}$, such that the diagram*

$$\begin{tikzcd} \widetilde{\mathcal{O}}_1 \arrow[r, "F"] \arrow[d, "\theta_{\mathcal{O}_1}"] & \widetilde{\mathcal{O}}_2 \arrow[d, "\theta_{\mathcal{O}_2}"] \\ \mathcal{O}_1 \arrow[r, "f"] & \mathcal{O}_2 \end{tikzcd} \tag{13}$$

is commutative, and for any $\sigma \in \Gamma_{\mathcal{O}_1}$, the equality

$$F \circ \sigma = \varphi(\sigma) \circ F \tag{14}$$

holds. The map F is defined by $\theta_{\mathcal{O}_1}$, $\theta_{\mathcal{O}_2}$, and f uniquely up to a transformation $F \to g \circ F$, where $g \in \Gamma_{\mathcal{O}_2}$. In the other direction, for any holomorphic map $F\colon \widetilde{\mathcal{O}}_1 \to \widetilde{\mathcal{O}}_2$ that satisfies (14) for some homomorphism $\varphi\colon \Gamma_{\mathcal{O}_1} \to \Gamma_{\mathcal{O}_2}$ there exists a uniquely defined holomorphic map between orbifolds $f\colon \mathcal{O}_1 \to \mathcal{O}_2$ such that diagram (13) is commutative. The holomorphic map F is an isomorphism if and only if f is a covering map between orbifolds. $\square$

With each holomorphic map $f\colon R_1 \to R_2$ between compact Riemann surfaces, one can associate two orbifolds $\mathcal{O}_1^f = (R_1, \nu_1^f)$ and $\mathcal{O}_2^f = (R_2, \nu_2^f)$ in a natural way, setting $\nu_2^f(z)$ equal to the least common multiple of local degrees of f at the points of the preimage $f^{-1}\{z\}$, and

$$\nu_1^f(z) = \frac{\nu_2^f(f(z))}{\deg_z f}.$$

By construction,

$$f\colon \mathcal{O}_1^f \to \mathcal{O}_2^f$$

is a covering map between orbifolds. It is easy to see that this covering map is minimal in the following sense. For any covering map $f\colon \mathcal{O}_1 \to \mathcal{O}_2$, we have:

$$\mathcal{O}_1^f \preceq \mathcal{O}_1, \quad \mathcal{O}_2^f \preceq \mathcal{O}_2.$$

The orbifolds $\mathcal{O}_1^f$ and $\mathcal{O}_2^f$ are good (see [12, Lemma 4.2]).

Theorem 2.4. *Let $f\colon R_1 \to R_2$ be a holomorphic map between compact Riemann surfaces and $\mathcal{O} = (R_2, \nu)$ an orbifold. Then f is a compositional left factor of $\theta_{\mathcal{O}}$ if and only if $\mathcal{O}_2^f \preceq \mathcal{O}$. Furthermore, for any decomposition $\theta_{\mathcal{O}} = f \circ \psi$, where $\psi\colon \widetilde{\mathcal{O}} \to R_1$ is a holomorphic map, the equality $\psi = \theta_{f^*\mathcal{O}}$ holds, and the map $f\colon f^*\mathcal{O} \to \mathcal{O}$ is a covering map between orbifolds. In particular, $\theta_{\mathcal{O}_2^f} = f \circ \theta_{\mathcal{O}_1^f}$.*

Proof. Since $\mathcal{O}_2^{\theta_{\mathcal{O}}} = \mathcal{O}$, the "only if" part follows from the chain rule. In the other direction, let ψ be the analytic continuation of $f^{-1} \circ \theta_{\mathcal{O}}$, where f^{-1} is a germ of the function inverse to f. It follows easily from the definitions and the condition $\mathcal{O}_2^f \preceq \mathcal{O}$ that ψ has no ramification. Therefore, since $\widetilde{\mathcal{O}}$ is simply connected, ψ is single-valued, and $\theta_{\mathcal{O}} = f \circ \psi$.

Finally, it follows from the equality $\theta_{\mathcal{O}} = f \circ \psi$ by Proposition 2.2 that

$$f\colon f^*\mathcal{O} \to \mathcal{O}, \qquad \psi\colon \widetilde{\mathcal{O}} \to f^*\mathcal{O}$$

are covering maps between orbifolds, implying that $\psi = \theta_{f^*\mathcal{O}}$, since $\widetilde{\mathcal{O}}$ is non-ramified and simply-connected. In particular, if $\mathcal{O} = \mathcal{O}_2^f$, then $f^*\mathcal{O}_2^f = \mathcal{O}_1^f$, so that $\psi = \theta_{\mathcal{O}_1^f}$. □

Corollary 2.5. *Let C and W be rational functions, and $\mathcal{O} = (\mathbb{CP}^1, \nu)$ an orbifold such that $\mathcal{O}_2^W \preceq \mathcal{O}$. Then any compositional left factor U of $C \circ W$ is a compositional left factor of $C \circ \theta_{\mathcal{O}}$. In particular, any compositional left factor of $C \circ W$ is a compositional left factor of $C \circ \theta_{\mathcal{O}_2^W}$.*

Proof. Indeed, the equalities $C \circ W = U \circ V$ and $\theta_{\mathcal{O}} = W \circ \psi$ imply the equality

$$C \circ \theta_{\mathcal{O}} = U \circ (V \circ \psi). \qquad \square$$

Corollary 2.6. *Let $f\colon R \to \mathbb{CP}^1$ be a holomorphic map between compact Riemann surfaces. Then $\chi(\mathcal{O}_2^f) > 0$ implies that $g(R) = 0$. On the other hand, $\chi(\mathcal{O}_2^f) = 0$ implies that $g(R) \leqslant 1$.*

Proof. If $\chi(\mathcal{O}_2^f) > 0$, then $\widetilde{\mathcal{O}_2^f} = \mathbb{CP}^1$. Thus, by Theorem 2.4, $\theta_{\mathcal{O}_1^f}\colon \mathbb{CP}^1 \to R$ is a holomorphic map, implying that $g(R) = 0$. Similarly, if $\chi(\mathcal{O}_2^f) = 0$, then $\theta_{\mathcal{O}_1^f}\colon \mathbb{C} \to R$ is a holomorphic map, implying that $g(R) \leqslant 1$, since otherwise lifting $\theta_{\mathcal{O}_1^f}$ to a map between universal coverings (in the usual sense) would result in a contradiction with the Liouville theorem. □

Corollary 2.7. *Let $f\colon R \to \mathbb{CP}^1$ be a holomorphic map between compact Riemann surfaces. Assume that $\mathcal{O}_2^f$ is defined by the conditions*

$$\nu_2^f(0) = n, \quad \nu_2^f(\infty) = n. \tag{15}$$

Then $g(R) = 0$, and $A = z^n \circ \mu$ for some Möbius transformation μ. On the other hand, if $\mathcal{O}_2^f$ is defined by the conditions

$$\nu_2^f(-1) = 2, \quad \nu_2^f(1) = 2, \quad \nu_2^f(\infty) = n, \tag{16}$$

then $g(R) = 0$, and either

$$f = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right) \circ \mu,$$

or $f = \pm T_n \circ \mu$ for some Möbius transformation μ.

Proof. Since by Theorem 2.4 the map f is a compositional left factor of $\theta_{\mathcal{O}_2^f}$, and the universal coverings for orbifolds given by (15) and (16) are rational functions

$$Z_n = z^n, \quad D_n = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right) \tag{17}$$

correspondingly, the statement follows from the well-known fact that any compositional left factor of Z_n has the form $Z_d \circ \mu$ for some Möbius transformation μ and $d|n$, while any compositional left factor of D_n has the form $\pm T_d \circ \mu$ or $D_d \circ \mu$ for some Möbius transformation μ and $d|n$ (see for example [15, Sections 4.1 and 4.2]). $\square$

2.2. Fiber products. Let $f\colon C_1 \to C$ and $g\colon C_2 \to C$ be holomorphic maps between compact Riemann surfaces. The collection

$$(C_1,\, f) \times_C (C_2,\, g) = \bigcup_{j=1}^{n(f,g)} \{R_j,\, p_j,\, q_j\},$$

where R_j are compact Riemann surfaces provided with holomorphic maps

$$p_j\colon R_j \to C_1, \quad q_j\colon R_j \to C_2, \quad 1 \leqslant j \leqslant n(f,\, g),$$

is called the *fiber product* of f and g if

$$f \circ p_j = g \circ q_j, \quad 1 \leqslant j \leqslant n(f,\, g),$$

and for any holomorphic maps $p\colon R \to C_1$, $q\colon R \to C_2$ between compact Riemann surfaces satisfying

$$f \circ p = g \circ q$$

there exist a uniquely defined index j and a holomorphic map $w\colon R \to R_j$ such that

$$p = p_j \circ w, \quad q = q_j \circ w.$$

The fiber product exists and is defined in a unique way up to natural isomorphisms.

In practical terms the fiber product is described by the following algebraic construction. Let us consider the algebraic curve

$$E = \{(x,\, y) \in C_1 \times C_2 \,|\, f(x) = g(y)\}. \tag{18}$$

Let us denote by V_j, $1 \leqslant j \leqslant n(f,\, g)$, irreducible components of E, by R_j, $1 \leqslant j \leqslant n(f,\, g)$, their desingularizations, and by

$$\pi_j\colon R_j \to V_j, \quad 1 \leqslant j \leqslant n(f,\, g),$$

the desingularization maps. Then the compositions

$$x \circ \pi_j\colon R_j \to C_1, \quad y \circ \pi_j\colon R_j \to C_2, \quad 1 \leqslant j \leqslant n(f,\, g),$$

extend to holomorphic maps

$$p_j\colon R_j \to C_1, \quad q_j\colon R_j \to C_2, \quad 1 \leqslant j \leqslant n(f,\, g),$$

and the collection $\bigcup_{j=1}^{n(f,g)}\{R_j,\, p_j,\, q_j\}$ is the fiber product of f and g.

Abusing notation we will call the Riemann surfaces R_j, $1 \leqslant j \leqslant n(f,\, g)$, irreducible components of the fiber product of f and g. The number of irreducible components $n(f,\, g)$ satisfies the inequality

$$n(f,\, g) \leqslant \mathrm{GCD}(\deg f,\, \deg g). \tag{19}$$

Indeed, the degree of every map

$$h_j = f \circ p_j = g \circ q_j, \quad 1 \leqslant j \leqslant n(f,\, g),$$

is divisible by LCM(deg f, deg g). On the other hand, calculating the degrees of projections of curve (18), we see that

$$\sum_j \deg p_j = \deg g, \qquad \sum_j \deg q_j = \deg f,$$

implying that

$$\sum_{j=1}^{n(f,g)} h_j = \deg f \deg g.$$

Therefore, (19) holds.

Theorem 2.8. *Let $f\colon C_1 \to C$, $g\colon C_2 \to C$, and $u\colon C_3 \to C_2$ be holomorphic maps between compact Riemann surfaces. Assume that*

$$(C_1,\, f) \times_C (C_2,\, g) = \bigcup_{j=1}^{n(f,g)} \{R_j,\, p_j,\, q_j\}$$

and

$$(R_j,\, q_j) \times_{C_2} (C_3,\, u) = \bigcup_{i=1}^{n(u,q_j)} \{R_{ij},\, p_{ij},\, q_{ij}\}, \qquad 1 \leqslant j \leqslant n(f,\, g).$$

Then

$$(C_1,\, f) \times_C (C_3,\, g \circ u) = \bigcup_{j=1}^{n(f,g)} \bigcup_{i=1}^{n(u,q_j)} \{R_{ij},\, p_j \circ p_{ij},\, q_{ij}\}.$$

Proof. It is clear that for j, $1 \leqslant j \leqslant n(f,\, g)$, and i, $1 \leqslant i \leqslant n(u,\, q_j)$, the diagram

$$\begin{tikzcd} R_{ij} \arrow[r, "p_{ij}"] \arrow[d, "q_{ij}"] & R_j \arrow[r, "p_j"] \arrow[d, "q_j"] & C_1 \arrow[d, "f"] \\ C_3 \arrow[r, "u"] & C_2 \arrow[r, "g"] & C \end{tikzcd}$$

commutes, so that

$$(g \circ u) \circ q_{ij} = f \circ (p_j \circ p_{ij}).$$

Assume now that p and q are holomorphic maps between compact Riemann surfaces such that

$$(g \circ u) \circ q = f \circ p.$$

By the universality property of the fiber product of g and f, this equality implies that

$$u \circ q = q_j \circ w, \qquad p = p_j \circ w$$

for some index j and holomorphic map w. In turn, by the universality property of the fiber product of u and q_j, the first from these equalities implies that

$$q = q_{ij} \circ \widetilde{w}, \qquad w = p_{ij} \circ \widetilde{w}$$

for some index i and holomorphic map $\widetilde{w}$. Thus,

$$p = p_j \circ p_{ij} \circ \widetilde{w}, \qquad q = q_{ij} \circ \widetilde{w}. \qquad \square$$

Corollary 2.9. *In the above notation, the fiber products* $(C_1, f) \times_C (C_2, g)$ *and* $(C_1, f) \times_C (C_3, g \circ u)$ *have the same number of irreducible components if and only if for every* j, $1 \leqslant j \leqslant n(f, g)$, *the fiber product* $(R_j, q_j) \times_{C_2} (C_3, u)$ *has a unique irreducible component.* □

Corollary 2.10. *Let* R *be a compact Riemann surface,* $U\colon R \to \mathbb{CP}^1$ *a holomorphic map, and* A *a rational function. Then there exists* $d_0 \geqslant 1$ *such that*

$$n(A^{\circ d}, U) = n(A^{\circ d_0}, U) \tag{20}$$

for all $d \geqslant d_0$.

Proof. Clearly, Theorem 2.8 implies that for every $d \geqslant 1$ the inequality

$$n(A^{\circ (d+1)}, U) \geqslant n(A^{\circ d}, U)$$

holds. On the other hand, by (19), for every $d \geqslant 1$ we have:

$$n(A^{\circ d}, U) \leqslant \mathrm{GCD}(\deg A^{\circ d}, \deg U) \leqslant \deg U.$$

Therefore, there exists $d_0 \geqslant 1$ such that (20) holds for all $d \geqslant d_0$. □

2.3. Functional equations and orbifolds. Orbifolds $\mathcal{O}_1^f$ and $\mathcal{O}_2^f$ defined above are useful for the study of the functional equation

$$f \circ p = g \circ q, \tag{21}$$

where

$$p\colon R \to C_1, \quad f\colon C_1 \to \mathbb{CP}^1, \quad q\colon R \to C_2, \quad g\colon C_2 \to \mathbb{CP}^1$$

are holomorphic maps between compact Riemann surfaces.

We say that holomorphic maps $p\colon R \to C_1$ and $q\colon R \to C_2$ *have no non-trivial common compositional right factor* if the equalities

$$p = \widetilde{p} \circ w, \quad q = \widetilde{q} \circ w,$$

where $w\colon R \to \widetilde{R}$, $\widetilde{p}\colon \widetilde{R} \to C_1$, $\widetilde{q}\colon \widetilde{R} \to C_2$ are holomorphic maps between compact Riemann surfaces, imply that $\deg w = 1$. If such p and q satisfy (21), then by the universality property of the fiber product

$$(C_1, f) \times_{\mathbb{CP}^1} (C_2, g) = \bigcup_{j=1}^{n(f,g)} \{R_j, p_j, q_j\}, \tag{22}$$

the equalities

$$p = p_j \circ w, \quad q = q_j \circ w$$

hold for some j, $1 \leqslant j \leqslant n(f, g)$, and an isomorphism $w\colon R_j \to R_j$.

A solution f, p, g, q of (21) is called *good* if the fiber product of f and g has a unique component, and p and q have no non-trivial common compositional right factor. Thus, good solutions correspond to fiber products (22) with $n(f, g) = 1$. In this notation, the following statement holds (see [12, Theorem 4.2]).

Theorem 2.11. *Let f, p, g, q be a good solution of* (21)*. Then the commutative diagram*

$$\begin{array}{ccc} \mathcal{O}_1^q & \xrightarrow{p} & \mathcal{O}_1^f \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle f} \\ \mathcal{O}_2^q & \xrightarrow{g} & \mathcal{O}_2^f \end{array}$$

consists of minimal holomorphic maps between orbifolds. □

Of course, vertical arrows in the above diagram are minimal holomorphic maps simply by definition. The meaning of the theorem is that the branching of f and q defines, to a certain extent, the branching of g and p and vice versa.

Below we will use the following criterion (see [12, Lemma 2.1]).

Lemma 2.12. *A solution f, p, g, q of* (21) *is good whenever any two of the following three conditions are satisfied*:

- *the fiber product of f and g has a unique component,*
- *p and q have no non-trivial common compositional right factor,*
- $\deg f = \deg q$, $\deg g = \deg p$. □

2.4. Generalized Lattès maps. Most of orbifolds considered in this paper are defined on $\mathbb{CP}^1$. For such orbifolds, we will omit the Riemann surface R in the definition of $\mathcal{O} = (R, \nu)$, meaning that $R = \mathbb{CP}^1$. Signatures of orbifolds on $\mathbb{CP}^1$ with non-negative Euler characteristics and corresponding $\Gamma_{\mathcal{O}}$ and $\theta_{\mathcal{O}}$ can be described explicitly as follows. If $\mathcal{O}$ is an orbifold distinct from the non-ramified sphere, then $\chi(\mathcal{O}) = 0$ if and only if the signature of $\mathcal{O}$ belongs to the list

$$\{2, 2, 2, 2\}, \quad \{3, 3, 3\}, \quad \{2, 4, 4\}, \quad \{2, 3, 6\}, \tag{23}$$

and $\chi(\mathcal{O}) > 0$ if and only if the signature of $\mathcal{O}$ belongs to the list

$$\{n, n\}, \quad n \geqslant 2, \quad \{2, 2, n\}, \quad n \geqslant 2, \quad \{2, 3, 3\}, \quad \{2, 3, 4\}, \quad \{2, 3, 5\}. \tag{24}$$

Groups $\Gamma_{\mathcal{O}} \subset \mathrm{Aut}(\mathbb{C})$ corresponding to orbifolds $\mathcal{O}$ with signatures (23) are generated by translations of $\mathbb{C}$ by elements of some lattice $L \subset \mathbb{C}$ of rank two and the rotation $z \to \varepsilon z$, where ε is an nth root of unity with n equal to 2, 3, 4, or 6, such that $\varepsilon L = L$ (see [11], or [5, Section IV.9.5]). Accordingly, the functions $\theta_{\mathcal{O}}$ may be written in terms of the corresponding Weierstrass functions as $\wp(z)$, $\wp'(z)$, $\wp^2(z)$, and $\wp'^2(z)$. Groups $\Gamma_{\mathcal{O}} \subset \mathrm{Aut}(\mathbb{CP}^1)$ corresponding to orbifolds $\mathcal{O}$ with signatures (24) are the well-known finite subgroups C_n, D_{2n}, A_4, S_4, A_5 of $\mathrm{Aut}(\mathbb{CP}^1)$, and the functions $\theta_{\mathcal{O}}$ are Galois coverings of $\mathbb{CP}^1$ by $\mathbb{CP}^1$ of degrees n, $2n$, 12, 24, 60, calculated for the first time by Klein in [8].

A *Lattès map* can be defined as a rational function A of degree at least two such that $A\colon \mathcal{O} \to \mathcal{O}$ is a covering self-map for some orbifold $\mathcal{O}$ (see [11]). Thus, A is a Lattès map if there exists an orbifold $\mathcal{O}$ such that for any $z \in \mathbb{CP}^1$ the equality

$$\nu(A(z)) = \nu(z) \deg_z A \tag{25}$$

holds. By formula (10), such $\mathcal{O}$ necessarily satisfies $\chi(\mathcal{O}) = 0$. Furthermore, for a given function A there might be at most one orbifold such that (25) holds (see [11] and [18, Theorem 6.1]).

Following [18], we say that a rational function A of degree at least two is a *generalized Lattès map* if there exists an orbifold $\mathcal{O}$, distinct from the non-ramified sphere, such that $A\colon \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic self-map between orbifolds; that is, for any $z \in \mathbb{CP}^1$, the equality

$$\nu(A(z)) = \nu(z)\,\mathrm{GCD}(\deg_z A,\, \nu(A(z))) \tag{26}$$

holds. By inequality (11), such $\mathcal{O}$ satisfies $\chi(\mathcal{O}) \geqslant 0$. Since (25) implies (26), any ordinary Lattès map is a generalized Lattès map. Notice that if $\mathcal{O}$ is the non-ramified sphere, then condition (26) trivially holds for any rational function A. We say that a rational function is *special* if it is either a Lattès map, or is conjugate to $z^{\pm n}$ or $\pm T_n$, where T_n is the Chebyshev polynomial.

In general, for a given function A there might be several orbifolds $\mathcal{O}$ satisfying (26), and even infinitely many such orbifolds. For example, $z^{\pm d}\colon \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map for any $\mathcal{O}$ defined by the conditions

$$\nu(0) = \nu(\infty) = n, \quad n \geqslant 2, \quad \mathrm{GCD}(d,\, n) = 1, \tag{27}$$

and $\pm T_d\colon \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map for any $\mathcal{O}$ defined by the conditions

$$\nu(-1) = \nu(1) = 2, \quad \nu(\infty) = n, \quad n \geqslant 1, \quad \mathrm{GCD}(d,\, n) = 1. \tag{28}$$

For odd d, additionally, $\pm T_d\colon \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map for $\mathcal{O}$ defined by

$$\nu(1) = 2, \quad \nu(\infty) = 2, \tag{29}$$

or

$$\nu(-1) = 2, \quad \nu(\infty) = 2. \tag{30}$$

Nevertheless, the following statement holds (see [18, Theorem 1.2]).

Theorem 2.13. *Let A be a rational function of degree at least two not conjugate to $z^{\pm d}$ or $\pm T_d$. Then there exists an orbifold $\mathcal{O}_0^A$ such that $A\colon \mathcal{O}_0^A \to \mathcal{O}_0^A$ is a minimal holomorphic map between orbifolds, and for any orbifold $\mathcal{O}$ such that $A\colon \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map between orbifolds, the relation $\mathcal{O} \preceq \mathcal{O}_0^A$ holds. Furthermore, $\mathcal{O}_0^{A^{\circ l}} = \mathcal{O}_0^A$ for any $l \geqslant 1$.* □

Clearly, generalized Lattès maps are exactly rational functions for which the orbifold $\mathcal{O}_0^A$ is distinct from the non-ramified sphere, completed by the functions $z^{\pm d}$ and $\pm T_d$ for which the orbifold $\mathcal{O}_0^A$ is not defined. Furthermore, ordinary Lattès maps are exactly rational functions for which $\chi(\mathcal{O}_0^A) = 0$ (see [18, Lemma 6.4]) and if A is a Lattès map, then the minimal holomorphic map $A\colon \mathcal{O}_0^A \to \mathcal{O}_0^A$ is a covering map by Proposition 2.1. Notice also that since a rational function A is conjugate to $z^{\pm d}$ or $\pm T_d$ if and only if some iterate $A^{\circ l}$ is conjugate to $z^{\pm ld}$ or $\pm T_{ld}$ (see for example [18, Lemma 6.3]), Theorem 2.13 implies that A is a generalized Lattès map if and only if some iterate $A^{\circ l}$ is a generalized Lattès map.

For exceptional functions $z^{\pm d}$ and $\pm T_d$, the orbifolds for which (26) holds are described as follows (see [18, Theorem 6.2]).

Theorem 2.14. *Let $\mathcal{O}$ be an orbifold distinct from the non-ramified sphere.*

(1) *The map $z^{\pm d}\colon \mathcal{O} \to \mathcal{O}$, $d \geqslant 2$, is a minimal holomorphic map between orbifolds if and only if $\mathcal{O}$ is defined by conditions (27).*

(2) *The map* $\pm T_d\colon \mathcal{O} \to \mathcal{O}$, $d \geqslant 2$, *is a minimal holomorphic map between orbifolds if and only if either* $\mathcal{O}$ *is defined by conditions* (28), *or* d *is odd and* $\mathcal{O}$ *is defined by conditions* (29) *or* (30). □

If A is a generalized Lattès map, then $c(\mathcal{O}_0^A)$ is a subset of the set $c(\mathcal{O}_2^A)$ consisting of critical values of A unless $\deg A \leqslant 4$. More generally, the following statement holds (see [18, Lemma 6.6]).

Lemma 2.15. *Let* A *be a rational function of degree at least five, and* $\mathcal{O}_1$, $\mathcal{O}_2$ *orbifolds distinct from the non-ramified sphere such that* $A\colon \mathcal{O}_1 \to \mathcal{O}_2$ *is a minimal holomorphic map between orbifolds. Assume that* $\chi(\mathcal{O}_1) \geqslant 0$. *Then* $c(\mathcal{O}_2) \subseteq c(\mathcal{O}_2^A)$.

3. Algebraic Curves $A^{\circ l}(x) - U(y) = 0$ with Components of Low Genus

In this section, we solve Problem 1.1. Our approach is based on the following theorem, which is a mild generalization of a result proved in [14].

Theorem 3.1. *Let* R *be a compact Riemann surface and* $W\colon R \to \mathbb{CP}^1$ *a holomorphic map of degree* n. *Then for any rational function* P *of degree* m *such that the fiber product of* P *and* W *consists of a unique component* E, *the inequality*

$$\chi(E) \leqslant \chi(R)(n-1) - \frac{m}{42} \tag{31}$$

holds, unless $\chi(\mathcal{O}_2^W) \geqslant 0$.

Proof. The proof of Theorem 3.1 in the case $R = \mathbb{CP}^1$ was given in [14, Section 3]. The proof in the general case is obtained in the same way with appropriate modifications. First of all, observe that if $q\colon E \to R$ is a holomorphic map of degree n between compact Riemann surfaces, then

$$\chi(\mathcal{O}_2^q) \geqslant \chi(E) + \chi(R)(1-n). \tag{32}$$

Indeed, it follows from (6) that

$$\chi(\mathcal{O}_2^q) \geqslant \chi(R) - c(q),$$

where $c(q)$ is the number of branch points of q. On the other hand, since the number $c(q)$ is less than or equal to the number of points $z \in E$ where $\deg_z q > 1$, the Riemann–Hurwitz formula

$$\chi(E) = \chi(R)n - \sum_{z \in E} (\deg_z q - 1)$$

implies that

$$c(q) \leqslant \chi(R)n - \chi(E).$$

Therefore, (32) holds.

Let $W \times_{\mathbb{CP}^1} P = \{E, p, q\}$. Since

$$P\colon \mathcal{O}_2^q \to \mathcal{O}_2^W$$

is a minimal holomorphic map between orbifolds by Theorem 2.11, it follows from Proposition 2.1 that

$$\chi(\mathcal{O}_2^q) \leqslant m\chi(\mathcal{O}_2^W). \tag{33}$$

On the other hand, (6) implies that if $\chi(\mathcal{O}) < 0$, then in fact

$$\chi(\mathcal{O}) \leqslant -\frac{1}{42}$$

(where the equality is attained for the collection of ramification indices (2, 3, 7)). Therefore, if $\chi(\mathcal{O}_2^W) < 0$, then it follows from (33) and (32) that

$$\chi(E) + \chi(R)(1-n) \leqslant -\frac{m}{42},$$

implying (31). □

Let us denote by $D = D\big[R_d,\, A,\, W_d,\, h_d\big]$ an infinite commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & R_3 \arrow[r, "h_3"] \arrow[d, "W_3"] & R_2 \arrow[r, "h_2"] \arrow[d, "W_2"] & R_1 \arrow[r, "h_1"] \arrow[d, "W_1"] & R_0 \arrow[d, "W_0"] \\
\cdots \arrow[r] & \mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1
\end{tikzcd}$$

consisting of holomorphic maps between compact Riemann surfaces. We say that D is *good* if for any $d_2 > d_1 \geqslant 0$, the maps

$$W_{d_1}, \quad h_{d_1+1} \circ h_{d_1+2} \circ \cdots \circ h_{d_2}, \quad A^{\circ(d_2-d_1)}, \quad W_{d_2}$$

form a good solution of equation (21). Notice that if D is good, then

$$\deg W_d = \deg W_0, \qquad d \geqslant 1, \tag{34}$$

by Lemma 2.12. We say that D is *preperiodic* if there exist $s_0 \geqslant 0$ and $l \geqslant 1$ such that for any $d \geqslant s_0$ the Riemann surfaces R_d and R_{d+l} are isomorphic and

$$W_d = W_{d+l} \circ \alpha_d$$

for some isomorphism

$$\alpha_d \colon R_d \to R_{d+l}.$$

Combined with the general properties of fiber products and generalized Lattès maps, Theorem 3.1 implies the following statement.

Theorem 3.2. *Let $D = D\big[R_d,\, A,\, W_d,\, h_d\big]$ be a diagram consisting of holomorphic maps of degree at least two. Assume that D is good and the sequence $g(R_d)$, $d \geqslant 0$, is bounded. Then $g(R_d) \leqslant 1$, $d \geqslant 0$, and, unless A is a Lattès map, $g(R_d) = 0$, $d \geqslant 0$. Furthermore, D is preperiodic and $A^{\circ l} \colon \mathcal{O}_2^{W_d} \to \mathcal{O}_2^{W_d}$ is a minimal holomorphic map between orbifolds for some $l \geqslant 1$ and all d big enough. In particular, A is a generalized Lattès map.*

Proof. Since the sequence $g(R_d)$, $d \geqslant 0$, is bounded from above, the sequence $\chi(R_d)$, $d \geqslant 0$, is bounded from below. Therefore, applying Theorem 3.1 for $W = W_d$, $d \geqslant 0$, and $P = A^{\circ j}$ with j big enough, we conclude that

$$\chi(\mathcal{O}_2^{W_d}) \geqslant 0, \qquad d \geqslant 0. \tag{35}$$

Hence, $g(R_d) \leqslant 1$, $d \geqslant 0$, by Corollary 2.6.

Let us show that the set of orbifolds $\mathcal{O}_2^{W_d}$, $d \geqslant 0$, contains only finitely many different orbifolds. Clearly, it is enough to show that the sequences $c(\mathcal{O}_2^{W_d})$, $d \geqslant 0$, and $\nu(\mathcal{O}_2^{W_d})$, $d \geqslant 0$, have only finitely many different elements. Since D is good, it follows from Theorem 2.11 that

$$A^{\circ(d_2-d_1)}\colon \mathcal{O}_2^{W_{d_2}} \to \mathcal{O}_2^{W_{d_1}} \tag{36}$$

is a minimal holomorphic map between orbifolds for any $d_2 > d_1 \geqslant 0$. In particular,

$$A\colon \mathcal{O}_2^{W_{d+1}} \to \mathcal{O}_2^{W_d}, \quad d \geqslant 0,$$

are minimal holomorphic maps. Therefore, if $\deg A > 4$, then by Lemma 2.15 every set $c(\mathcal{O}_2^{W_d})$, $d \geqslant 0$, is a subset of the set $c(\mathcal{O}_2^A)$, and hence, the sequence $c(\mathcal{O}_2^{W_d})$, $d \geqslant 0$, has only finitely many different elements. Moreover, this is true if $\deg A \leqslant 4$. Indeed, the inequality $\deg A \geqslant 2$ implies the inequality $\deg A^{\circ 3} > 4$, and hence, every set $c(\mathcal{O}_2^{W_d})$, $d \geqslant 0$, is a subset of the set $c(A^{\circ 3})$, since

$$A^{\circ 3}\colon \mathcal{O}_2^{W_{d+3}} \to \mathcal{O}_2^{W_d}, \quad d \geqslant 0,$$

also are minimal holomorphic maps. Finally, by (35), possible signatures of the orbifolds $\mathcal{O}_2^{W_d}$, $d \geqslant 0$, belong to lists (23) and (24), and it follows from equality (34) and Corollary 2.7 that if $\nu(\mathcal{O}_2^{W_d}) = \{n, n\}$, $n \geqslant 2$, or $\nu(\mathcal{O}_2^{W_d}) = \{2, 2, n\}$, $n \geqslant 2$, then either $n = \deg W_0$ or $n = \deg W_0/2$. Therefore, the sequence $\nu(\mathcal{O}_2^{W_d})$, $d \geqslant 0$, also has only finitely many different elements.

Since the set $\mathcal{O}_2^{W_d}$, $d \geqslant 0$, contains only finitely many different orbifolds, there exist an orbifold $\mathcal{O}$ with $\chi(\mathcal{O}) \geqslant 0$ and a monotonically increasing sequence $d_k \to \infty$ such that $\mathcal{O}_2^{W_{d_k}} = \mathcal{O}$, $k \geqslant 0$. Moreover, by Theorem 2.4, the equalities

$$\theta_{\mathcal{O}} = W_{d_k} \circ \theta_{\mathcal{O}_1^{W_{d_k}}}, \quad k \geqslant 0, \tag{37}$$

hold. By the classification given in Section 2.4, the group $\Gamma_{\mathcal{O}}$ is finitely generated, and therefore, it has at most finitely many subgroups of any given index. Since (37) and (34) imply that every group $\Gamma_{\mathcal{O}_1^{W_{d_k}}}$, $k \geqslant 0$, has index $\deg W_0$ in $\Gamma_{\mathcal{O}}$, we conclude that the set of groups $\Gamma_{\mathcal{O}_1^{W_{d_k}}}$, $k \geqslant 0$, contains only finitely many different groups. Therefore,

$$\Gamma_{\mathcal{O}_1^{W_{d_{k_i}}}} = \Gamma_{\mathcal{O}_1^{W_{d_{k_j}}}}$$

for some $k_i > k_j$ and hence

$$\theta_{\mathcal{O}_1^{W_{d_{k_i}}}}(x) = \alpha \circ \theta_{\mathcal{O}_1^{W_{d_{k_j}}}}(y)$$

for some isomorphism $\alpha\colon R_{d_{k_j}} \to R_{d_{k_i}}$, implying by (37) the equality

$$W_{d_{k_j}} = W_{d_{k_i}} \circ \alpha. \tag{38}$$

Since R_d, W_d, h_d, $d \geqslant 1$, are defined by R_{d-1} and W_{d-1} in a unique way up to natural isomorphisms, (38) implies that the preperiodicity condition holds for $l = d_{k_i} - d_{k_j}$ and $s_0 = d_{k_j}$.

Finally, setting $d_2 = d + l$ and $d_1 = d$ in (36), we see that if $d \geqslant s_0$, then

$$A^{\circ l}\colon \mathcal{O}_2^{W_d} \to \mathcal{O}_2^{W_d}$$

is a minimal holomorphic map. Therefore, since the inequality $\deg W_d \geqslant 2$ implies that $\mathcal{O}_2^{W_d}$ cannot be the non-ramified sphere, $A^{\circ l}$ is a generalized Lattès map, and hence, A is also a generalized Lattès map. Moreover, unless A is a Lattès map,

$\chi(\mathcal{O}_2^{W_d}) > 0$, $d \geqslant s_0$. Therefore, $g(R_d) = 0$, $d \geqslant s_0$, by Corollary 2.6, implying that $g(R_d) = 0$ for all $d \geqslant 0$, since $g(R_{d+1}) \geqslant g(R_d)$, $d \geqslant 0$. □

Four theorems below provide a solution of Problem 1.1. The first theorem imposes no restrictions on the function A and relates Problem 1.1 with semiconjugacies. The other three provide more precise information for different classes of A. In particular, Theorem 3.4 implies Theorem 1.2 stated in the introduction. In fact, we examine a more general version of Problem 1.1, in which U is allowed to be a holomorphic map $U\colon R \to \mathbb{CP}^1$, where R is a compact Riemann surface, and instead of curves (1) the fiber products of U and $A^{\circ d}$, $d \geqslant 1$, are considered.

Let us denote by $g_d = g_d(A, U)$, $d \geqslant 1$, the minimal number g such that the fiber product of U and $A^{\circ d}$ has a component of genus g.

Theorem 3.3. *Let R be a compact Riemann surface, $U\colon R \to \mathbb{CP}^1$ a holomorphic map of degree at least two, and A a rational function of degree at least two. Then the sequence g_d, $d \geqslant 1$, is bounded if and only if there exist a compact Riemann surface S of genus 0 or 1 and holomorphic maps $F\colon S \to S$ and $W\colon S \to \mathbb{CP}^1$ such that the diagram*

$$\begin{CD} S @>F>> S \\ @VWVV @VVWV \\ \mathbb{CP}^1 @>A^{\circ l_1}>> \mathbb{CP}^1 \end{CD} \tag{39}$$

commutes for some $l_1 \geqslant 1$, the fiber product of W and $A^{\circ l_1}$ consists of a unique component, $A^{\circ l_1}\colon \mathcal{O}_2^W \to \mathcal{O}_2^W$ is a minimal holomorphic map between orbifolds, and U is a compositional left factor of $A^{\circ l_2} \circ W$ for some $l_2 \geqslant 0$. In particular, if A is not a generalized Lattès map, then g_d, $d \geqslant 1$, is bounded if and only if U is a compositional left factor of $A^{\circ l}$ for some $l \geqslant 1$.

Proof. To prove the sufficiency, observe that (39) and

$$A^{\circ l_2} \circ W = U \circ V \tag{40}$$

imply that

$$A^{\circ(l_2+l_1k)} \circ W = U \circ V \circ F^{\circ k}, \qquad k \geqslant 0.$$

Therefore, for every $d \geqslant 1$, there exist holomorphic maps of the form

$$\varphi_d = A^{\circ s_d} \circ W, \qquad \psi_d = V \circ F^{\circ r_d},$$

where $s_d \geqslant 0$, $r_d \geqslant 0$, satisfying

$$A^{\circ d} \circ \varphi_d = U \circ \psi_d.$$

By the universality property of the fiber product, this implies that for every $d \geqslant 1$, there exist a component $\{E, p, q\}$ of $A^{\circ d} \times U$ and a holomorphic map $w\colon S \to E$ such that

$$\varphi_d = p \circ w, \qquad \psi_d = q \circ w.$$

Clearly, for such E we have:

$$g(E) \leqslant g(S) \leqslant 1.$$

Let us now prove the necessity. Let d_0 be the number such that (20) holds for all $d \geqslant d_0$, and let

$$(\mathbb{CP}^1,\, A^{\circ(d_0+k)}) \times_{\mathbb{CP}^1} (R,\, U) = \bigcup_{j=1}^{s} \{R_{j,k},\, W_{j,k},\, H_{j,k}\}, \qquad k \geqslant 0, \tag{41}$$

where $s = n(A^{\circ d_0},\, U)$. It follows from the universality property of the fiber product and equality (20) that for every $k \geqslant 0$ and j, $1 \leqslant j \leqslant s$, there exists a uniquely defined j' such that

$$H_{j',k+1} = H_{j,k} \circ h$$

for some holomorphic map $h\colon R_{j',k+1} \to R_{j,k}$, and without loss of generality we may assume that the numeration in (41) is chosen in such a way that $j = j'$. Thus, we can assume that for every j, $1 \leqslant j \leqslant s$, there exist holomorphic maps $h_{j,k}$, $k \geqslant 1$, such that

$$H_{j,k} = H_{j,0} \circ h_{j,1} \circ h_{j,2} \circ \cdots \circ h_{j,k}$$

and the diagram

$$\begin{tikzcd}
\cdots & R_{j,3} \arrow[r, "h_{j,3}"] \arrow[d, "W_{j,3}"] & R_{j,2} \arrow[r, "h_{j,2}"] \arrow[d, "W_{j,2}"] & R_{j,1} \arrow[r, "h_{j,1}"] \arrow[d, "W_{j,1}"] & R_{j,0} \arrow[d, "W_{j,0}"] \\
& \mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1
\end{tikzcd} \tag{42}$$

commutes. Moreover, this diagram is good by Corollary 2.9. Finally, since

$$g(R_{j,k+1}) \geqslant g(R_{j,k}), \qquad k \geqslant 0,$$

it follows from the boundness of the sequence g_d, $d \geqslant 1$, that for at least one j, $1 \leqslant j \leqslant s$, the sequence $g(R_{j,k})$, $k \geqslant 0$, is bounded. In particular, for such j we can apply Theorem 3.2 to diagram (42), unless $\deg W_{j,0} = 1$.

By construction, for each j, $1 \leqslant j \leqslant s$, the diagram

$$\begin{tikzcd}
\cdots & R_{j,3} \arrow[r, "h_{j,3}"] \arrow[d, "W_{j,3}"] & R_{j,2} \arrow[r, "h_{j,2}"] \arrow[d, "W_{j,2}"] & R_{j,1} \arrow[r, "h_{j,1}"] \arrow[d, "W_{j,1}"] & R_{j,0} \arrow[r, "H_{j,0}"] \arrow[d, "W_{j,0}"] & R \arrow[d, "U"] \\
& \mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1 \arrow[r, "A^{\circ d_0}"] & \mathbb{CP}^1
\end{tikzcd}$$

commutes. Fix now j such that the sequence $g(R_{j,k})$, $k \geqslant 0$, is bounded. If $\deg W_{j,0} = 1$, then $R = \mathbb{CP}^1$ and the equality

$$A^{\circ d_0} = U \circ H_{j,0} \circ W_{j,0}^{-1}$$

implies that U is a compositional left factor of $A^{\circ d_0}$. Therefore, in this case the theorem is true for

$$S = \mathbb{CP}^1, \qquad W = z, \qquad F = A, \qquad l_1 = 1, \qquad l_2 = d_0.$$

On the other hand, if $\deg W_{j,0} \geqslant 2$, then by Theorem 3.2 there exist $l \geqslant 1$, $s_0 \geqslant 0$, and an isomorphism

$$\alpha\colon R_{j,s_0} \to R_{j,s_0+l}$$

such that

$$W_{j,s_0} = W_{j,s_0+l} \circ \alpha,$$

and

$$A^{\circ l} \colon \mathcal{O}_2^{W_{j,s_0}} \to \mathcal{O}_2^{W_{j,s_0}}$$

is a minimal holomorphic map. Thus, (39) holds for

$$S = R_{j,s_0}, \quad W = W_{j,s_0}, \quad F = h_{j,s_0+1} \circ h_{j,s_0+2} \circ \cdots \circ h_{j,s_0+l} \circ \alpha,$$

and A is a generalized Lattès map. Finally, U is a compositional left factor of $A^{\circ l_2} \circ W$ for $l_2 = d_0 + s_0$, since

$$A^{\circ(d_0+s_0)} \circ W_{j,s_0} = U \circ H_{j,0} \circ h_{j,1} \circ h_{j,2} \circ \cdots \circ h_{j,s_0}. \qquad \square$$

Notice that in the proof of the sufficiency we did not use the assumptions that the fiber product of W and $A^{\circ l_1}$ has one component and $A^{\circ l_1} \colon \mathcal{O}_2^W \to \mathcal{O}_2^W$ is a minimal holomorphic map between orbifolds. Thus, the theorem implies that if U satisfy (39) and (40) for some W, F, and V, then it satisfies (39) and (40) for W, F, and V that obey these conditions (cf. [18, Section 3]).

Theorem 3.4. *Let R be a compact Riemann surface, $U \colon R \to \mathbb{CP}^1$ a holomorphic map of degree at least two, and A a non-special rational function of degree at least two. Then the sequence g_d, $d \geqslant 1$, is bounded if and only if $R = \mathbb{CP}^1$ and U is a compositional left factor of $A^{\circ l} \circ \theta_{\mathcal{O}_0^A}$ for some $l \geqslant 1$. In particular, if A is not a generalized Lattès map, then g_d, $d \geqslant 1$, is bounded if and only if U is a compositional left factor of $A^{\circ l}$ for some $l \geqslant 1$.*

Proof. If the sequence g_d, $d \geqslant 1$, is bounded, then by Theorem 3.3 there exist a compact Riemann surface S of genus 0 or 1 and a holomorphic map $W \colon S \to \mathbb{CP}^1$ such that U is a compositional left factor of $A^{\circ l_2} \circ W$ for some l_2, and

$$A^{\circ l_1} \colon \mathcal{O}_2^W \to \mathcal{O}_2^W$$

is a minimal holomorphic map for some l_1. On the other hand, since A is not conjugate to $z^{\pm n}$ or $\pm T_n$, the orbifold $\mathcal{O}_0^A$ is well-defined and

$$\mathcal{O}_2^W \preceq \mathcal{O}_0^{A^{\circ l_1}} = \mathcal{O}_0^A,$$

by Theorem 2.13. Thus, U is a compositional left factor of the holomorphic map $A^{\circ l_2} \circ \theta_{\mathcal{O}_0^A}$ by Corollary 2.5. Moreover, since A is not a Lattès map, $\chi(\mathcal{O}_A^0) > 0$ and $\theta_{\mathcal{O}_0^A}$ and U are rational functions.

In the other direction, since $\chi(\mathcal{O}_0^A) > 0$, Proposition 2.3 implies that there exists a rational function F such that the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{F} & \mathbb{CP}^1 \\ \downarrow{\scriptstyle \theta_{\mathcal{O}_0^A}} & & \downarrow{\scriptstyle \theta_{\mathcal{O}_0^A}} \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array}$$

commutes. Arguing now as in Theorem 3.3, we conclude that if U is a compositional left factor of $A^{\circ l} \circ \theta_{\mathcal{O}_0^A}$, then the sequence g_d, $d \geqslant 1$, is bounded. $\square$

Theorem 3.5. *Let R be a compact Riemann surface, $U\colon R \to \mathbb{CP}^1$ a holomorphic map of degree at least two, and A a Lattès map. Then the sequence g_d, $d \geqslant 1$, is bounded if and only if U is a compositional left factor of $\theta_{\mathcal{O}_0^A}$.*

Proof. Arguing as in Theorem 3.4, we conclude that if the sequence g_d, $d \geqslant 1$, is bounded, then U is a compositional left factor of $A^{\circ l} \circ \theta_{\mathcal{O}_0^A}$ for some $l \geqslant 1$. Thus, to prove the necessity, we must only show that if A is a Lattès map, then any compositional left factor of $A^{\circ l} \circ \theta_{\mathcal{O}_0^A}$, $l \geqslant 1$, is a compositional left factor of $\theta_{\mathcal{O}_0^A}$. Recall that for a Lattès map A the equality $\chi(\mathcal{O}_0^A) = 0$ holds and $A\colon \mathcal{O}_0^A \to \mathcal{O}_0^A$ is a covering map between orbifolds (see the remarks after Theorem 2.13). Therefore, by Proposition 2.3, the function F in diagram (13) is an isomorphism, implying that (13) takes the form

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F=az+b} & \mathbb{C} \\ \Big\downarrow{\scriptstyle \theta_{\mathcal{O}_0^A}} & & \Big\downarrow{\scriptstyle \theta_{\mathcal{O}_0^A}} \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1, \end{array} \tag{43}$$

where $a,\, b \in \mathbb{C}$, $a \neq 0$. Thus, for every $d \geqslant 1$ the equality

$$\theta_{\mathcal{O}_0^A} = A^{\circ d} \circ \theta_{\mathcal{O}_0^A} \circ (F^{-1})^{\circ d}$$

holds, implying the necessary statement.

Assume now that $\theta_{\mathcal{O}_0^A} = U \circ \psi$, where $\psi\colon \mathbb{C} \to R$ and $U\colon R \to \mathbb{CP}^1$ are holomorphic maps between Riemann surfaces. Since diagram (43) commutes, for every $d \geqslant 1$ the equality

$$A^{\circ d} \circ \theta_{\mathcal{O}_0^A} = U \circ (\psi \circ F^{\circ d})$$

holds, implying that the map $\psi_d\colon \mathbb{C} \to \mathbb{CP}^1 \times R$ given by

$$\psi_d\colon z \to (\theta_{\mathcal{O}_0^A},\, \psi \circ F^{\circ d})$$

is a meromorphic parametrization of an irreducible component of the algebraic curve

$$E = \{(x,\, y) \in \mathbb{CP}^1 \times R \,|\, A^{\circ d}(x) = U(y)\}.$$

Since an algebraic curve possessing a parametrization by meromorphic functions on $\mathbb{C}$ has genus at most one, this implies that the sequence g_d, $d \geqslant 1$, is bounded. ☐

Theorem 3.6. *Let R be a compact Riemann surface, $U\colon R \to \mathbb{CP}^1$ a holomorphic map of degree at least two, and A a rational function of degree at least two.*

(1) *If $A = z^m$, then the sequence g_d, $d \geqslant 1$, is bounded if and only if $R = \mathbb{CP}^1$ and $U = z^s \circ \mu$, $s \geqslant 2$, where μ is a Möbius transformation,*

(2) *If $A = T_m$, then the sequence g_d, $d \geqslant 1$, is bounded if and only if $R = \mathbb{CP}^1$ and either $U = \pm T_s \circ \mu$, $s \geqslant 2$, or*

$$U = \frac{1}{2}\left(z^s + \frac{1}{z^s}\right) \circ \mu, \qquad s \geqslant 2,$$

where μ is a Möbius transformation.

Proof. Let us consider the case $A = T_m$. In the case $A = z^m$ the proof is similar. For brevity, we will use the notation D_n introduced in (17). Let us prove the necessity. Applying Theorem 3.3 and keeping its notation, we observe first that if $\deg W = 1$, then U is a compositional left factor of $T_{m^{l_2}}$. Therefore, since any compositional factor of T_l has the form $T_d \circ \mu$ for some $d \,|\, l$ and Möbius transformation μ, in this case the statement is true.

Let us assume now that $\deg W > 1$. Since

$$T_{m^{l_1}} : \mathcal{O}_2^W \to \mathcal{O}_2^W$$

is a minimal holomorphic map between orbifolds, it follows from Theorem 2.14 that $\mathcal{O}_2^W$ is defined by one of conditions (28), (29), (30). Further, any compositional left factor of the map $A^{\circ l_2} \circ W$ is a compositional left factor of the map $A^{\circ l_2} \circ \theta_{\mathcal{O}_2^W}$, by Corollary 2.5. Since the universal coverings of the orbifolds given by (28), (29), (30) are the function D_n and the functions $-T_2$, T_2, correspondingly, this implies that U is a compositional left factor either of the function

$$T_{m^{l_2}} \circ D_n = D_{nm^{l_2}}$$

or of the function

$$T_{m^{l_2}} \circ \pm T_2 = \pm T_{2m^{l_2}}.$$

Since any compositional left factor of D_l has the form $\pm T_s \circ \mu$ or $D_s \circ \mu$ for some Möbius transformation μ and $s|l$, this proves the necessity.

Finally, since

$$T_m \circ D_s = D_s \circ z^m$$

for any $s \geqslant 2$, and $\pm T_s \circ \mu$ and $D_s \circ \mu$ are compositional left factors of D_s, the sufficiency can be proved as in Theorem 3.3. $\square$

4. Arithmetic of Orbits of Rational Functions

4.1. Normalizations and definition fields. Recall that for a non-constant holomorphic map between compact Riemann surfaces $X : C \to \mathbb{CP}^1$, its normalization N_X is defined as a holomorphic map of the lowest possible degree between compact Riemann surfaces $N_X : S_X \to \mathbb{CP}^1$, such that N_X is a Galois covering and

$$N_X = X \circ H \tag{44}$$

for some holomorphic map $H : S_X \to C$. From the algebraic point of view the passage from X to N_X corresponds to the passage from the field extension $\mathcal{M}(C)/X^*\mathbb{C}(z)$ to its Galois closure. The corresponding Galois group G_X may be identified with the monodromy group of the covering $X : C \to \mathbb{CP}^1$, or, in terms of the normalization, with the group $\mathrm{Aut}(S_X)$. The surface S_X is defined up to isomorphism. For fixed S_X, the function N_X is defined in a unique way while the function H in (44) is defined up to the change $H \to H \circ \mu$, where $\mu \in \mathrm{Aut}(S_X)$.

Theorem 4.1. *Let C be a compact Riemann surface, and let $A\colon \mathbb{CP}^1 \to \mathbb{CP}^1$, $B\colon C \to C$, $X\colon C \to \mathbb{CP}^1$ be non-constant holomorphic maps such that the diagram*

$$\begin{tikzcd} C \arrow[r, "B"] \arrow[d, "X"] & C \arrow[d, "X"] \\ \mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1 \end{tikzcd} \tag{45}$$

commutes and the fiber product of A and X consists of a unique component. Then there exist holomorphic maps $F\colon S_X \to S_X$, $H\colon S_X \to C$, and a group automorphism $\varphi\colon \mathrm{Aut}(S_X) \to \mathrm{Aut}(S_X)$ such that the equality $N_X = X \circ H$ holds, the diagram

$$\begin{tikzcd} S_X \arrow[r, "F"] \arrow[d, "H"] & S_X \arrow[d, "H"] \\ C \arrow[r, "B"] \arrow[d, "X"] & C \arrow[d, "X"] \\ \mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1, \end{tikzcd} \tag{46}$$

commutes, and for any $\sigma \in \mathrm{Aut}(S_X)$ the equality

$$F \circ \sigma = \varphi(\sigma) \circ F$$

holds.

Proof. The proof is based on the following geometric description of G_X and S_X (see, for example [6, §I.G] or [7, Section 2.2]). Let L be the n-fold fiber product of $X\colon C \to \mathbb{CP}^1$ with itself, that is, the algebraic curve in C^n defined by the equation

$$X(z_1) = X(z_2) = \cdots = X(z_n),$$

where $n = \deg X$. Let us denote by Δ the big diagonal of C^n, which consists of points where at least two coordinates coincide, and by L_0 the Zariski closure of $L \setminus \Delta$ in L. Then all irreducible components $V_1, V_2, \ldots, V_r$ of L_0 are isomorphic, and the group G_X can be identified with the subgroup of S_n consisting of all permutations $\sigma \in S_n$, such that

$$(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) \in V_j$$

for some j, $1 \leqslant j \leqslant r$, if and only if

$$(x_1, x_2, \ldots, x_n) \in V_j.$$

Furthermore, if V is any irreducible component of L_0, and $\widetilde{V} \xrightarrow{\eta} V$ is the desingularization map, then $\widetilde{V} = S_X$ and the map N_X is given by the composition

$$\widetilde{V} \xrightarrow{\eta} V \xrightarrow{\pi_i} C \xrightarrow{X} \mathbb{CP}^1,$$

where π_i is the projection to any coordinate.

Define the maps $\mathcal{B}\colon C^n \to C^n$, $\mathcal{A}\colon (\mathbb{CP}^1)^n \to (\mathbb{CP}^1)^n$, and $\mathcal{X}\colon C^n \to (\mathbb{CP}^1)^n$ by the formulas

$$\begin{aligned}
\mathcal{A}&\colon (z_1, z_2, \dots, z_n) \to (A(z_1), A(z_2), \dots, A(z_n)),\\
\mathcal{B}&\colon (z_1, z_2, \dots, z_n) \to (B(z_1), B(z_2), \dots, B(z_n)),\\
\mathcal{X}&\colon (z_1, z_2, \dots, z_n) \to (X(z_1), X(z_2), \dots, X(z_n)).
\end{aligned}$$

Clearly, the diagram

$$\begin{tikzcd}
C^n \arrow[r, "\mathcal{B}"] \arrow[d, "\mathcal{X}"] & C^n \arrow[d, "\mathcal{X}"] \\
(\mathbb{CP}^1)^n \arrow[r, "\mathcal{A}"] & (\mathbb{CP}^1)^n
\end{tikzcd} \tag{47}$$

commutes, and by construction, $L = \mathcal{X}^{-1}(\Delta_0)$, where Δ_0 is the usual diagonal in $(\mathbb{CP}^1)^n$ which consists of points where all coordinates coincide. Therefore, since $\mathcal{A}(\Delta_0) = \Delta_0$, it follows from (47) that $\mathcal{B}(L) \subseteq L$.

Let us show that

$$\mathcal{B}(L_0) \subseteq L_0. \tag{48}$$

Since the fiber product of A and X consists of a unique component, it follows from Lemma 2.12 that the maps X and B have no non-trivial common compositional right factor, implying that

$$[X^*\mathbb{C}(z),\, B^*\mathcal{M}(C)] = \mathcal{M}(C).$$

By the primitive element theorem, $B^*\mathcal{M}(C) = \mathbb{C}[h]$ for some $h \in B^*\mathcal{M}(C)$, so that

$$\mathcal{M}(C) = X^*\mathbb{C}(z)[h].$$

Clearly, this equality implies that for all but finitely many points $z_0 \in \mathbb{CP}^1$, the function h takes n distinct values on the set $X^{-1}\{z_0\}$. Since $h \in B^*\mathcal{M}(C)$, this implies, in turn, that for all but finitely many points $z_0 \in \mathbb{CP}^1$ the map B takes n distinct values on the set $X^{-1}\{z_0\}$, or equivalently that (48) holds.

Let V be an irreducible component of L_0. Then (48) implies that $\mathcal{B}(V) \subseteq V'$, where V' is another irreducible component of L_0. Clearly, we can complete the diagram

$$\begin{tikzcd}
V \arrow[r, "\mathcal{B}"] \arrow[d, "\pi_i"] & V' \arrow[d, "\pi_i"] \\
C \arrow[r, "B"] \arrow[d, "X"] & C \arrow[d, "X"] \\
\mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1,
\end{tikzcd}$$

where π_i is the projection to any coordinate, to the diagram

$$\begin{tikzcd} S_X \arrow[r, "F_0"] \arrow[d, "\eta"] & S_X \arrow[d, "\eta'"] \\ V \arrow[r, "\mathcal{B}"] \arrow[d, "\pi_i"] & V' \arrow[d, "\pi_i"] \\ C \arrow[r, "B"] \arrow[d, "X"] & C \arrow[d, "X"] \\ \mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1, \end{tikzcd}$$

where $F_0\colon S_X \to S_X$ is a holomorphic map and

$$N_X = X \circ \pi_i \circ \eta' = X \circ \pi_i \circ \eta. \tag{49}$$

Since (49) implies that

$$\pi_i \circ \eta' = \pi_i \circ \eta \circ \mu$$

for some $\mu \in \mathrm{Aut}(S_X)$, we conclude that diagram (46) commutes for $F = \mu \circ F_0$ and $H = \pi_i \circ \eta$.

Since $X \circ H$ is a Galois covering, it is easy to see that for any holomorphic map $F'\colon S_X \to S_X$ that, along with F, satisfies (46), there exists $g \in \mathrm{Aut}(S_X)$ such that $F' = g \circ F$. In particular, for any $\sigma \in \mathrm{Aut}(S_X)$ the equality

$$F \circ \sigma = g_\sigma \circ F$$

holds for some $g_\sigma \in \mathrm{Aut}(S_X)$, and it is easy to see that the map

$$\varphi\colon \sigma \to g_\sigma$$

is a group homomorphism. Finally, since $\mathrm{Aut}(S_X) = G_X$, if $\mathrm{Ker}\,\varphi \neq e$, then there exists a non-identical permutation $\sigma \in G_X$ such that

$$\mathcal{B}(z_{\sigma(1)}, z_{\sigma(2)}, \dots, z_{\sigma(n)}) = \mathcal{B}(z_1, z_2, \dots, z_n)$$

for all $(z_1, z_2, \dots, z_n) \in V$. Since this contradicts the fact that for all but finitely many points $z_0 \in \mathbb{CP}^1$, the map B takes n distinct values on the set $X^{-1}\{z_0\}$, we conclude that φ is an automorphism. □

Notice that since the only compact Riemann surfaces admitting self maps of degree $d > 1$ are the Riemann sphere and tori, Theorem 4.2 implies that if A, B and X are rational functions of degree at least two such that diagram (45) commutes and the fiber product of A and X consists of a unique component, then $g(S_X) \leqslant 1$ (cf. [16]).

Corollary 4.2. *Let C be a compact Riemann surface and let $A\colon \mathbb{CP}^1 \to \mathbb{CP}^1$, $B_1\colon C \to C$, $B_2\colon C \to C$, $X\colon C \to \mathbb{CP}^1$ be non-constant holomorphic maps such that the diagrams*

$$\begin{tikzcd} C \arrow[r, "B_1"] \arrow[d, "X"] & C \arrow[d, "X"] \\ \mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1, \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "B_2"] \arrow[d, "X"] & C \arrow[d, "X"] \\ \mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1 \end{tikzcd}$$

commute and the fiber product of A and X consists of a unique component. Then for the integer

$$r = |G_X||\mathrm{Aut}(G_X)|$$

the equality $B_1^{\circ r} = B_2^{\circ r}$ holds.

Proof. Applying Theorem 4.1, we can find F_1, F_2, and H such that the diagrams

$$\begin{tikzcd} S_X \arrow[r, "F_1"] \arrow[d, "H"] & S_X \arrow[d, "H"] \\ C \arrow[r, "B_1"] \arrow[d, "X"] & C \arrow[d, "X"] \\ \mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1, \end{tikzcd} \qquad \begin{tikzcd} S_X \arrow[r, "F_2"] \arrow[d, "H"] & S_X \arrow[d, "H"] \\ C \arrow[r, "B_2"] \arrow[d, "X"] & C \arrow[d, "X"] \\ \mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1 \end{tikzcd}$$

commute. Furthermore, considering instead of the functions A, B_1, B_2, F_1, F_2 their $|\mathrm{Aut}(G_X)|$-th iterates, we may assume that the corresponding automorphisms φ_i, $i = 1, 2$, of S_X are the identical automorphisms, that is, that F_i, $i = 1, 2$, commute with G_X. Under this assumption, we must show that

$$B_1^{\circ|G_X|} = B_2^{\circ|G_X|}. \tag{50}$$

Since

$$F_2 = g \circ F_1$$

for some $g \in G_X$, we have:

$$F_2^{\circ|G_X|} = (g \circ F_1)^{\circ|G_X|} = g^{\circ|G_X|} \circ F_1^{\circ|G_X|} = F_1^{\circ|G_X|},$$

implying (50).

Corollary 4.3. *Let E be an algebraic curve over $\mathbb{C}$, and $X\colon E \to \mathbb{CP}^1$, $B\colon E \to E$, $A\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ dominant morphisms such that the diagram*

$$\begin{tikzcd} E \arrow[r, "B"] \arrow[d, "X"] & E \arrow[d, "X"] \\ \mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1 \end{tikzcd} \tag{51}$$

commutes and the fiber product of A and X consists of a unique component. Assume that the curve E and the morphisms X, A are defined over some number field K.

Then for the integer

$$r = |G_X||\mathrm{Aut}(G_X)|$$

the iterate $B^{\circ r}$ is defined over K.

Proof. It is clear that B is defined over $\bar{\mathbb{Q}}$ and that for any $\gamma \in \mathrm{Gal}\,(\bar{\mathbb{Q}}/K)$ the function ${}^{\gamma}B$ satisfies (51) along with B. Thus,

$$B^{\circ r} = ({}^{\gamma}B)^{\circ r},$$

by Corollary 4.2. Since

$${}^{\gamma}(B^{\circ r}) = ({}^{\gamma}B)^{\circ r},$$

this implies that

$${}^{\gamma}(B^{\circ r}) = B^{\circ r}$$

for any $\gamma \in \mathrm{Gal}\,(\bar{\mathbb{Q}}/K)$, and hence, $B^{\circ r}$ is defined over K. □

4.2. Proof of Theorem 1.3. Without loss of generality, we may assume that x_0 is not A-preperiodic, since otherwise the theorem is obviously true. As in Section 3, we will consider the fiber products of $A^{\circ d}$, $d \geqslant 1$, and U, but now regarding them as algebraic curves

$$\mathcal{E}_d \colon A^{\circ d}(x) - U(y) = 0$$

in $(\mathbb{CP}^1)^2$. We observe that $\mathcal{E}_d$, $d \geqslant 2$, is the preimage of $\mathcal{E}_{d-1}$ under the map $f \colon (\mathbb{CP}^1)^2 \to (\mathbb{CP}^1)^2$ defined by

$$f \colon (x,\, y) \to (A(x),\, y).$$

Let us denote by π_x and π_y the projection maps to x and y in $(\mathbb{CP}^1)^2$.

Let E_d, $d \geqslant 1$, be a sequence of irreducible components of $\mathcal{E}_d$, $d \geqslant 1$, such that the diagram

$$\begin{tikzcd}
\dots & E_3 \arrow[r, "f_3"] \arrow[d, "\pi_{x,3}"] & E_2 \arrow[r, "f_2"] \arrow[d, "\pi_{x,2}"] & E_1 \arrow[r, "\pi_{y,1}"] \arrow[d, "\pi_{x,1}"] & \mathbb{CP}^1 \arrow[d, "U"] \\
 & \mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1 \arrow[r, "A"] & C\mathbb{P}^1,
\end{tikzcd} \tag{52}$$

where f_d, $\pi_{x,d}$, $\pi_{y,d}$, $d \geqslant 1$, denotes the restriction of f, π_x, π_y on the curve E_d, commutes. It is clear that to prove Theorem 1.3, it is enough to show that if the set I consisting of $i \in \mathbb{N}$ such that the curve E_i contains a K-point of the form $(x_0,\, y)$ is infinite, then I is a finite union of arithmetic progressions.

For any K-point $(x_0,\, y)$ of E_i, the point $(A^s(x_0),\, y)$, where $s \leqslant i - 1$, is the K-point of E_{i-s}. Since by assumption x_0 is not A-preperiodic, this implies that if I is infinite, then every curve E_d, $d \geqslant 1$, has infinitely many K-points, implying by the Faltings theorem that $g(E_d) \leqslant 1$, $d \geqslant 1$. Let us consider, along with diagram

(52), the diagram

$$\begin{tikzcd}
 & \widetilde{E}_3 \arrow[r, "F_3"] \arrow[d, "\eta_3"] & \widetilde{E}_2 \arrow[r, "F_2"] \arrow[d, "\eta_2"] & \widetilde{E}_1 \arrow[d, "\eta_1"] & \\
\cdots & E_3 \arrow[r, "f_3"] \arrow[d, "\pi_{x,3}"] & E_2 \arrow[r, "f_2"] \arrow[d, "\pi_{x,2}"] & E_1 \arrow[r, "\pi_{y,1}"] \arrow[d, "\pi_{x,1}"] & \mathbb{CP}^1 \arrow[d, "U"] \\
 & \mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1,
\end{tikzcd}$$

where $\eta_d\colon \widetilde{E}_d \to E_d$, $d \geqslant 1$, are desingularization maps, and $F_d\colon \widetilde{E}_d \to \widetilde{E}_{d-1}$, $d \geqslant 2$, are holomorphic maps between compact Riemann surfaces. Since Corollary 2.10 and Corollary 2.9 imply that there exists d_0 such that the diagram

$$D[\widetilde{E}_d,\, A,\, \pi_{x,d} \circ \eta_d,\, F_d], \qquad d \geqslant d_0,$$

is good, applying Theorem 3.2, we conclude that there exist $s_0 \geqslant d_0$ and $l \geqslant 1$ such that for any $d \geqslant s_0$, the Riemann surfaces $\widetilde{E}_d$ and $\widetilde{E}_{d+l}$ are isomorphic and

$$(\pi_{x,d} \circ \eta_d) = \pi_{x,d+l} \circ \eta_{d+l} \circ \widetilde{\alpha}_d, \tag{53}$$

for some isomorphism $\widetilde{\alpha}_d\colon \widetilde{E}_d \to \widetilde{E}_{d+l}$.

Since $\widetilde{\alpha}_d$ descends to an automorphism $\alpha_d\colon E_d \to E_{d+l}$ that makes the diagram

$$\begin{tikzcd}
\widetilde{E}_d \arrow[r, "\widetilde{\alpha}_d"] \arrow[d, "\eta_d"] & \widetilde{E}_{d+l} \arrow[d, "\eta_{d+l}"] \\
E_d \arrow[r, "\alpha_d"] & E_{d+l}
\end{tikzcd}$$

commutative, it follows from (53) that for every $d \geqslant s_0$ the equality

$$\pi_{x,d} = \pi_{x,d+l} \circ \alpha_d \tag{54}$$

holds and the diagram

$$\begin{tikzcd}
E_d \arrow[r, "R_d"] \arrow[d, "\pi_{x,d}"] & E_d \arrow[d, "\pi_{x,d}"] \\
\mathbb{CP}^1 \arrow[r, "A^{\circ l}"] & \mathbb{CP}^1,
\end{tikzcd} \tag{55}$$

where

$$R_d = f_{d+1} \circ \cdots \circ f_{d+l-1} \circ f_{d+l} \circ \alpha_d,$$

commutes. By Corollary 4.3, for every $d \geqslant s_0$ there exists r such that $R_d^{\circ r}$ is defined over K. Moreover, since r is defined in terms of the monodromy group of $\pi_{x,d}$, it follows from equality (54) that for all $d \geqslant s_0$ from the same class by modulo l we can take the same r. Therefore, considering the least common multiple of the corresponding r for all such classes, without loss of generality we may assume that $R_d^{\circ r}$ is defined over K for all $d \geqslant s_0$.

Let us assume now that E_{i_0} contains a K-point of the form (x_0, y) for $i_0 \geqslant s_0$. Since (55) implies that for every $k \geqslant 1$ the equality

$$A^{\circ lk}(x_0) = \pi_{x,i_0} \circ R_{i_0}^{\circ k}(x_0, y)$$

holds, setting $R = lr$, we have:

$$A^{\circ(i_0+Rk)}(x_0) = A^{i_0} \circ A^{\circ Rk}(x_0) = A^{i_0} \circ \pi_{x,i_0} \circ R_{i_0}^{\circ rk}(x_0, y) = \\ = U \circ \pi_{y,1} \circ f_2 \circ f_3 \circ \cdots \circ f_{i_0} \circ R_{i_0}^{\circ rk}(x_0, y).$$

Therefore, since $R_d^{\circ r}$ is defined over K, all the numbers $A^{\circ(i_0+Rk)}(x_0)$, $k \geqslant 1$, belong to $U(K)$. This shows that the set of $i \in \mathbb{N}$ such that the curve E_i contains a K-point of the form (x_0, y) is a union of a finite set and a finite number of arithmetic progressions with denominator R.

Finally, if A is not a generalized Lattès map, then arguing as in Theorem 3.3 we conclude that there exists $s_0 \geqslant 0$ such that $\deg \pi_{x,d} = 1$ for all $d \geqslant s_0$. Therefore, (54) and (55) hold for $l = 1$. Moreover, since $\deg \pi_{x,d} = 1$, the map R_d is defined over K, so that $A^{\circ(i_0+k)}(x_0) \in U(K)$ for all $k \geqslant 0$. □

4.3. Example. In conclusion, we illustrate some of the constructions and results of this paper with the following example:

$$A = 144\,\frac{z\,(z+3)}{(z-9)^2}, \qquad U = z^2, \qquad z_0 = 1, \qquad k = \mathbb{Q}.$$

The function A is obtained from a one-parameter series introduced in the paper [3] where the value of the parameter is equal to one. It is shown in [3] that

$$I = \{0, 2\} \cup \{1 + 2m\colon m \geqslant 0\},$$

so, by Theorem 1.3, the function A should be a generalized Lattès map. Specifically, $A\colon \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map for the orbifold $\mathcal{O}$ defined by the equalities

$$\nu(0) = 2, \qquad \nu(-3) = 2.$$

Indeed,

$$A^{-1}(0) = \{0, -3\},$$

and the multiplicity of A at the points 0 and -3 equals one, and hence, (26) holds at $z = 0$ and $z = -3$. Moreover,

$$A^{-1}(-3) = -9/7,$$

and the multiplicity of A at $-9/7$ equals two, and hence, (26) holds at $z = -9/7$. On the other hand, for any point z distinct from 0, -3, and $-9/7$, equality (26) also holds, since for such a point, $\nu(z) = 1$ and $\nu(A(z)) = 1$.

To simplify formulas, let us consider instead of the functions A and U the functions

$$\widetilde{A} = \mu \circ A \circ \mu^{-1}, \qquad \widetilde{U} = \mu \circ U,$$

where

$$\mu = \frac{z}{z+3},$$

so that

$$\widetilde{A} = 48\,\frac{z}{(4\,z+3)^2}, \qquad \widetilde{U} = \frac{z^2}{z^2+3}.$$

Then $\widetilde{A}\colon \widetilde{\mathcal{O}} \to \widetilde{\mathcal{O}}$ is a minimal holomorphic map for the orbifold $\widetilde{\mathcal{O}}$ defined by the equalities

$$\widetilde{\nu}(0) = 2, \quad \widetilde{\nu}(\infty) = 2,$$

and the functions F and $\theta_{\widetilde{\mathcal{O}}}$ from Proposition 2.3, which make the diagram

$$\begin{tikzcd}
\mathbb{CP}^1 \arrow[r, "F"] \arrow[d, "\theta_{\widetilde{\mathcal{O}}}"] & \mathbb{CP}^1 \arrow[d, "\theta_{\widetilde{\mathcal{O}}}"] \\
\mathbb{CP}^1 \arrow[r, "\widetilde{A}"] & \mathbb{CP}^1
\end{tikzcd}$$

commutative, have the form

$$F = 4\,\frac{\sqrt{3}z}{4\,z^2+3}, \qquad \theta_{\widetilde{\mathcal{O}}} = z^2.$$

Further, the diagram

$$\begin{tikzcd}
\mathbb{CP}^1 \arrow[r, "F"] \arrow[d, "(\theta_{\widetilde{\mathcal{O}}}{,}V)"] & \mathbb{CP}^1 \arrow[d, "(\theta_{\widetilde{\mathcal{O}}}{,}V)"] & \\
E \arrow[r, "R"] \arrow[d, "\pi_x"] & E \arrow[r, "\pi_y"] \arrow[d, "\pi_x"] & \mathbb{CP}^1 \arrow[d, "\widetilde{U}"] \\
\mathbb{CP}^1 \arrow[r, "\widetilde{A}"] & \mathbb{CP}^1 \arrow[r, "\widetilde{A}"] & C\mathbb{P}^1,
\end{tikzcd}$$

where

$$V = 12\,\frac{z}{4\,z^2-3}$$

and the morphism $R\colon E \to E$ is defined by

$$x \to \widetilde{A}(x), \quad y \to -48\,\frac{\sqrt{3}\big((4\,xy-3\,y)^2+108\big)(4\,xy-3\,y)}{\big((4\,xy-3\,y)^2-36\big)\big((4\,xy-3\,y)^2-324\big)},$$

commutes. Finally, the function $\bar{F}$ Galois conjugated to F satisfies $\bar{F} = -F$, and the function

$$F^{\circ 2} = \bar{F}^{\circ 2} = 16\,\frac{\big(4\,z^2+3\big)z}{16\,z^4+88\,z^2+9}$$

as well as the morphism $R^{\circ 2}$ have rational coefficients.

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