

## Conservative polynomials and yet another action of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on plane trees

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RÉSUMÉ. Dans cet article nous étudions une action  $D$  du groupe de Galois absolu  $\Gamma = \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  sur des arbres planaires bicolores. A l'encontre de l'action similaire fournie par la théorie des “dessins d'enfants” de Grothendieck, l'action  $D$  est induite par l'action de  $\Gamma$  sur des classes d'équivalence de polynômes conservateurs qui sont les exemples les plus simples de fonctions rationnelles finies postcritiques. Nous établissons les propriétés principales de l'action  $D$  et la comparons avec l'action de Grothendieck.

ABSTRACT. In this paper we study an action  $D$  of the absolute Galois group  $\Gamma = \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  on bicolored plane trees. In distinction with the similar action provided by the Grothendieck theory of “Dessins d'enfants” the action  $D$  is induced by the action of  $\Gamma$  on equivalence classes of conservative polynomials which are the simplest examples of postcritically finite rational functions. We establish some basic properties of the action  $D$  and compare it with the Grothendieck action.

### 1. Introduction

The remarkable Belyi theorem [1] states that an algebraic curve  $X$  defined over  $\mathbb{C}$  is defined over  $\bar{\mathbb{Q}}$  if and only if there is a holomorphic function  $\beta : X \rightarrow \mathbb{CP}^1$  which is ramified only over  $\{0, 1, \infty\}$ . A pair consisting of such  $X$  and  $\beta$  is called a Belyi pair. The isomorphism classes of Belyi pairs are in one-to-one correspondence with the classes of isotopical equivalence of bicolored graphs embedded into topological models of Riemann surfaces. Namely, if  $(X, \beta)$  is a Belyi pair then the corresponding graph  $\Omega$  is the preimage of the segment  $[0, 1]$  under the map  $\beta : X \rightarrow \mathbb{CP}^1$  where white (resp. black) vertices of  $\Omega$  are preimages of 0 (resp. 1).

The absolute Galois group  $\Gamma = \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  acts naturally on isomorphism classes of Belyi pairs and this action descends to an action on equivalence classes of bicolored graphs. The study of this action, which we will denote by  $G$ , is the subject of the Grothendieck theory of “Dessins d'enfants” (see

e.g. [4] and the bibliography there). Note that the action  $G$  is highly non-trivial even when restricted on the bicolored plane trees (which are in one-to-one correspondence with the equivalence classes of the polynomial Belyi functions on the Riemann sphere). In particular, the action of  $\Gamma$  on plane bicolored trees is faithful ([8]).

The Belyi pairs are an example of “rigid” analytical objects which are completely defined up to an equivalence by some “combinatorial” data. Another example of such objects is postcritically finite rational functions that is the rational functions for which orbits of critical points under iterations are finite. Indeed, by the result due to Thurston [2] such functions, apart from a very special family, are uniquely defined, up to a conjugacy by a Möbius transformation, by purely combinatorial data. In particular, conjugacy classes of postcritically finite polynomials can be classified by means some rather complicated combinatorial objects called Hubbard trees (see [6], [7], [5]).

The finiteness of conjugacy classes of postcritically finite polynomials of a given degree makes possible to define an action of  $\Gamma$  on these classes similarly to the action  $G$ . Note that this action, which we will denote by  $D$ , is faithful [5]. Furthermore, there is some interplay between two theories since each equivalence class of polynomial Belyi functions contains postcritically finite polynomials (see [5]).

The simplest examples of postcritically finite polynomials are polynomials with all critical points fixed, called conservative, and in this paper we study the action  $D$  restricted to such polynomials. This particular case seems to be especially interesting since, in distinction with the general case, for conservative polynomials the corresponding combinatorial data can be described quite transparently. Namely, by the result due to Tischler [11] the equivalence classes of conservative polynomials considered as holomorphic dynamical systems on  $\mathbb{C}$  are in one-to-one correspondence with the classes of isotopical equivalence of bicolored plane trees. So, the absolute Galois group  $\Gamma$  acts on bicolored plane trees in two different ways: one action is induced by the action on the polynomial Belyi functions and the other one by the action on the conservative polynomials!

The paper is organized as follows. In the second section we reproduce the Tishler correspondence between conservative polynomials and bicolored plane trees and provide some examples. We write explicitly equations for defining a conservative polynomial  $C(z)$  corresponding to a tree  $\lambda$  and show how  $C(z)$  reflects the symmetries of  $\lambda$ . Besides, we prove that any conservative polynomial is indecomposable (that is can not be represented as a composition of two non-linear polynomials) unless it is equivalent to the polynomial  $z^n$  for composite  $n \in \mathbb{N}$ .

In the third section we establish some properties of the action  $D$  and compare it with the action  $G$ . In particular, we show that the list of valencies of “white” (but not “black” !) vertices of a tree  $\lambda$  and the symmetry group of  $\lambda$  are combinatorial Galois invariants of the action  $D$ . Then we give several examples of calculations of Galois orbits. Finally, we describe all trees which are defined uniquely by the list of valencies of “white” vertices and calculate the corresponding polynomials.

## 2. Conservative polynomials and plane trees

Recall that a *plane tree* is a tree embedded into the plane and that a *bicolored tree* is a tree vertices of which are colored in two colors in such a way that any edge connects vertices of different colors. Two bicolored plane trees  $\lambda$  and  $\tilde{\lambda}$  are called *equivalent* if there exists an orientation preserving homeomorphism  $h$  of  $\mathbb{C}$  such that  $\tilde{\lambda} = h(\lambda)$  and  $h$  preserves the colors of vertices.

A complex polynomial  $C(z)$  is called *conservative* if all its critical points are fixed that is if the equality  $C'(\zeta) = 0$ ,  $\zeta \in \mathbb{C}$ , implies that  $C(\zeta) = \zeta$ . A conservative polynomial  $C(z)$  is called *normalised* if  $C(z)$  is monic and  $C(0) = 0$ . Two conservative polynomials  $\tilde{C}(z)$ ,  $C(z)$  are called *equivalent* if there exists a complex polynomial  $A(z)$  of degree one such that  $\tilde{C} = A^{-1} \circ C \circ A$ .

Conservative polynomials were introduced by Smale [10] in connection with his “mean value conjecture”. Motivated by Smale’s conjecture Kostrikin proposed in [3] several conjectures concerning conservative polynomials. In particular, on the base of numerical experiments Kostrikin conjectured that the number of normalised conservative polynomials of degree  $d$  is finite and is equal to  $C_{2d-2}^{d-1}$ . This conjecture was proved by Tischler in the paper [11]. Moreover, in this paper a one-to-one correspondence between equivalence classes of conservative polynomials and equivalence classes of plane bicolored trees was established. Below we reproduce the Tischler construction.

Let  $\zeta$  be a critical point of  $C(z)$  and  $d \geq 2$  be the multiplicity of  $C(z)$  at  $\zeta$ . Then one can show (see [11]) that the immediate attractive basin  $B_\zeta$  of  $\zeta$  is a disk and that there is an analytic conjugation of  $C(z)$  on  $B_\zeta$  to  $z \rightarrow z^d$  on the unit disk  $D$  such that the conjugating map  $\varphi_\zeta : D \rightarrow B_\zeta$  extends continuously to the closed unit disk  $\bar{D}$ . Let  $S$  be a union of  $d - 1$  radial segments which are forward invariant under the map  $z \rightarrow z^d$  on  $\bar{D}$  and  $S_\zeta$  be the image of  $S$  under the map  $\varphi_\zeta$  (see Fig. 1 where  $d = 4$ ). We consider  $S_\zeta$  as a bicolored graph with a unique white vertex which is the image of zero and  $d - 1$  black vertices which are the images of end-points of  $S$ .

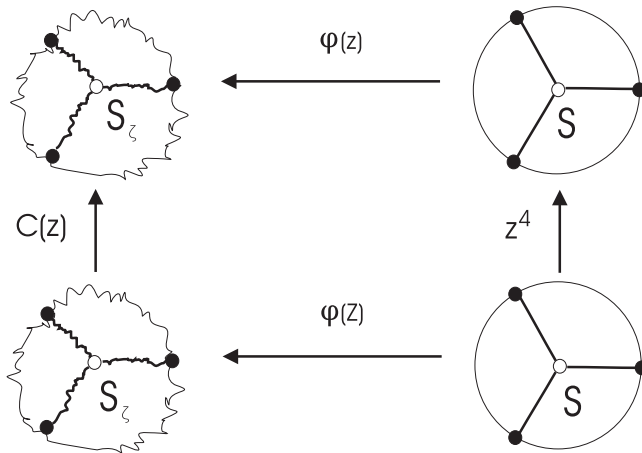


FIGURE 1

Define now a bicolored graph  $\lambda_C$  as a union  $\lambda_C = \cup_{i=1}^p S_{\zeta_i}$ , where  $\zeta_i$ ,  $1 \leq i \leq p$ , are all finite critical points of  $C(z)$ .

Clearly, the valency of a white vertex  $v$  of  $\lambda_C$  coincides with the multiplicity of the point  $v$  with respect to the map  $C'(z) : \mathbb{C} \rightarrow \mathbb{C}$ . Note also that by construction the graph  $\lambda_C$  is a forward invariant of  $C(z)$  and white (resp. black) vertices of  $\lambda_C$  are attractive (resp. repelling) fixed points of  $C(z)$ .

It turns out that the graph  $\lambda_C$  is actually a tree. Moreover, the following theorem proved by Tischler is true.

**Theorem 2.1** ([11]). *The map  $C \rightarrow \lambda_C$  descends to a bijection between equivalence classes of conservative polynomials of degree  $d$  and equivalence classes of bicolored plane trees with  $d - 1$  edges. Furthermore, the number of normalised conservative polynomials of degree  $d$  is  $\binom{2d-2}{d-1}$ .*

The simplest example of a conservative polynomial is the polynomial  $z^d$  with a unique critical point  $\zeta = 0$ . Clearly, the corresponding tree is a  $d - 1$ -edged star.

Another example (cf. [10],[3]) is the polynomial

$$f_d(z) = z^d + (d/d - 1)z,$$

$d \geq 2$ . Since all zeros of  $f'_d(z)$  are simple all white vertices of the corresponding tree  $\lambda_d$  are of valency 1 and therefore  $\lambda_d$  is also a  $d - 1$ -edged star but *with the bicolouring changed*. On the right side of Fig. 2 the dynamics

of  $f_6(z)$  are shown: the points for which the limits of iterations have the same value are painted by the same color<sup>1</sup>.

Note that we face here a phenomenon which is absent in the “Dessins d’enfants” theory: the calculation of a polynomial corresponding to a tree  $\lambda$  and the calculation of a polynomial corresponding to a tree  $\lambda^*$  obtained from  $\lambda$  by the change of the bicoloring are essentially different questions apparently not connected between themselves.

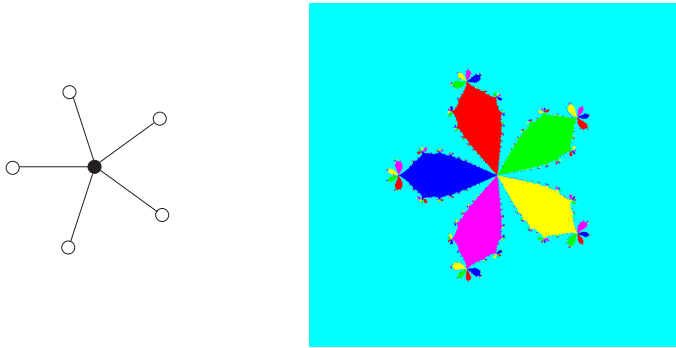


FIGURE 2

Starting from a tree  $\lambda$  we can find a conservative polynomial  $C(z)$  from the corresponding equivalence class as follows. Let  $\alpha = \langle \alpha_1, \alpha_2, \dots, \alpha_p \rangle$  be the sequence of valencies of white vertices of  $\lambda$  in decreasing order. Denote by  $a_1, a_2, \dots, a_p$  unknown coordinates of white vertices of  $\lambda$  coinciding with zeros of  $C'(z)$ . Clearly, without loss of generality we can assume that  $a_1 = 0$ .

Set  $C(z)$  equal to the indefinite integral

$$\int n z^{\alpha_1} (z - a_2)^{\alpha_2} \dots (z - a_p)^{\alpha_p} dz$$

normalised by the condition  $C(0) = 0$ . Then the system of equations to determine  $a_2, a_3, \dots, a_p$  is

$$(2.1) \quad C(a_j) = a_j, \quad 2 \leq j \leq p.$$

Observe that system (2.1) depends only on the sequence  $\alpha$  which is called *the type* of  $\lambda$ . Therefore, solutions of (2.1) along with a polynomial corresponding to  $\lambda$  contain all polynomials corresponding to trees of type  $\alpha$ . Note also that system (2.1) may have solutions for which the numbers  $a_1, a_2, \dots, a_p$  are *not* mutually distinct. Such a solution also corresponds to a tree  $\tilde{\lambda}$  but the type of this tree is distinct from  $\alpha$ . Geometrically,  $\tilde{\lambda}$  is

<sup>1</sup>To prepare dynamical pictures for this paper we used C. McMullen’s programs available on <http://www.math.harvard.edu/~ctm/programs.html>

obtained from some tree  $\lambda$  of type  $\alpha$  by “merging” some number of white vertices of  $\lambda$ .

Let us describe now following [11] the combinatorial datum which permits to determine a normalised conservative polynomial corresponding to a tree  $\lambda$  uniquely. First, the condition  $f(0) = 0$  corresponds to the choice of a vertex  $v$  of  $\lambda$  which we place at the origin. Furthermore, since the polynomial  $C(az)/a$ ,  $a \in \mathbb{C}$  is monic if and only if  $a^{d-1} = 1$ , where  $d = \deg C(z)$ , if  $v$  is already chosen then the corresponding normalised polynomial is defined up to a change  $z \rightarrow \varepsilon z$ ,  $\varepsilon^{d-1} = 1$ .

Let  $S_\infty \subset \mathbb{CP}^1$  be a graph defined like the graphs  $S_{\zeta_i}$ ,  $1 \leq i \leq p$ , with the only difference that we start from the infinite critical point of  $C(z)$ . It is shown in [11] that each edge of  $S_\infty$  ends at some black vertex of  $\lambda_C$  and that the number of edges ending at a black vertex  $w$  is equal to the multiplicity of  $w$ . More precisely, each angle formed by two adjacent edges emerging from a black vertex of  $\lambda$  contains exactly one edge of  $S_\infty$  (see Fig. 3).

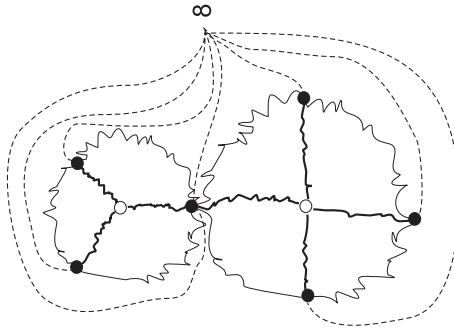


FIGURE 3

Clearly, a choice of a  $d - 1$ -th root of unity corresponds to a choice of an edge of  $S_\infty$  and therefore a normalised conservative polynomial corresponding to a tree  $\lambda$  is determined uniquely by fixing a vertex  $v$  (black or white) of  $\lambda$  and an “angle”  $\varphi$  adjacent to a black vertex of  $\lambda$ .

As an example consider the set of bicolored trees with 4 vertices. It is easy to see that there are exactly three such trees which are shown on Fig. 4.

On the other hand, by the Tischler theorem there exist  $\binom{6}{3} = 20$  different normalised conservative polynomials of degree 4 : to the first tree shown on Fig. 4 correspond 12 different polynomials while to the second and to the third ones correspond only 4 different polynomials.

In general, since collections  $(\lambda, v_1, \varphi_1)$  and  $(\lambda, v_2, \varphi_2)$  are equivalent if and only if there exists an orientation preserving homeomorphism  $\chi$  of  $\mathbb{C}$

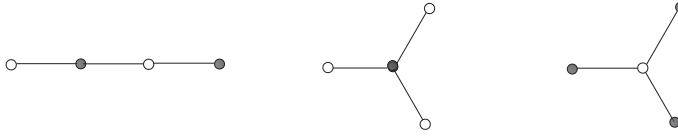


FIGURE 4

such that  $\chi(\lambda) = \lambda$  and  $\chi(v_1) = v_2$ ,  $\chi(\varphi_1) = \varphi_2$ , it is easy to see that to a plane tree  $\lambda$  with  $d$  vertices correspond  $d(d-1)/|\text{Aut } \lambda|$  different normalised conservative polynomials, where  $\text{Aut } \lambda$  denotes the group of symmetries of  $\lambda$ . Furthermore, the following statement holds.

**Proposition 2.1.** *A tree  $\lambda$  has a symmetry of order  $k$  if and only if the corresponding class of conservative polynomials contains a polynomial of the form  $C(z) = zR(z^k)$ , where  $R(z)$  is a polynomial.*

*Proof.* Indeed, if  $C(z)$  is a normalised conservative polynomial of degree  $d$  corresponding to a collection  $(\lambda, v, \varphi)$  and  $v$  is placed at the origin then to a collection  $(\lambda, v, \tilde{\varphi})$  corresponds a conservative polynomial  $C(\varepsilon z)/\varepsilon$ , where  $\varepsilon$  is some  $d-1$ -th root of unity. Therefore, a tree  $\lambda$  has a symmetry of order  $k$  with the center at  $v$  if and only if  $C(z) = C(\varepsilon z)/\varepsilon$  for any  $k$ -th root of unity  $\varepsilon$ . This condition is equivalent to the condition that the polynomial  $C(z)/z$  is invariant with respect to any rotation of the form  $z \rightarrow \varepsilon z$ , where  $\varepsilon$  is a  $k$ -th root of unity. In its turn the last condition is equivalent to the condition that  $C(z) = zR(z^k)$  for some polynomial  $R(z)$ .

Since many constructions of the ‘‘Dessins d’enfants’’ theory (for instance, the Belyi theorem) make use compositions of functions, and such compositions survive under the Galois action, it is natural to examine compositional properties of the conservative polynomials. It turns out that conservative polynomials are essentially indecomposable.

**Proposition 2.2.** *All conservative polynomials not equivalent to  $z^n$  for composite  $n$  are indecomposable.*

*Proof.* Indeed, suppose that  $C(z) = C_1(C_2(z))$ , where  $C_1(z), C_2(z)$  are polynomials such that  $\deg C_1(z), \deg C_2(z) > 1$ . Let  $\zeta \in \mathbb{C}$  be a critical point of the polynomial  $C_1(z)$ . Then the chain rule implies that any point  $\mu \in \mathbb{C}$  such that  $C_2(\mu) = \zeta$  is a critical point of  $C(z)$ . Furthermore, if  $C_2(z)$  is not equal to  $A(z - \mu)^l + \zeta$  for some  $A, \mu, \zeta \in \mathbb{C}$  and integer  $l \geq 2$ , then there exist  $\mu_1, \mu_2 \in \mathbb{C}$ ,  $\mu_1 \neq \mu_2$ , such that  $C_2(\mu_1) = C_2(\mu_2) = \zeta$ . For these points we have  $C'(\mu_1) = C'(\mu_2) = 0$  and  $C(\mu_1) = C(\mu_2) = C_1(\zeta)$ . Since  $\mu_1 \neq \mu_2$  this contradicts to the condition that  $\mu_1, \mu_2$  are fixed points of  $C(z)$ .

Therefore,  $C_2(z) = A(z - \mu)^l + \zeta$ . In particular, for any  $\tilde{\zeta} \in \mathbb{C}$ ,  $\tilde{\zeta} \neq \zeta$  there exist mutually distinct  $\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_l \in \mathbb{C}$  such that  $C_2(\tilde{\mu}_i) = \tilde{\zeta}$ ,  $1 \leq i \leq l$ . Thus if  $\tilde{\zeta}$  is a critical point of  $C_1(z)$  distinct from  $\zeta$  then as above we obtain a contradiction since  $C'(\tilde{\mu}_1) = C'(\tilde{\mu}_2) = 0$  and  $C(\tilde{\mu}_1) = C(\tilde{\mu}_2) = C_1(\tilde{\zeta})$ . Hence,  $\zeta$  is a unique finite critical point of  $C_1(z)$ . This implies that  $C_1(z) = B(z - \zeta)^k + \nu$  for some  $B, \nu \in \mathbb{C}$  and integer  $k \geq 2$  and hence  $C(z) = C_1(C_2(z))$  is equivalent to  $z^n$  for composite  $n$ .

### 3. Galois group action

In each equivalence class of conservative polynomials there exist polynomials with algebraic coefficients. Indeed, it follows from the Tischler theorem that system (2.1) has only a finite number of solutions. Therefore, since equations (2.1) have rational coefficients, all these solutions are algebraic.

Furthermore, the group  $\Gamma$  acts on the set of conservative polynomials with algebraic coefficients in a natural way: it is easy to see that if  $C(z)$  is a conservative polynomial with algebraic coefficients and  $\sigma \in \Gamma$  then the polynomial  $C^\sigma(z)$  obtained from  $C(z)$  by the action of  $\sigma$  on coefficients of  $C(z)$  again is a conservative polynomial. Moreover, since

$$(A^{-1} \circ P \circ A)^\sigma = (A^\sigma)^{-1} \circ P^\sigma \circ A^\sigma,$$

this action descends to an action on equivalence classes. Hence, by the Tischler theorem, we obtain an action  $D$  of  $\Gamma$  on bicolored plane trees.

The type  $\alpha$  of a tree  $\lambda$  is an invariant of the action  $D$  since the sequence  $\alpha$  coincides with the sequence of multiplicities of zeros of  $C'(z)$ . Furthermore, like the ‘‘Dessins d’enfants’’ theory the Galois orbit of  $\lambda$  often coincides with the set of all trees of type  $\alpha$ . Moreover, in view of Proposition 2.1 the symmetry group of  $\lambda$  is also a Galois invariant of the action  $D$  since if  $C(z)$  has the form  $C(z) = zR(z^k)$  for some polynomial  $R(z)$  then the Galois conjugated polynomial also has such a form.

Finally, it is easy to see (see e. g. Example 1 below) that the action  $D$  is distinct from the action  $G$ . Summing up we obtain the following statement.

**Theorem 3.1.**  *$D$  is a well-defined action of the group  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  on bicolored plane trees distinct from the action  $G$ . The type and the symmetry group of a tree  $\lambda$  are combinatorial Galois invariants of the action  $D$ .*

Define a field of modules  $k_\lambda^D$  of a tree  $\lambda$  as a fixed field of the stabilizer of  $\lambda$  with respect to the action  $D$ . Since there exists only a finite number of trees of a given type the field  $k_\lambda^D$  is a number field whose degree over  $\mathbb{Q}$  is equal to the length of the orbit containing  $\lambda$ . Note that the result proved in [9] implies that for any bicolored plane tree  $\lambda$  there exists a conservative polynomial  $C(z)$  from the corresponding equivalence class such that  $C(z) \in k_\lambda^D[z]$ .



**Example 1.** As the first example consider the trees of type  $\langle 3, 1, 1 \rangle$ . There exist two trees of this type which are shown on Fig. 5.

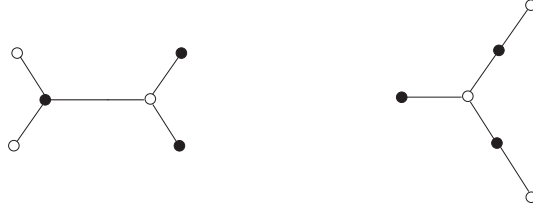


FIGURE 5

Place the white vertex of valency 3 at zero. Then

$$C'(z) = az^3(z^2 + cz + b)$$

for some  $a, b, c \in \mathbb{C}$  such that the polynomial  $z^2 + cz + b$  has two different roots distinct from 0. Furthermore,  $c \neq 0$  since otherwise  $C(z) = \tilde{C}(z^2)$  for some polynomial  $\tilde{C}(z)$  in contradiction with proposition 2.2. Therefore, we can suppose that  $c = 1$ .

Since  $C(0) = 0$  we have

$$C(z) = a(z^6/6 + z^5/5 + bz^4/4)$$

and a calculation shows that

$$(3.1) \quad C(z) - z = A(z)(z^2 + z + b) + B(z),$$

where

$$A(z) = az^4/6 + az^3/30 + (ab/12 - a/30)z^2 + (-7ab/60 + a/30)z - ab^2/12 + 3ab/20 - a/30,$$

$$B(z) = (a/30 - 1 + ab^2/5 - 11ab/60)z + ab^3/12 - 3ab^2/20 + ab/30.$$

Since roots  $\beta_1, \beta_2$  of  $z^2 + z + b$  are fixed points of  $C(z)$  and  $\beta_1 \neq \beta_2$  it follows from (3.1) taking into account the equality  $\deg B(z) = 1$  that  $B(z) \equiv 0$ . This gives us the system

$$\begin{cases} a/30 - 1 + ab^2/5 - 11ab/60 = 0, \\ ab^3/12 - 3ab^2/20 + ab/30 = 0. \end{cases}$$

Solving this system we conclude that either

$$(3.2) \quad a = 30, \quad b = 0,$$

or

$$a = -5055/8 \pm 795\sqrt{41}/8, \quad b = 9/10 \pm \sqrt{41}/10.$$

All three solutions above correspond some conservative polynomials but in case when (3.2) holds one of the roots of  $z^2 + dz + b$  coincides with zero.

This means that solution (3.2) actually corresponds to the tree  $\lambda_{4,1}$  (see Fig. 8 below).

After rejecting solution (3.2), the remaining solutions are Galois conjugated, and therefore the trees shown on Fig. 5 form a two-element Galois orbit with the field of modulus  $\mathbb{Q}(\sqrt{41})$ .

In this example we can see an important difference between the actions  $G$  and  $D$ . Namely, for the action  $G$  not only the list of valencies of “white” vertices but also the list of valencies of “black” vertices is Galois invariant. Therefore, “expected” orbits of the action  $D$  are much longer than the ones with respect to the action  $G$ . In particular, since both of the trees shown on Fig. 5 are determined uniquely by the lists of valencies of “white” and “black” vertices, each of them forms an one-element Galois orbit with respect to the action  $G$ .

**Example 2.** Consider now the trees of type  $\langle 2, 1, 1 \rangle$ . Again there exist two trees of this type: one of them is shown on Fig. 6 and the other one on Fig. 7. Nevertheless, since the first tree has a symmetry of order 2 and the second one does not have symmetries it follows from Theorem 3.1 that each of these trees forms a one-element Galois orbit with respect to the action  $G$ .

In order to calculate a conservative polynomial corresponding to the first tree set

$$C'(z) = az^2(z^2 + bz - 1).$$

In other words we suppose that the white vertex of valency 2 is located at zero and that the product of the coordinates of two white vertices of valency 1 equals -1. Observe now that in view of Proposition 2.1 we necessarily have  $b = 0$ . In particular, the coordinates of white vertices of valency 1 are  $\pm 1$ .

Therefore,  $C(z) = az^5/5 - az^3/3$  and the conditions  $C(1) = 1$ ,  $C(-1) = -1$  reduce to the equality  $-2a/15 = 1$ . Hence,

$$C(z) = -3z^5/2 + 5z^3/2.$$

On the right side of Fig. 6 the dynamics of  $C(z)$  are shown.

In order to calculate a polynomial corresponding to the second tree observe that in this case Proposition 2.1 implies that the sum of coordinates of two white vertices of valency 1 is necessarily distinct from zero so we can set this sum equal 2.

Then

$$C'(z) = az^2(z^2 + 2z + b), \quad C(z) = a(z^5/5 + z^4/2 + bz^3/3)$$

and as in the first example the remainder

$$B(z) = (14ab/15 - 4a/5 - 1 - 2ab^2/15)z + 11ab^2/30 - 2ab/5$$

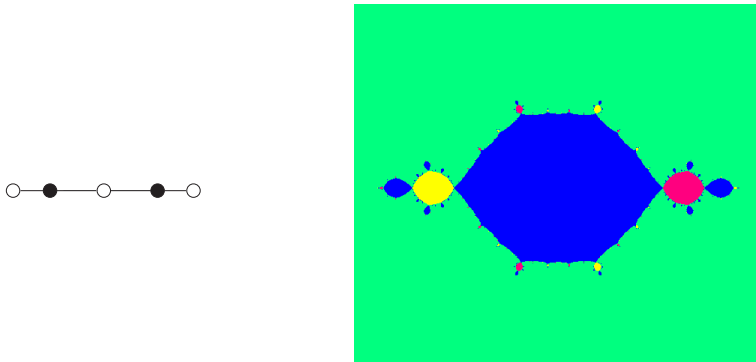


FIGURE 6

after the division of  $C(z) - z$  by  $z^2 + 2z + b$  equals zero. Solving the corresponding system

$$(3.3) \quad \begin{cases} 14ab/15 - 4a/5 - 1 - 2ab^2/15 = 0, \\ 11ab^2/30 - 2ab/5 = 0 \end{cases}$$

and rejecting the solution

$$a = -5/4, \quad b = 0$$

corresponding to the tree  $\lambda_{2,1}$  we conclude that

$$a = 605/36, \quad b = 12/11$$

and therefore

$$C(z) = 121z^5/36 + 605z^4/72 + 55z^3/9.$$

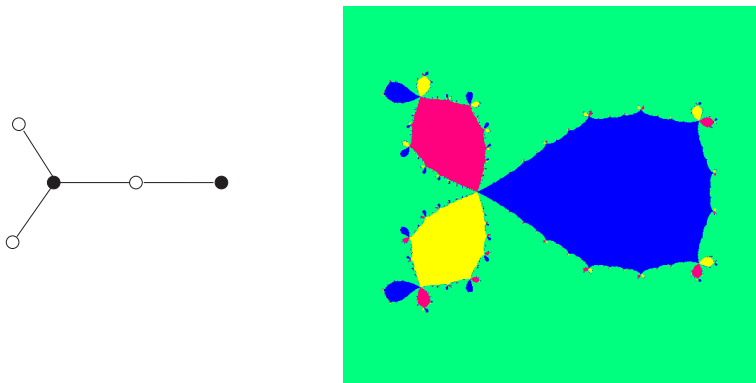


FIGURE 7

**Example 3.** Consider finally the tree  $\lambda_{r,s}$  shown on Fig. 8, where  $r, s$  are some integers  $\geq 1$ .

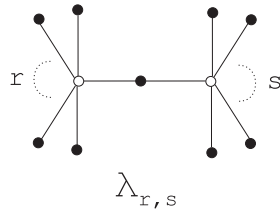


FIGURE 8

Clearly, we can place white vertices to the points 0, 1. Then

$$C'(z) = cz^r(1-z)^s,$$

where  $c \in \mathbb{C}$  is a parameter to define. Integrating and taking into account that  $C(0) = 0$ , we conclude that

$$C(z) = cz^{r+1} {}_2F_1(-s, r+1, 2+r, z).$$

Finally, since

$$C(1) = c \frac{r!s!}{(r+s+1)!},$$

the condition  $C(1) = 1$  implies that

$$c = \frac{(r+s+1)!}{r!s!}.$$

Note that any tree  $\lambda_{r,s}$  is determined uniquely by its type. Since in view of Theorem 3.1 such a tree necessarily forms a one-element Galois orbit it is interesting to know how many trees possess this property. The proposition below shows that such a phenomenon is rather exceptional.

**Proposition 3.1.** *Suppose that a tree  $\lambda$  is determined uniquely by its type. Then  $\lambda$  is either a star with black center, or a star with white center, or the tree  $\lambda_{r,s}$  for some  $r, s \geq 1$ .*

*Proof.* Let  $\alpha_1, \alpha_2, \dots, \alpha_p$  (resp.  $\beta_1, \beta_2, \dots, \beta_q$ ) be the list of valencies of white (resp. black) vertices of  $\lambda$ . If  $q = 1$  then  $\lambda$  is a star with the black center so we will suppose that  $q > 1$ .

Show first that if  $\beta_2 > 1$  then there exists more than one tree of type  $\alpha$ . Indeed, let  $v_1$  (resp.  $v_2$ ) be a black vertex of  $\lambda$  of valency  $\beta_1$  (resp.  $\beta_2$ ) and let  $f$  be a path connecting  $v_1$  and  $v_2$ . Consider the following operation. Cut off a branch  $b$  of  $\lambda$  growing from the vertex  $v_2$  (that is a maximal subtree  $b$  of  $\lambda$  for which  $v_2$  is a vertex of valency 1) such that  $b$  does not contain  $f$ . Then glue  $b$  to the vertex  $v_1$  (see Fig. 9). Since  $\beta_2 > 1$  we always can

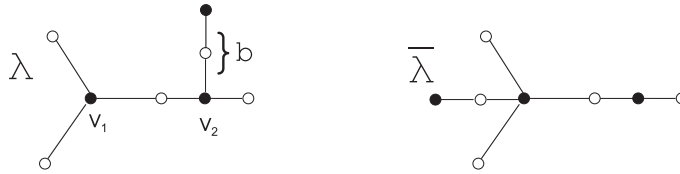


FIGURE 9

perform such an operation and though generally there are many ways for doing it in any case the obtained tree  $\bar{\lambda}$  can not be isotopically equivalent to  $\lambda$  since the maximal valency of a black vertex of  $\bar{\lambda}$  is greater than the corresponding valency of  $\lambda$ . On the other hand, the list of valencies of white vertices of  $\bar{\lambda}$  remains the same. So, in the following we will suppose that  $\beta_2 = 1$ .

If  $\beta_1 > 2$  then cutting off any branch  $b$  of  $\lambda$  growing from  $v_1$  which does not contain  $f$  and gluing  $b$  to the vertex  $v_2$  we again obtain a tree  $\bar{\lambda}$  which is not isotopically equivalent to  $\lambda$  since the corresponding lists of valencies of black vertices  $\{\beta_1, 1, 1, \dots, 1\}$  and  $\{\beta_1 - 1, 2, 1, \dots, 1\}$  can not coincide due to the condition  $\beta_1 > 2$ .

Therefore, either  $\beta_1 = 2$  and then  $\lambda = \lambda_{r,s}$ , or  $\beta_1 = 1$  and then  $\lambda$  is a star with the white center.

**Acknowledgments.** I am grateful to G. Shabat and A. Eremenko for discussions. Also I would like to thank the Max Planck Institut für Mathematik for its support and hospitality.

### References

- [1] G. BELYI, *On Galois extensions of a maximal cyclotomic field*. Math. USSR, Izv. **14** (1980), 247–256.
- [2] A. DOUADY, J. HUBBARD, *A proof of Thurston’s topological characterization of rational functions*. Acta Math. **171**, No.2 (1993), 263–297.
- [3] A. KOSTRIKIN, *Conservative polynomials*. In “Stud. Algebra Tbilisi”, 115–129, 1984.
- [4] S. LANDO, A. ZVONKIN, *Graphs on Surfaces and Their Applications*. Encyclopedia of Mathematical Sciences **141**(II), Berlin: Springer, 2004.
- [5] K. PILGRIM, *Dessins d’enfants and Hubbard trees*, Ann. Sci. École Norm. Sup. (4) **33** (2000), no. 5, 671–693.
- [6] A. POIRIER, *On postcritically finite polynomials, part 1: critical portraits*. Preprint, arxiv:math. DS/9305207.
- [7] A. POIRIER, *On postcritically finite polynomials, part 2: Hubbard trees*. Preprint, arxiv:math. DS/9307235.
- [8] L. SCHNEPS, *Dessins d’enfants on the Riemann sphere*. In “The Grothendieck Theory of Dessins D’enfants” (L. Shneps eds.), Cambridge University Press, London mathematical society lecture notes series **200** (1994), 47–77.
- [9] J. SILVERMAN, *The field of definition for dynamical systems on  $\mathbb{P}^1$* . Compos. Math. **98**, No.3 (1995), 269–304.

- [10] S. SMALE, *The fundamental theorem of algebra and complexity theory*. Bull. Amer. Math. Soc. **4** (1981), 1–36.
- [11] D. TISCHLER, *Critical points and values of complex polynomials*. J. of Complexity **5** (1989), 438–456.

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