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Elliptic polynomials

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1. Classical Chebyshev polynomials, which we can define, for example, with the help of the equality $T_n(\cos\varphi) = \cos n\varphi$, possess many remarkable properties. The best known is that, among all the real polynomials of degree n with leading coefficient 1, these polynomials, after a suitable normalization, have least deviation from 0 on the segment $[-1,1]$ (see [1], Ch. 2). As is shown below, we can also describe these polynomials to within linear equivalence in purely geometric terms as polynomials for which the inverse image of a certain segment is homeomorphic to a segment. In this note we take this property of Chebyshev polynomials as a basis for defining their generalization—elliptic polynomials, that is, complex polynomials for which the inverse image of a certain segment is homeomorphic to the union of two segments.

2. We denote by $a_1, a_2, \ldots, a_k$ all the critical values of the polynomial $P(z) \in \mathbb{C}[z]$ which lie within the segment $[u_1, u_2]$, and also put $a_0 = u_1$, $a_{k+1} = u_2$. In studying the geometry of the set $P^{-1}[u_1, u_2]$ it is convenient to regard it as embedded in the plane graph G_p whose vertices are the inverse images of the points a_i, $i = 0, \ldots, k+1$, and whose edges are the inverse images of the segments $[a_i, a_{i+1}]$, $i = 0, \ldots, k$. The degree of each vertex with coordinate x is equal to the multiplicity with which the polynomial takes its value at this point if $P(x) \in \{u_1, u_2\}$, and equal to double the multiplicity if $P(x) \in \{a_1, \ldots, a_k\}$.

Lemma 1. *The graph G_p consists of $S_p = \sum_{i=0}^{k+1} \#\{P^{-1}(a_i)\} - (k+1)\deg P$ connectivity components each of which is a tree. $S_p = 1$ if and only if all the critical values of $P(z)$ belong to the segment $[u_1, u_2]$, and $S_p = 2$ if and only if all the critical values of $P(z)$ except for one belong to $[u_1, u_2]$, and the inverse image of the single critical value not lying on $[u_1, u_2]$ contains only one critical point at which the polynomial takes its value with multiplicity two.*

Theorem 1. *Let $P(z) \in \mathbb{C}[z]$ be a polynomial of degree n for which the inverse image of a certain segment is homeomorphic to a segment. Then there exist $a, b, \widetilde{a}, \widetilde{b} \in \mathbb{C}$ such that $\widetilde{a}P(az+b) + \widetilde{b} = T_n(z)$, where $T_n(z)$ is the nth Chebyshev polynomial.*

Definition 1. The polynomial $P(z)$ is called *elliptic* if there exists a segment $[u_1, u_2]$ such that $P^{-1}[u_1, u_2]$ is the union of two different sets (which may be intersecting) each of which is homeomorphic to a segment.

For definiteness we shall assume everywhere from now on that $[u_1, u_2] = [-1, 1]$.

Theorem 2. *A polynomial $P(z)$ of degree n is an elliptic polynomial if and only if it satisfies the equation*

$$P^2(z) - \left(\frac{P'(z)}{n(z-x)}\right)^2 R(z) = 1, \tag{1}$$

where $R(z)$ is a 4th degree polynomial with leading coefficient 1 and without multiple roots, and x is a certain complex number.

We note that the roots of $R(z) = (z-\alpha)(z-\beta)(z-\gamma)(z-\delta)$ are precisely the coordinates of the vertices of odd degree of the graph G_p.

Definition 2. An elliptic polynomial is called *primitive* if it cannot be put in the form of a composition $\pm T_d\big(Q(z)\big)$, where $d, \deg Q(z) > 1$.

Definition 3. Two polynomials $P(z)$ and $Q(z)$ are called (linearly) *equivalent* if $a, b \in \mathbb{C}$ exist such that $P(z) = \pm Q(az+b)$.

Suppose that we have an equivalence class of elliptic polynomials of degree n. We choose representations $P(z)$ in it so that $\alpha+\beta+\gamma+\delta=0$ and suppose that

$$R(z)=(z-\alpha)(z-\beta)(z-\gamma)(z-\delta)=z^4-6Az^2+4Bz+C.$$

From the numbers A, B, C we now construct the elliptic curve

$$L_p:\ w^2=4v^3-g_2v-g_3,\quad g_2=3A^2+C,\quad g_3=-AC+A^3-B^2,$$

and the point $N_p=(v,w)=(A,B)$ on it. If $\widetilde{P}(z)$ is another representation in the same class for which $\alpha+\beta+\gamma+\delta=0$, then $\widetilde{P}(z)=\pm P(\lambda z)$, where $\lambda\in\mathbb{C}^\times$, and hence $\widetilde{R}(z)=z^4-6A\lambda^2z^2+4B\lambda^3z+C\lambda^4$. Therefore, as we easily verify, the curve $L_{\widetilde{p}}$ is isomorphic to the curve L_p and the point $N_{\widetilde{p}}$ in this isomorphism goes into the point N_p. Thus we have a well-defined map F_n from the set of equivalence classes of elliptic polynomials of degree n into the set of isomorphism classes of elliptic curves with a distinguished point.

We note that the map

$$(x,y)=\left(\frac{1}{2}\left(\frac{B+w}{A-v}\right),2v+A-\frac{1}{4}\left(\frac{B+w}{A-v}\right)^2\right)$$

establishes a birational isomorphism between the curve L_p and the curve $y^2=R(x)$. We can therefore consider the pair (L_p,N_p) as an elliptic curve $y^2=R(x)$ together with the point at infinity.

Theorem 3. *The map F_n effects a bijective correspondence between the set V_n of equivalence classes of primitive elliptic polynomials of degree n and the set W_n of isomorphism classes of pairs (L,N), where L is an elliptic curve over $\mathbb{C}$ and N is a point of proximate order n on it.*

3. In similar fashion to Chebyshev polynomials, elliptic polynomials under certain natural restrictions possess interesting extremal properties.

Theorem 4. *Let $P(z)=a_nz^n+\cdots+a_1z+a_0$ be an elliptic polynomial such that $R(z)=(z-\alpha)(z-\beta)(z-\gamma)(z-\delta)$ has only real roots, $\alpha<\beta<\gamma<\delta$. Then among all the real polynomials of degree n with leading coefficient 1 the polynomial $\widetilde{P}(z)=P(z)/a_n$ has the least deviation from 0 on the set $[\alpha,\beta]\cup[\gamma,\delta]$. The absolute value of the deviation is equal to $1/|a_n|$.*

Example. Consider the third-order point $(v,w)=(3,4)$ on the elliptic curve $w^2=4v^3-60v+88$. We can easily verify that the polynomials in (1) take the forms:

$$R(z)=z^4-18z^2+16z+33,\qquad P(z)=\frac{1}{8}(z^3+3z^2-9z-19),\qquad x=1.$$

Now, using Theorem 4, we deduce that, among all the real polynomials of degree 3 with leading coefficient 1, the polynomial $\widetilde{P}(z)=z^3+3z^2-9z-19$ deviates least from zero on the set $\left[-1-2\sqrt{3},-1\right]\cup\left[-1+2\sqrt{3},3\right]$. The absolute value of the deviation is 8.

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Remark. After completion of this paper, A. P. Veselov drew the author's attention to the fact that the connection between polynomials satisfying (1) and isomorphism classes of elliptic curves with a distinguished point of finite order has already been studied in [6] from a somewhat different point of view.

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