Elliptic polynomials

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Elliptic polynomials

F. B. Pakovich

1. Classical Chebyshev polynomials, which we can define, for example, with the help of the equality $T_n(\cos \phi) = \cos n\phi$, possess many remarkable properties. The best known is that, among all the real polynomials of degree $n$ with leading coefficient 1, these polynomials, after a suitable normalization, have least deviation from 0 on the segment $[-1,1]$ (see [1], Ch. 2). As is shown below, we can also describe these polynomials to within linear equivalence in purely geometric terms as polynomials for which the inverse image of a certain segment is homeomorphic to a segment. In this note we take this property of Chebyshev polynomials as a basis for defining their generalization—elliptic polynomials, that is, complex polynomials for which the inverse image of a certain segment is homeomorphic to the union of two segments.

2. We denote by $a_1, a_2, \ldots, a_k$ all the critical values of the polynomial $P(z) \in \mathbb{C}[z]$ which lie within the segment $[u_1, u_2]$, and also put $a_0 = u_1$, $a_{k+1} = u_2$. In studying the geometry of the set $P^{-1}[u_1, u_2]$ it is convenient to regard it as embedded in the plane graph $G_p$ whose vertices are the inverse images of the points $a_i$, $i = 0, \ldots, k+1$, and whose edges are the inverse images of the segments $[a_i, a_{i+1}], i = 0, \ldots, k$. The degree of each vertex with coordinate $x$ is equal to the multiplicity with which the polynomial takes its value at this point if $P(x) \in \{u_1, u_2\}$, and equal to double the multiplicity if $P(x) \in \{a_1, \ldots, a_k\}$.

Lemma 1. The graph $G_p$ consists of $S_p = \sum_{i=0}^{k+1} \# \{P^{-1}(a_i)\} + (k + 1)\deg P$ connectivity components each of which is a tree. $S_p = 1$ if and only if all the critical values of $P(z)$ belong to the segment $[u_1, u_2]$, and $S_p = 2$ if and only if all the critical values of $P(z)$ except for one belong to $[u_1, u_2]$, and the inverse image of the single critical value not lying on $[u_1, u_2]$ contains only one critical point at which the polynomial takes its value with multiplicity two.

Theorem 1. Let $P(z) \in \mathbb{C}[z]$ be a polynomial of degree $n$ for which the inverse image of a certain segment is homeomorphic to a segment. Then there exist $a, b, a, b \in \mathbb{C}$ such that $aP(az + b) + b = T_n(z)$, where $T_n(z)$ is the $n$th Chebyshev polynomial.

Definition 1. The polynomial $P(z)$ is called elliptic if there exists a segment $[u_1, u_2]$ such that $P^{-1}[u_1, u_2]$ is the union of two different sets (which may be intersecting) each of which is homeomorphic to a segment.

For definiteness we shall assume everywhere from now on that $[u_1, u_2] = [-1,1]$.

Theorem 2. A polynomial $P(z)$ of degree $n$ is an elliptic polynomial if and only if it satisfies the equation

\[ P^2(z) - \left( \frac{P'(z)}{n(z-x)} \right)^2 R(z) = 1, \]

where $R(z)$ is a 4th degree polynomial with leading coefficient 1 and without multiple roots, and $x$ is a certain complex number.

We note that the roots of $R(z) = (z-a)(z-\beta)(z-\gamma)(z-\delta)$ are precisely the coordinates of the vertices of odd degree of the graph $G_p$.

Definition 2. An elliptic polynomial is called primitive if it cannot be put in the form of a composition $\pm T_d(Q(z))$, where $d, \deg Q(z) > 1$.

Definition 3. Two polynomials $P(z)$ and $Q(z)$ are called (linearly) equivalent if $a, b \in \mathbb{C}$ exist such that $P(z) = \pm Q(az + b)$. 
Suppose that we have an equivalence class of elliptic polynomials of degree \( n \). We choose representations \( P(z) \) in it so that \( \alpha + \beta + \gamma + \delta = 0 \) and suppose that

\[
R(z) = (z - \alpha)(z - \beta)(z - \gamma)(z - \delta) = z^4 - 6Az^2 + 4Bz + C.
\]

From the numbers \( A, B, C \) we now construct the elliptic curve

\[
L_p : w^2 = 4v^3 - g_2v - g_3, \quad g_2 = 3A^2 + C, \quad g_3 = -AC + A^3 - B^2,
\]

and the point \( N_p = (v, w) = (A, B) \) on it. If \( \tilde{P}(z) \) is another representation in the same class for which \( \alpha + \beta + \gamma + \delta = 0 \), then \( \tilde{P}(z) = \pm P(\lambda z) \), where \( \lambda \in \mathbb{C}^* \), and hence \( \tilde{R}(z) = z^4 - 6A\lambda^2 z^2 + 4B\lambda^3 z + C\lambda^4 \). Therefore, as we easily verify, the curve \( L_{\tilde{p}} \) is isomorphic to the curve \( L_p \) and the point \( N_{\tilde{p}} \) in this isomorphism goes into the point \( N_p \). Thus we have a well-defined map \( F_n \) from the set of equivalence classes of elliptic polynomials of degree \( n \) into the set of isomorphism classes of elliptic curves with a distinguished point.

We note that the map

\[
(x, y) = \left( \frac{1}{2} \left( \frac{B + w}{A - v} \right), 2v + A - \frac{1}{4} \left( \frac{B + w}{A - v} \right)^2 \right)
\]

establishes a birational isomorphism between the curve \( L_p \) and the curve \( y^2 = R(x) \). We can therefore consider the pair \( (L_p, N_p) \) as an elliptic curve \( y^2 = R(x) \) together with the point at infinity.

**Theorem 3.** The map \( F_n \) effects a bijective correspondence between the set \( V_n \) of equivalence classes of primitive elliptic polynomials of degree \( n \) and the set \( W_n \) of isomorphism classes of pairs \( (L, N) \), where \( L \) is an elliptic curve over \( \mathbb{C} \) and \( N \) is a point of proximate order \( n \) on it.

**3.** In similar fashion to Chebyshev polynomials, elliptic polynomials under certain natural restrictions possess interesting extremal properties.

**Theorem 4.** Let \( P(z) = a_n z^n + \cdots + a_1 z + a_0 \) be an elliptic polynomial such that \( R(z) = (z - \alpha)(z - \beta)(z - \gamma)(z - \delta) \) has only real roots, \( \alpha < \beta < \gamma < \delta \). Then among all the real polynomials of degree \( n \) with leading coefficient \( 1 \) the polynomial \( \tilde{P}(z) = P(z)/a_n \) has the least deviation from \( 0 \) on the set \( [\alpha, \beta] \cup [\gamma, \delta] \). The absolute value of the deviation is equal to \( 1/|a_n| \).

**Example.** Consider the third-order point \( (v, w) = (3, 4) \) on the elliptic curve \( w^2 = 4v^3 - 60v + 88 \). We can easily verify that the polynomials in (1) take the forms:

\[
R(z) = z^4 - 18z^2 + 16z + 33, \quad P(z) = \frac{1}{8}(z^3 + 3z^2 - 9z - 19), \quad x = 1.
\]

Now, using Theorem 4, we deduce that, among all the real polynomials of degree 3 with leading coefficient 1, the polynomial \( \tilde{P}(z) = z^3 + 3z^2 - 9z - 19 \) deviates least from zero on the set \( [-1 - 2\sqrt{3}, -1] \cup [-1 + 2\sqrt{3}, 3] \). The absolute value of the deviation is 8.

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**Remark.** After completion of this paper, A. P. Veselov drew the author’s attention to the fact that the connection between polynomials satisfying (1) and isomorphism classes of elliptic curves with a distinguished point of finite order has already been studied in [6] from a somewhat different point of view.
Bibliography


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